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COMPOSITES OF EFFECTIVE DESCENT MAPS

by Jan REITERMAN +, *Manuela SOBRAL* & Walter THOLEN**

Résumé. Etant donné un morphisme $q : E \to X$ de descente effective dans une catégorie avec des produits fibrés, nous montrons que les données de descente par rapport à n'importe quel morphisme $\varphi : X \to B$ sont equivalentes aux données de descente par rapport au composé φq. On en déduit que la classe des morphismes de descente effective est fermée sous l'action de la composition.

Abstract

For an effective descent morphism $q : E \to X$ of a category C with pullbacks and coequalizers it is shown that descent data with respect to any morphism $\varphi : X \to B$ are equivalent to descent data with respect to the composite φq. Consequently, the class of effective descent morphisms is closed under composition.

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Keywords: descent data, effective descent map, universal regular epimorphism

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Introduction

It was shown in [S-T] that, in a category C with pullbacks and coequalizers, the class $\mathcal{E}$ of effective descent maps is stable under pullback. Furthermore, closedness of $\mathcal{E}$ under composition was derived under the following pullback condition:

(P) pullbacks of regular epimorphisms along $\mathcal{E}$-morphisms are regular epimorphisms.

In fact, with the help of (P) it was shown that certain general sufficient conditions guaranteeing that the composite of monadic functors be monadic are satisfied. Since, in general, composites of monadic functors are far from being monadic (see, for example, **Cat** $\rightarrow$ **Graph** $\rightarrow$ **Set**), it seemed plausible that additional conditions, like (P), are needed to obtain that $\mathcal{E}$ be closed under composition. However, condition (P) is certainly not necessary for $\mathcal{E}$ being closed under composition: one can check quite easily that (P) fails in **Top** (see Remark (2) of Section 4 below), yet from the criteria for effective descent maps in **Top** given in [R-T] it follows immediately that $\mathcal{E}$ is closed under composition. Hence the problem was stated in [S-T] whether there is any (decent) category in which $\mathcal{E}$ fails to be closed under composition. In this paper we prove that *the class $\mathcal{E}$ is closed under composition in every category C with pullbacks and coequalizers*, thus give a straight "No" answer to the problem stated in [S-T].

The composition theorem for effective descent maps is derived here from a more general result:

Invariance Theorem For Descent Data. *Let $q : E \rightarrow X$ be an effective descent map of a category C with pullbacks and coequalizers. Then the category* $\mathrm{Des}(\varphi)$ *of descent data with respect to any map $\varphi : X \rightarrow B$ is canonically equivalent to the category* $\mathrm{Des}(\varphi q)$.

Although we present here a "direct" proof of the Invariance Theorem, we wish to point out that the idea for its proof occurred to us only after viewing $\mathrm{Des}(\varphi)$ as the category $\mathcal{C}^{\mathrm{Eq}(\varphi)}$ of internal category actions with respect to the $\mathcal{C}$-category $\mathrm{Eq}(\varphi)$ (i.e., the equivalence relation defined by φ), not as the Eilenberg-Moore category with respect to the change-of-base functor $\varphi^* : \mathcal{C}/B \to \mathcal{C}/X$. We refer the interested reader to the papers [J-T2] and [J-T3] for a more detailed description of these different aspects of descent data.

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2. Preliminaries

Let $p : E \to B$ be a morphism of a category $\mathcal{C}$ with pullbacks. *Descent data* for a $\mathcal{C}$-object $(C, \gamma : C \to E)$ over E are given by a morphism ξ in $\mathcal{C}$ which makes the diagrams

$$\begin{tikzcd}
C \arrow[r, "\langle \gamma, 1_C \rangle"] \arrow[d, "1_C"'] & E \times_B C \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] \arrow[dl, "\xi"'] & C \arrow[d, "p\gamma"] \\
C \arrow[r, "\gamma"'] & E \arrow[r, "p"'] & B
\end{tikzcd} \tag{1}$$

$$\begin{tikzcd}
E \times_B (E \times_B C) \arrow[r, "1_E \times_B \xi"] \arrow[d, "1_E \times_B \pi_2"'] & E \times_B C \arrow[d, "\xi"] \\
E \times_B C \arrow[r, "\xi"'] & C
\end{tikzcd} \tag{2}$$

commute. These are the objects of the category $\mathrm{Des}(p)$ a morphism $h : (C, \gamma; \xi) \to (C', \gamma'; \xi')$ of which is a $\mathcal{C}$-morphism which makes

$$\begin{tikzcd}
E\times_B C \arrow[rr, "1_E\times_B h"] \arrow[d, "\xi"'] & & E\times_B C' \arrow[d, "\xi'"] \\
C \arrow[rr, "h"] \arrow[dr, "\gamma"'] & & C' \arrow[dl, "\gamma'"] \\
& E &
\end{tikzcd} \tag{3}$$

commute. There is a *comparison functor*

$$\Phi^p : \mathcal{C}/B \to \mathrm{Des}(p)$$

with $\Phi^p(A, \alpha : A \to B) = (E \times_B A, \pi_1; 1_E \times_B \pi_2)$. The morphism p is called a *descent morphism* if Φ^p is full and faithful, and it is an *effective descent morphism* or *effective for descent* in $\mathcal{C}$ if Φ^p is an equivalence of categories. It is well known that p is a descent morphism if and only if p is a *universal regular epimorphism* of $\mathcal{C}$, i.e., if every pullback of p is a regular epimorphism (as defined in [K]). For details, see [J-T1] and [J-T2].

It was an important observation of Bénabou and Roubaud [B-R] and of Beck (unpublished) that the category $\mathrm{Des}(p)$ is the *Eilenberg-Moore category* of the monad induced by the change-of-base functor $p^* : \mathcal{C}/B \to \mathcal{C}/E$ and its left adjoint $p_!$, even in the context of an arbitrary bifibred category satisfying the Beck-Chevalley Condition (for details, see [P1], [P2], [J-T2]). However, for our purposes it is more helpful to view $\mathrm{Des}(p)$ as the *category of actions of the internal category* $\mathrm{Eq}(p)$ *in* $\mathcal{C}$, i.e.,

$$\mathrm{Des}(p) = \mathcal{C}^{\mathrm{Eq}(p)}.$$

Here $\mathrm{Eq}(p)$ is given by the diagram

$$\begin{tikzcd}
(E\times_B E)\times_E(E\times_B E) \arrow[r, "\pi_{1,4}"] & E\times_B E \arrow[r, shift left=3, "p_2"] \arrow[r, shift right=3, "p_1"'] & E \arrow[l, "\Delta" description]
\end{tikzcd}$$

so that the projections p_2 and p_1 play the role of domain and codomain while $\pi_{1,4}$ and $\Delta_E = \langle 1_E, 1_E \rangle$ give composition and identity operation, respectively. The point is that the comparison functor Φ^p is nothing but the *composition functor*

$$\mathcal{C}^{(p,p')} : \mathcal{C}^{\mathrm{Eq}(1_B)} \to \mathcal{C}^{\mathrm{Eq}(p)}$$

induced by the obvious internal functor $(p, p') : \mathrm{Eq}(p) \to \mathrm{Eq}(1_B)$, with $p' = pp_1 = pp_2$. Similarly, the forgetful functor $U^p : \mathrm{Des}(p) \to \mathcal{C}/E$ is induced by $(1_E, \Delta_E) : \mathrm{Eq}(1_E) \to \mathrm{Eq}(p)$. For details, see [J-T3].

3. The Invariance Theorem

We consider morphisms $q : E \to X$ and $\varphi : X \to B$ in $\mathcal{C}$ and compare descent data w.r.t. q with descent data w.r.t. $p = \varphi q$, as follows:

Proposition *There are canonical functors V^φ and Π^q which make the diagram*

$$\begin{tikzcd}
 & \mathcal{C}/B \arrow[dl, "\Phi^\varphi"'] \arrow[dr, "\Phi^p"] & \\
\mathrm{Des}(\varphi) \arrow[rr, "\Pi^q"] \arrow[d, "U^\varphi"'] & & \mathrm{Des}(p) \arrow[d, "V^\varphi"] \\
\mathcal{C}/X \arrow[rr, "\Phi^q"] & & \mathrm{Des}(q)
\end{tikzcd} \tag{4}$$

commute in CAT, *up to canonical isomorphism.*

Proof. Diagram (4) is induced by the following commutative diagram of internal functors.

$$\begin{tikzcd}
& \mathrm{Eq}(1_B) & \\
\mathrm{Eq}(\varphi) \arrow[ur, "{(\varphi,\varphi')}"] & & \mathrm{Eq}(p) \arrow[ul, "{(p,p')}"'] \arrow[ll, "{(q,q\times_B q)}"'] \\
\mathrm{Eq}(1_X) \arrow[u, "{(1_X,\Delta_X)}"] & & \mathrm{Eq}(q) \arrow[u, "{(1_E,1_E\times_\varphi 1_E)}"'] \arrow[ll, "{(q,q')}"']
\end{tikzcd} \tag{5}$$

From the explicit description of the pseudofunctor $\mathbf{cat}(\mathcal{C}) \to \mathbf{CAT}$, $\Box \mapsto$ $\mapsto \mathcal{C}^{\Box}$ ([J-T3]), one obtains an explicit description of the functors V^{φ} and Π^{q}, as follows:

$$\begin{array}{rcl}
V^{\varphi}:\ \mathrm{Des}(p) & \to & \mathrm{Des}(q) \\
(C,\gamma;\xi) & \mapsto & (C,\gamma;\xi(1_E\times_\varphi 1_C)) \\
h & \mapsto & h
\end{array}$$

$$\begin{array}{rcl}
\Pi^{q}:\ \mathrm{Des}(\varphi) & \to & \mathrm{Des}(p) \\
(D,\delta;\zeta) & \mapsto & (E\times_X D,\pi_1;\xi) \\
g & \mapsto & 1_E\times_X g
\end{array}$$

with $\xi = \langle\hat{\pi}_1, \zeta(q\times_B \pi_2)\rangle$ given by the following commutative diagram:

$$\begin{tikzcd}
E\times_B(E\times_X D) \arrow[rr, "q\times_B \pi_2"] \arrow[dr, "\xi"] \arrow[dd, "\hat{\pi}_1"'] & & X\times_B D \arrow[d, "\zeta"] \\
& E\times_X D \arrow[r, "\pi_2"] \arrow[dl, "\pi_1"] & D \arrow[d, "\delta"] \\
E \arrow[rr, "q"] & & X
\end{tikzcd} \tag{6}$$

With $C := E \times_X D$ one has a decomposition of $q \times_B \pi_2$ as

$$E \times_B C \xrightarrow{q \times_B 1_C} X \times_B C \xrightarrow{1_X \times_B \pi_2} X \times_B D.$$

The first factor is a pullback of q, and so is the second, as a pullback of π_2 which is itself a pullback of q. Hence, if q is a descent morphism and therefore a universal regular epimorphism, then $q \times_B \pi_2$ is the composite of two universal regular epimorphisms in $\mathcal{C}$ and therefore a (universal) regular epimorphism (this latter fact follows from Lemma 1.7 of [G-U]; see also Section 4 below). With this observation we are ready to prove the main result of the paper:

Theorem *The functor* Π^q : Des)(φ) $\to$ Des(p) *is full and faithful if q is a descent morphism. It is an equivalence of categories if q is an effective descent morphism.*

Proof Clearly, faithfulness of the functor Φ^q implies the same properly for Π^q. So we must show only that Π^q is full if Φ^q is full. For that one considers a morphism

$$h : \Pi^q(D, \delta; \zeta) \to \Pi^q(D', \delta'; \zeta')$$

in Des(p) and obtains a morphism $g : (D, \delta) \to (D', \delta')$ in $\mathcal{C}/X$ with

$$\Phi^q(g) = 1_E \times_X g = h = V^\varphi(h) : \Phi^q(D, \delta) \to \Phi^q(D', \delta').$$

We claim that

$$g : (D, \delta; \zeta) \to (D', \delta'; \zeta')$$

is a map of Des(φ) so that then $\Pi^q(g) = h$. With $C := E \times_X D$ and $\xi := \langle \hat{\pi}_1, \zeta(q \times_B \pi_2) \rangle$ (and similarly for the primed data), we must show that if the frontface of

$$\begin{tikzcd}
& X \times_B D \arrow[rr, "1_X \times_B g"] \arrow[dd, "\zeta"] & & X \times_B D' \arrow[dd, "\zeta'"] \\
E \times_B C \arrow[ur, "q \times_B \pi_2"] \arrow[rr, "1_E \times_B h"] \arrow[dd, "\xi"'] & & E \times_B C' \arrow[ur, "q \times_B \pi_2'"] \arrow[dd, "\xi'"] & \\
& D \arrow[rr, "g"] & & D' \\
C \arrow[ur, "\pi_2"] \arrow[rr, "h"'] & & C' \arrow[ur, "\pi_2'"'] &
\end{tikzcd}$$

commutes, then also the backface commutes. But this is obvious since $q \times_B \pi_2$ is epic.

To complete the proof, we must show that Π^q is surjective on objects up to isomorphism if Φ^q is an equivalence of categories. For an object $(C, \gamma; \xi) \in \mathrm{Des}(p)$ we find, by hypothesis, an object $(D, \delta) \in \mathcal{C}/X$ with

$$\Phi^q(D, \delta) = (E \times_X D, \pi_1; 1_E \times_X \pi_2) \cong (C, \gamma; \xi(1_E \times_\varphi 1_C)) = V^\varphi(C, \gamma; \xi).$$

Without loss of generality, we may assume strict equality here. From the left face of (7) we see that we should construct a morphism ζ with

$$(*) \qquad\qquad \zeta(q \times_B \pi_2) = \pi_2 \xi$$

such that $(D, \delta; \zeta)$ becomes an object of $\mathrm{Des}(\varphi)$. Then necessarily $\Pi^q(D, \delta; \zeta) = (C, \gamma; \xi)$, and the proof will be complete.

But $q \times_B \pi_2$ is a regular epimorphism so that it suffices to show that

$$(**) \qquad\qquad (q \times_B \pi_2)x = (q \times_B \pi_2)y \text{ implies } \pi_2 \xi x = \pi_2 \xi y$$

for all $x, y : T \to E \times_B C$, in order to obtain ζ with $(*)$.

In order to show $(**)$, first we note that, by hypothesis one has $\gamma = \pi_1$ and $\xi t = 1_E \times_X \pi_2$, with $t := 1_E \times_\varphi 1_C$.

$$\begin{tikzcd}
 & C = E \times_X D \arrow[rrr, "\pi_2"] & & & D \arrow[dr, "\varphi\delta"] & \\
E \times_X C \arrow[ur, "\check{\pi}_2"] \arrow[dr, "\check{\pi}_1"'] \arrow[rr, "t"] & & E \times_B C \arrow[ul, "\hat{\pi}_2"'] \arrow[dl, "\hat{\pi}_1"] \arrow[r, "q \times_B \pi_2"] & X \times_B D \arrow[ur, "\tilde{\pi}_2"'] \arrow[dr, "\tilde{\pi}_1"] & & B \\
 & E \arrow[rrr, "q"] & & & X \arrow[ur, "\varphi"'] &
\end{tikzcd} \tag{7}$$

From the hypothesis of (**), $(q \times_B \pi_2)x = (q \times_B \pi_2)y$, one obtains

$$q\hat{\pi}_1 x = q\hat{\pi}_1 y \text{ and } \pi_2\hat{\pi}_2 x = \pi_2\hat{\pi}_2 y.$$

With $q\pi_1 = \delta\pi_2$, the second identity implies $q\pi_1\hat{\pi}_2 x = q\pi_1\hat{\pi}_2 y$ which gives the morphism

$$u = \langle \pi_1\hat{\pi}_2 y, \hat{\pi}_2 x\rangle : T \to E \times_X C.$$

From $\pi_1(1_E \times_X \pi_2)u = \check{\pi}_1 u = \pi_1\hat{\pi}_2 y$ and $\pi_2(1_E \times_X \pi_2)u = \pi_2\hat{\pi}_2 x = \pi_2\hat{\pi}_2 y$ one obtains the identity $(1_E \times_X \pi_2)u = \hat{\pi}_2 y$ which we shall use later. First we construct morphisms

$$a = \langle \hat{\pi}_1 x, tu\rangle, \; b = \langle \hat{\pi}_1 y, (1_E \times_B \xi)a\rangle : T \to E \times_B (E \times_B C).$$

Indeed,

$$p\hat{\pi}_1 x = \varphi q\hat{\pi}_1 x = \varphi q\hat{\pi}_1 y = p\hat{pi}_1 y = p\pi_1\hat{\pi}_2 y = p\check{\pi}_1 u = p\hat{\pi}_1 tu$$

and

$$p\hat{\pi}_1 y = p\hat{\pi}_1 x = p\overline{\pi}_1 a = p\hat{\pi}_1(1 \times_B \xi)a.$$

$$\begin{tikzcd}
E\times_B(E\times_B C) \arrow[r, "\overline{\pi}_2"] \arrow[dd, "\overline{\pi}_1"'] & E\times_B C \arrow[r, "\hat{\pi}_2"] \arrow[d, "\hat{\pi}_1"'] & C = E\times_X D \arrow[d, "p\pi_1"] \\
 & E \arrow[r, "p"] \arrow[d, "p"] & B \\
E \arrow[r, "p"] & B &
\end{tikzcd} \tag{8}$$

Now x, y factor jointly through a, b, respectively. In fact, since

$$\begin{aligned}
\hat{\pi}_1(1_E\times_B\hat{\pi}_2)a &= \overline{\pi}_1 a = \hat{\pi}_1 x, \\
\hat{\pi}_2(1_E\times_B\hat{\pi}_2)a &= \hat{\pi}_2\overline{\pi}_2 a = \hat{\pi}_2 tu = \check{\pi}_2 u = \hat{\pi}_2 x,
\end{aligned}$$

one has $(1_E\times_B\hat{\pi}_2)a = x$, and since

$$\begin{aligned}
\hat{\pi}_1(1_E\times_B\hat{\pi}_2)b &= \overline{\pi}_1 b = \hat{\pi}_1 y, \\
\hat{\pi}_2(1_E\times_B\hat{\pi}_2)b &= \hat{\pi}_2\overline{\pi}_2 b = \hat{\pi}_2(1_E\times_B\xi)a = \xi\overline{\pi}_2 a = \xi tu = (1_E\times_X\pi_2)u = \hat{\pi}_2 y,
\end{aligned}$$

one obtains $(1\times_B\hat{\pi}_2)b = y$. Finally we note that there is a morphism

$$v = \langle\hat{\pi}_1(1_E\times_B\xi)b,\ \hat{\pi}_2(1_E\times_B\xi)b\rangle : T \to E\times_X C$$

since

$$\begin{aligned}
q\hat{\pi}_1(1_E\times_B\xi)b &= q\overline{\pi}_1 b = q\hat{\pi}_1 y = q\hat{\pi}_1 x = q\overline{\pi}_1 a \\
&= q\hat{\pi}_1(1_E\times_B\xi)a = q\pi_1\xi\overline{\pi}_2 b = q\pi_1\hat{\pi}_2(1_E\times_B\xi)b.
\end{aligned}$$

It obviously satisfies the identity $tv = (1_E\times_B\xi)b$. Now (**) follows:

$$\begin{aligned}
\pi_2\xi x &= \pi_2\xi(1_E \times_B \hat{\pi}_2)a \\
&= \pi_2\xi(1_E \times_B \xi)a \quad \text{(from (2))} \\
&= \pi_2\xi\overline{\pi}_2 b \\
&= \pi_2\hat{\pi}_2(1_E \times_B \xi)b \\
&= \pi_2\check{\pi}_2 v \\
&= \pi_2(1_E \times_X \pi_2)v \\
&= \pi_2\xi t v \\
&= \pi_2\xi(1_E \times_B \xi)b \\
&= \pi_2\xi(1_E \times_B \hat{\pi}_2)b \quad \text{(from (2))} \\
&= \pi_2\xi y.
\end{aligned}$$

In order to complete the proof of the theorem, we must show that ζ gives descent data for (D, δ) with respect of φ. That is, we must check the commutativity of:

$$\begin{tikzcd}
D \arrow[r, "\langle\delta, 1_D\rangle"] \arrow[d, "1_D"'] & X \times_B D \arrow[d, "\tilde{\pi}_1"] \arrow[dl, "\zeta"'] \\
D \arrow[r, "\delta"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times_B (X \times_B D) \arrow[r, "1_X \times_B \zeta"] \arrow[d, "1_X \times_B \tilde{\pi}_2"'] & X \times_B D \arrow[d, "\zeta"] \\
X \times_B D \arrow[r, "\zeta"] & D
\end{tikzcd}
\quad (9)$$

In fact, since π_2 and $q \times_B \pi_2$ are epic, the commutativity of the left diagram follows from the commutativity of (1) and from $(*)$:

$$\begin{aligned}
\zeta\langle\delta, 1_D\rangle\pi_2 &= \zeta\langle q\pi_1, \pi_2\rangle = \zeta(q \times_B \pi_2)\langle\pi_1, 1_C\rangle \\
&= \pi_2\xi\langle\gamma, 1_C\rangle = \pi_2, \\
\delta\zeta(q \times_B \pi_2) &= \delta\pi_2\xi = q\pi_1\xi = \gamma\xi = q\hat{\pi}_1 = \tilde{\pi}_1(q \times_B \pi_2).
\end{aligned}$$

Finally, when composing $\xi(1_E \times_B \xi) = \xi(1_E \times_B \hat{\pi}_2)$ with π_2 from the left, we obtain

$$\zeta(q \times_B \pi_2)(1_E \times_B \xi) = \zeta(q \times_B \pi_2)(1_E \times_B \hat{\pi}_2).$$

But since

$$\begin{array}{lrcl} & (q \times_B \pi_2)(1_E \times_B \xi) & = & (1_X \times_B (\pi_2 \xi))(q \times_B 1_{E \times_B C}) \\ & & = & (1_X \times_B \zeta)(1_X \times_B (q \times_B \pi_2))(q \times_B 1) \\ & & = & (1_X \times_B \zeta)(q \times_B (q \times_B \pi_2)) \\ \text{and} & (q \times_B \pi_2)(1_E \times_B \hat{\pi}_2) & = & (q \times_B (\tilde{\pi}_2(q \times_B \pi_2)) \\ & & = & (1_X \times_B \tilde{\pi}_2)(q \times_B (q \times_B \pi_2)) \end{array}$$

with $q \times_B (q \times_B \pi_2)$ epic, this yields $\zeta(1_X \times_B \zeta) = \zeta(1_X \times_B \tilde{\pi}_2)$.□

4. The Composition and Cancellation Theorem

We denote by

- $\mathcal{S}$ the class of split epimorphisms
- $\mathcal{E}$ the class of effective descent morphisms
- $\mathcal{D}$ the class of descent morphisms
- $\mathcal{R}$ the class of regular epimorphisms,

in a category $\mathcal{C}$ with pullbacks. Then the inclusions

$$\mathcal{S} \subseteq \mathcal{E} \subseteq \mathcal{D} \subseteq \mathcal{R}$$

hold and are, in general, proper (for $\mathcal{S} \subseteq \mathcal{E}$, see [J-T1], [J-T3]; for properness of $\mathcal{E} \subseteq \mathcal{D}$ in case $\mathcal{C}=$ **Top**, see [R-T]).

Let now

$$p = (E \xrightarrow{q} X \xrightarrow{\varphi} B)$$

be any morphisms in $\mathcal{C}$. Trivially one has $(q, \varphi \in \mathcal{S} \Rightarrow p \in \mathcal{S})$ and $(p \in \mathcal{S} \Rightarrow \varphi \in \mathcal{S})$, but in neither rule can one trade $\mathcal{S}$ for $\mathcal{R}$, in general. However, one still has the composition rule $(q \in \mathcal{S}, \varphi \in \mathcal{R} \Rightarrow p \in \mathcal{R})$ and the cancellation rule $(p \in \mathcal{R}, q$ epic $\Rightarrow \varphi \in \mathcal{R})$, see [K]. It was observed in [B] that one even has the strict cancellation rule $(p \in \mathcal{R} \Rightarrow \varphi \in \mathcal{R})$ provided the composition rule $(q \in \mathcal{R}, \varphi \in \mathcal{S} \Rightarrow p \in \mathcal{R})$ holds in the category, but that even under this hypothesis $\mathcal{R}$ may fail to be closed under composition.

By contrast, the class $\mathcal{D}$ of universal regular epimorphisms behaves very decently. It was shown in [G-U] (Lemma 1.6) that if q is a regular epimorphism whose pullbacks are epic, then its composite φq with $\varphi \in \mathcal{R}$ still belongs to $\mathcal{R}$. Together with the cancellation rule $(p \in \mathcal{R}, q$ epic $\Rightarrow \varphi \in \mathcal{R})$, this immediately proves the $\mathcal{D}$-version of the following Theorem. Its $\mathcal{E}$-version follows from the Invariance Theorem and its Proposition.

Theorem *In a category with pullbacks both the class $\mathcal{D}$ of descent maps and the class $\mathcal{E}$ of effective descent maps satisfy the following composition-cancellation rule: given a composite $p = \varphi \cdot q$, then*

(a) $$if\ q \in \mathcal{D}\ then\ p \in \mathcal{D} \Leftrightarrow \varphi \in \mathcal{D}$$

and

(b) $$if\ q \in \mathcal{E}\ then\ p \in \mathcal{E} \Leftrightarrow \varphi \in \mathcal{E}. \ \square$$

Remark $\mathcal{S}$ and $\mathcal{D}$ are trivially stable under pullback. It was shown in [S-T] that also $\mathcal{E}$ is stable under pullback, provided $\mathcal{C}$ has pullbacks and coequalizers. Of course, in general $\mathcal{R}$ is not stable under pullback, not even stable under pullback along $\mathcal{S}$ morphisms: there are quotient maps p in **Top** and spaces Z for which $p \times 1_Z$ fails to be a quotient map, see [D-K]. This shows in particular that $\mathcal{R}$ is not stable under pullback along $\mathcal{E}$-morphism, i.e., that condition (P) of the Introduction fails in **Top**.

5. Relativization of results with respect to a class E

Let E be a class of morphisms stable under pullback in C. We may then consider the E-modification of the descent problem, given by the corresponding subfibration of the basic fibration (cf. [J-T2]): one considers the full subcategories $\mathsf{E}(B)$ of C/B and $\mathrm{Des}_{\mathsf{E}}(p)$ of $\mathrm{Des}(p)$ of objects (C,γ) and $(C,\gamma;\xi)$ with $\gamma \in \mathsf{E}$, respectively; then $p : E \to B$ is an (effective) E-descent morphism if the restricted comparison functor

$$\Phi^p : \mathsf{E}(B) \to \mathrm{Des}_{\mathsf{E}}(p)$$

is full and faithful (an equivalence of categories). Now, an easy inspection of the proof of the Invariance Theorem of Section 3 yields that it remains true verbatim, with the obvious restricted functor

$$\Pi^p : \mathrm{Des}_{\mathsf{E}}(\varphi) \to \mathrm{Des}_{\mathsf{E}}(p),$$

and with (effective) E-descent instead of (effective) descent. This then gives the following relativised composition-cancellation rule:

Corollary *Let $q : E \to X$ be a descent morphism and an effective* E*-descent morphism of C. Then any morphism $\varphi : X \to B$ is an effective* E*-descent morphism if and only if the composite φq is an effective* E*-descent morphism.* □

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