HORST HERRLICH

On the failure of Birkhoff's theorem for locally small based equational categories of algebras

Cahiers de topologie et géométrie différentielle catégoriques, tome 34, n° 3 (1993), p. 185-192
<http://www.numdam.org/item?id=CTGDC_1993__34_3_185_0>

CAHIERS DE TOPOLOGIE
ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

VOL. XXXIV-3 (1993)

ON THE FAILURE OF BIRKHOFF'S THEOREM FOR LOCALLY SMALL BASED EQUATIONAL CATEGORIES OF ALGEBRAS

by Horst HERRLICH

Dedicated to the memory of Jan Reiterman, who left us far too early

Résumé

Dans la catégorie **Alg**(P) des P-algèbres (oú P: Set $\longrightarrow$ **Set** est le foncteur 'parties' covariant), on construit une sous-catégorie de Birkhoff qui n'est pas équationnelle.[1]

Introduction

As in [1] a foundation distinguishing between sets, classes and conglomerates will be adapted.

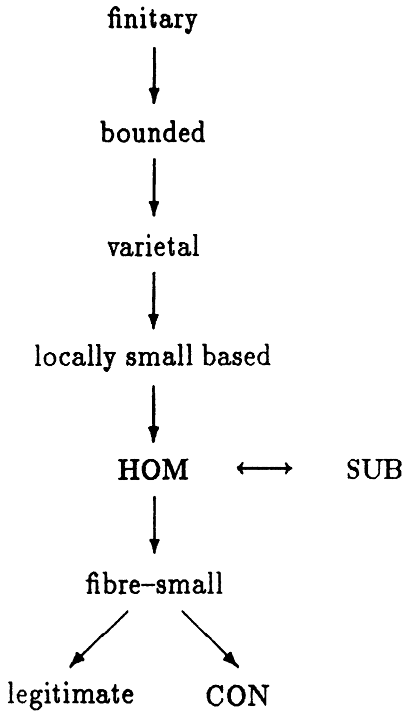
A pair (Ω, E) consisting of a family $\Omega = (k_i)_{i \in I}$ of sets k_i (or ordinals, or cardinals – whatever the reader prefers), indexed by a class I, and a conglomerate E of Ω-equations is called an **equational theory**; and the corresponding quasicategory $\mathbf{Alg}(\Omega, E)$, consisting of those Ω-algebras that satisfy the equations in E, is called (by abuse of language) an **equational category**. Disregarding the foundational problems involved, these concepts have seen the light of day in the fundamental paper [3] by Birkhoff. The **finitary** case, i.e., the case where I is required to be a set and each k_i is required to be finite, has since grown into a special branch of mathematics, called universal algebra. Many

[1]Math.Subj.Class. (1991): 03C05, 08A65, 08C05, 18C05.
Key Words: Equational categories of algebras, Birkhoff subcategories.

results remain true in the **bounded** case, where I is required to be a set (Słominski [22]). A more striking observation is due to Linton [11], who realized that the **varietal** case, i.e., the case where the forgetful functor $\mathbf{Alg}(\Omega, E) \longrightarrow \mathbf{Set}$ is required to be adjoint, still allows a sufficiently rich theory (see, e.g., Manes [12]). The categories **HComp** of compact Hausdorff spaces, **Fram** of frames, and **JCPos** of complete lattices and join–preserving maps are prime examples of equational categories that are varietal but not bounded. Unfortunately, the equational categories **CBoo** of complete Boolean algebras (Gaifman [4], Hales [6]), **CDLat** of complete distributive lattices (Garcia and Nelson [5]), $\mathbf{Alg}(F)$ of F–structured algebra for many functors $F\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ (Kůrková–Pohlová and Koubek [10]), and $\Lambda - \mathbf{JCPos}$ of **JCPos**–objects with one added unary operation (Reiterman [18]) fail to be varietal. (If, for a moment, the base category **Set** is replaced by the category **FSet** of finite sets (cf. Reiterman [16]), it becomes even more apparent that the concept of varietal equational categories is too narrow to cover all interesting cases: categories of finite groups and the like are not varietal over **FSet**). Such observations led Reiterman [13]–[18] and Rosický [19]–[21] to some penetrating studies of equational theories and categories in a more general realm. However, the general concept of equational categories itself, though useful as a frame for more restricted notions, is quite obviously too wide. At least, $\mathbf{Alg}(\Omega, E)$ should be required to be **legitimate**, i.e., isomorphic to a category (equivalently: $\mathbf{Alg}(\Omega, E)$ should not be bigger than a proper class; equivalently: none of the fibres of $\mathbf{Alg}(\Omega, E)$ should be bigger than a proper class). Slightly more restrictive, however still too general, is the requirement that $\mathbf{Alg}(\Omega, E)$ be **fibre–small**. In [13] Reitermann introduced smallness conditions

LSB (= locally small based), HOM, SUB, and CON

to make precise the idea that algebras, resp. homomorphisms, resp. subalgebras, resp. congruence relations can be described "locally" by sets of operations only. In particular, he demonstrated that between these concepts the following implications (and no others) hold:



He suggested that "reasonable" equational theories and categories ought to be **locally small based** and demonstrated that all the above examples of nonvarietal equational categories are locally small based. He demonstrated also that such phenomena as "operational stability", "equational completeness " and "canonical algebraicity" do not carry over from varietal theories to locally small based ones (Reiterman [18]).

The perhaps most famous result in universal algebra is Birkhoff's theorem, which states that a full subcategory of a finitary equational category $\mathrm{Alg}(\Omega, E)$ is equational, i.e., describable by means of Ω-equations; if and only if it is a **Birkhoff–subcategory**, i.e., closed under the formation of products, subalgebras and homomorphic images (Birkhoff [3]). The "only if" part trivially carries over to arbitrary equational categories. The "if" part carries over to the bounded case (Słominski [22]) and even to the varietal case. However, the Birkhoff theorem fails for locally small based categories (Reiterman [15], Example 4.6). Unfortunately, Reiterman's presentation of his example is rather sketchy and it is quite cumbersome to follow his verifications. Much simpler counterexamples exist in the general case (Rosický [19], Example 6.1, Herrlich [7]). In these examples all the operations are unary (Rosický), resp. nullary (Herrlich). Such simple examples don't exist for locally small based categories, since a locally small based equational theory, in which the arities (considered as cardinals) have a common upper bound, must be varietal (Reiterman [13], III 1. Lemma). It is the author's hope that the following example of a non-equational Birkhoff subcategory of the category **Alg**(P) of P-algebras is somewhat easier to understand than the above-mentioned example by Reiterman.

Example

Consider the equational theory (Ω, E), where

(a) $\Omega = (M)_{M \in U}$ with U being the universe (= class of all sets), and

(b) $E = \{E_f \mid M \xrightarrow{f} N$ is a surjective function between sets$\}$ with E_f being the equation $\qquad \omega_N((x_n)_{n \in N}) \approx \omega_M((x_{f(m)})_{m \in M})$.

The above equations express the fact that $\omega_N((x_n)_{n\in N})$ depends only on the set $\{x_n \mid n \in N\}$. Thus the concrete quasicategory $\mathbf{Alg}(\Omega, E)$ is concretely isomorphic to the construct $\mathbf{Alg}(P)$, where $P\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ is the covariant power-set functor. Consequently (Ω, E) and $\mathbf{Alg}(\Omega, E)$ are not only legitimate but even locally small based (cf. Reiterman [18], 3.1), but obviously fail to be varietal. Via transfinite induction one can define derived 0-ary operations ψ_α for all ordinals α as follows:

$$\psi_\alpha = \begin{cases} \omega_1(\psi_\beta) & \text{, provided that } \beta \text{ is a direct predecessor of } \alpha, \\ \omega_\alpha((\psi_\beta)_{\beta<\alpha}) & \text{, provided that } \alpha \text{ has no direct predecessor.} \end{cases}$$

Let $\mathbf{A}$ be the full subcategory of $\mathbf{Alg}(\Omega, E)$, consisting of those algebras, whose derived nullary operations ψ_α satisfy the following condition:

$$\exists \alpha \ \ \forall \beta \geq \alpha \ \ \psi_\beta = \psi_\alpha.$$

Clearly **A** is a Birkhoff subcategory of **Alg**(Ω, E). To show that **A** is not equational in **Alg**(Ω, E) the following will be proved:

Claim: If an Ω–equation $t \approx t'$ is satisfied by all **A**–objects, then it is satisfied by all **Alg**(Ω, E)–objects.

Assume that an Ω–equation $t \approx t'$ is not satisfied by some **Alg**(Ω, E)–object. Since each of the terms t and t' involves only a set of operation symbols, there exists a regular cardinal number γ such that no operation symbol ω_M with $\gamma \leq \mathrm{Card}M$ occurs in t or t'. Consider the equational theory $(\Omega_\gamma, E_\gamma)$, where

(a) $\Omega_\gamma = (M)_{M \in U_\gamma}$ with U_γ being the class of all sets M with $\mathrm{Card}M < \gamma$, and

(b) $E_\gamma = \{E_f \mid M \xrightarrow{f} N$ is a surjective function between members of $U_\gamma\}$ and the E_f are defined as before.

Let $P_\gamma\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ be the subfunctor of the power–set functor that associates with every set X the set $P_\gamma X = \{Y \in PX \mid \mathrm{Card}Y < \gamma\}$. Then **Alg**$(\Omega_\gamma, E_\gamma)$ is varietal and concretely isomorphic to **Alg**(P_γ). Let $(A, (\omega_M)_{M \in U})$ be an **Alg**(Ω, E)–object that does not satisfy the Ω–equation $t \approx t'$. Then $(A, (\omega_M)_{M \in U_\gamma})$ is an **Alg**$(\Omega_\gamma, E_\gamma)$– object that does not satisfy the Ω_γ–equation $t \approx t'$. Thus there is a free **Alg**$(\Omega_\gamma, E_\gamma)$–algebra $(F, (\varphi_M)_{M \in U_\gamma})$ that does not satisfy $t \approx t'$. Add an element ∞ to F and define an **Alg**(Ω, E)–object $B = (F \cup \{\infty\}, (\bar{\varphi}_M)_{M \in U})$ as follows:

$$\bar{\varphi}_M((x_m)_{m \in M}) = \begin{cases} \varphi_N((n)_{n \in N}) & \text{, if } \infty \notin N = \{x_m \mid m \in M\} \in U_\gamma \\ \infty & \text{, otherwise} \end{cases}$$

Then B does not satisfy the equation $t \approx t'$. However, by construction, the map $\psi\colon \mathrm{Ord} \longrightarrow F \cup \{\infty\}$, which associate with every ordinal α the α–th derived 0–ary operation ψ_α of B, has the following properties

(a) ψ, restricted to the set $\{\alpha \in \mathrm{Ord} \mid \alpha < \gamma\}$ of all ordinals less than γ, is injective, and

(b) ψ, restricted to the class $\{\alpha \in \mathrm{Ord} \mid \gamma \leq \alpha\}$, is constant with value ∞.

Thus B is an **A**–object that fails to satisfy the equation $t \approx t'$.
This proves the above claim. Consequently **A** is not equational in **Alg**(Ω, E). In fact: the equational hull of **A** in **Alg**(Ω, E) is **Alg**(Ω, E) itself.

Remarks:

1. The arguments given above provide a new (and as it occurs to me simpler) proof of the fact that the Birkhoff subcategory exhibited by Reiterman [15] is not equational.

2. Every Birkhoff subcategory of **Alg**(Ω, E) is regular epireflective in **Alg**(Ω, E) and thus **implicational**, i.e., definable by Ω–**implications** (Banaschewski and Herrlich [2]). In the general case an illegitimate collection of Ω–implications may be necessary (Herrlich [7]), but for legitimate equational categories **Alg**(Ω, E) a class of Ω–implications suffices. In the example discussed above one may choose the following particularly simple Ω–implications Imp(α, β) for each ordinal α and each limit ordinal β with $\alpha < \beta$:

$$\text{Imp}(\alpha, \beta) \qquad\qquad \text{If } \psi_\alpha = \psi_\beta, \text{ then } \psi_\alpha = \psi_{\alpha+1}.$$

References

[1] J. Adámek, H. Herrlich and G.E. Strecker, Abstract and Concrete Categories, Wiley 1990

[2] B. Banaschewski and H. Herrlich, Subcategories defined by implications, Houston J. Math **2** (1976) 149–171

[3] G. Birkhoff, On the structure of abstract algebras, Proc. Cambr. Phil. Soc. 31 (1935) 433–454

[4] H. Gaifman, Infinite Boolean polynomicals I, Fund. Math. 54 (1964) 230–250

[5] O. Garcia and E. Nelson, On the non–existence of free complete distributive lattices, Order **1** (1985) 399–403

[6] A.W. Hales, On the non–existence of free complete Boolean algebras, Fund. Math. **54** (1964) 45–66

[7] H. Herrlich, Remarks on categories of algebras defined by a proper class of operations, Quaestiones Math. **13** (1990) 385–393

[8] J.R. Isbell, A note on complete closure algebras, Math. Syst. Theory **3** (1970) 310–312

[9] J.R. Isbell, General functorial semantics I, Amer. J. Math. **94** (1972) 535–596

[10] V. Kůrková–Pohlová and V. Koubek, When a generalized algebraic category is monadic, Comment. Math. Univ. Carolinae **15** (1974) 577–587

[11] F.E.J. Linton, Some aspects of equational categories, Proc. Conf. Cat. Algebra, La Jolla 1965, Springer 1966, 84–94

[12] E.G. Manes, Algebraic Theories, Springer 1976

[13] J. Reiterman, Large algebraic theories with small algebras, Bull. Austral. Math. Soc. **19** (1978) 371–380

[14] J. Reiterman, One more categorical model of universal algebra, Math. Z. **161** (1978) 137–146

[15] J. Reiterman, Structure–semantics relation for non–varietal equational theories, Alg. Univ. **10** (1980)8 399–401

[16] J. Reiterman, The Birkhoff theorem for finite algebras, Alg. Univ. **14** (1982) 1–10

[17] J. Reiterman, Algebraic theories and varieties of functor algebras, Fund. Math. **118** (1983) 59–68

[18] J. Reiterman, On locally small based algebraic theories, Comment. Math. Univ. Carolinae **27** (1986) 325–340

[19] J. Rosický, On algebraic categories, Coll. Math. Soc. J. Bolyai **29**, Univ. Algebra, Erztergom 1977, 663–690

[20] J. Rosický, Equational categories, Cahiers Topol. Geom. Diff. **22** (1981) 85–95

[21] J. Rosický, Varieties of infinitary universal algebras, Alg. Univ. **20** (1985) 123–126

[22] J. Słominski, The theory of abstract algebras with infinitary operations, Rozprawy Mat. 18 (1959)

Horst Herrlich
Feldhäuser Str. 69
28865 Lilienthal
Germany