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LOCAL TO GLOBAL PROPERTIES IN THE THEORY OF FIBRATIONS

by Peter I. BOOTH

Résumé A partir des résultats constatés dans la communication fondamentale de Dold "Partitions of Unity in the Theory of Fibrations" *Ann. of Math.* 1963, 223-255, nous tirons des conclusions générales au niveau de fibrations enrichies, et ainsi á bien des théories individuelles des fibrations. Quatre définitions raisonnables de fibrations enrichies se montrent équivalentes. On discute des fibrations avec sections en détail.

0 Introduction.

The results of [Do] concerning the covering homotopy property (CHP), the weak covering homotopy property (WCHP) and fibre homotopy equivalence (FHE) are basic to the theory of fibrations. Analagous results play equally important roles in various parallel theories including those of sectioned fibrations (= ex-fibrations), principal fibrations, and equivariant fibrations of various types.

In this paper we give a systematic procedure for developing Dold-type results in theories of fibrations, presenting a technique that reduces questions in other theories to questions covered by the results of Dold's paper. Such theories are specified by the categories $\mathcal{E}$ of enriched topological spaces in which their fibres are constrained to lie. For example Hurewicz fibrations have fibres in the category $\mathcal{T}$ of spaces and maps, whilst for sectioned fibrations they are in $\mathcal{T}^0$, the category of pointed spaces (spaces with base points) and pointed maps (base point preserving maps). If G is a topological monoid then principal G-fibrations have fibres in $G\mathcal{T}^{he}$, the category whose objects are G-spaces that are G-homotopy equivalent to G and whose morphisms are G-homotopy equivalences.

For each pair of objects P and Q in such a category $\mathcal{E}$ we will use $\mathcal{E}(P, Q)$ to denote the set of all $\mathcal{E}$-maps from P to Q (see section 1). Given that the maps $q: Y \to B$ and $r: Z \to C$ have fibres that are enriched spaces, we define and discuss (section 2) a functional space $Y \Box Z$ with underlying set

$\bigcup_{b\in B, c\in C} \mathcal{E}(q^{-1}(b), r^{-1}(c))$, and a functional projection $q\Box r: Y\Box Z \to B \times C$ It is a result of [Mo2], and our theorem 3.4, that q satisfies a suitably defined $\mathcal{E}$-covering homotopy property ($\mathcal{E}$CHP) if and only if $q\Box q$ satisfies the CHP. We give analogous results relating the $\mathcal{E}$WCHP and the $\mathcal{E}$FHE property to the WCHP and the FHE property, respectively, i.e. theorems 4.6 and 5.2, respectively. These are used to obtain generalizations, i.e. enriched versions, of theorems concerning the CHP [Do, thm. 4.8], WCHP [Do, thms. 5.12 and 6.4] and FHE [Do, thms. 3.3, 6.1 and 6.3], in our chapters 3, 4, 5 and 6. Our generalization of [Do, thm. 6.4], i.e. our theorem 6.3, asserts that four potential definitions of the $\mathcal{E}$-fibration concept are frequently equivalent. This result allows great flexibility in applications, in particular it will be very useful in a sequel to this paper [Bo2].

It is shown in section 7 that Hurewicz fibrations with closed cofibration sections are examples of $\mathcal{T}^0$-fibrations, and hence covered by our theory; section 8 gives a variety of other examples, including, of course, the basic theories of Dold, Hurewicz and principal fibrations. The list of specific cases that we discuss is, however, just a sample. Many sorts of diagrams of spaces and/or G-spaces are objects for appropriate enriched categories and hence generate associated theories of enriched fibrations.

We do not, of course, claim that our theory is comprehensive in the sense of covering every conceivable theory of fibrations. Any effort to produce such a theory would surely be a case of trying the impossible, for some topologist would always be liable to introduce a theory that sat just outside any such approach. Further, any attempted development along those lines would entail a considerable complication of our basic ideas, something that would certainly disguise the simplicity of the underlying concepts involved. Our work should rather be viewed as *a prototype for a minimal pattern of argument, one that can be used to develop a treatment of the passage from local to global properties for almost any notion of fibration or generalized fibration.* A case in point is the theory of equivariant fibrations. The author prefers to develop the analagous equivariant theory elsewhere, rather than including it here and thereby compromising on the straightforwardness of the basic relationships presented.

The $\mathcal{E}$CHP idea is, of course, the $\mathcal{F}$-fibration concept of [Ma]. Generalized versions of [Do, thms. 3.3 and 4.8], and of [Do, thm. 6.3] for the enriched CHP case, are given in [Ma], i.e. as theorems 1.5, 3.8 and 2.6 of that memoir, respectively. They are derived by methods quite different from ours. Some corresponding results for sectioned fibrations are developed in [E].

We find it convenient, for the remainder of this paper, to work with *compactly-generated spaces* or *cg-spaces* [V, section 5, example (ii)], i.e. spaces having the final (= weak) topology relative to all incoming maps from compact Hausdorff spaces. Any space in $\mathcal{T}$ can be *cg-ified*, i.e. retopologized as a cg-space, by giving it this final topology. Colimits of cg-spaces, such as quotients and topological sums, are themselves cg-spaces; limits such as subspaces and product spaces, as well as function spaces, must be cg-ified to ensure that they are spaces of this type. *All spaces referred to in the pages that follow should be assumed to be cg-spaces.* In particular the categories $\mathcal{T}$, $\mathcal{T}^0$ and $G\mathcal{T}^{he}$ should be taken as containing only such spaces.

A space B will be said to be *weak Hausdorff* if the diagonal subset $\Delta_B = \{(b,b)|b \in B\}$ is closed in the (of course cg-ified) product space $B \times B$. *All spaces denoted by B and used later should be assumed to be weak Hausdorff.* Hausdorff spaces are, of course, weak Hausdorff. Given a point $c \in B$, the continuity of the inclusion $B \to B \times B$, $b \to (b,c)$, where $b \in B$, ensures that $\{c\}$ is closed in B, and so weak Hausdorff spaces are T_1.

Given spaces X, Y and Z, $\mathcal{T}(Y,Z)$ will denote the function space of maps from Y to Z equipped with the (of course cg-ified) compact-open topology, and we have the following *exponential law* [V, thm. 3.6].

(0.1) There is a bijective correspondence between maps $f: X \times Y \to Z$ and maps $g: X \to \mathcal{T}(Y,Z)$, determined by the rule $f(x,y) = g(x)(y)$, where $x \in X$ and $y \in Y$.

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1 Enriched spaces and enriched overspaces.

Throughout this paper we will make use of a category $\mathcal{E}$, with objects that will be known as *enriched spaces* or *$\mathcal{E}$-spaces*, and morphisms as *$\mathcal{E}$-maps.* An *$\mathcal{E}$-homeomorphism* is an isomorphism in $\mathcal{E}$, i.e. an $\mathcal{E}$-map $f: P \to Q$ such that there is an $\mathcal{E}$-map $g: Q \to P$ with $gf = 1_P$ and $fg = 1_Q$.

A *category of enriched spaces* ($\mathcal{E}$, U) will consist of a category $\mathcal{E}$ and a faithful functor $U: \mathcal{E} \to \mathcal{T}$, which satisfy the condition that if P is an $\mathcal{E}$-space and $f': UP \to Q'$ is a homeomorphism into a space Q', then there is a unique $\mathcal{E}$-space Q and $\mathcal{E}$-homeomorphism $f: P \to Q$ such that $UQ = Q'$ and $Uf = f'$.

If $f: P \to Q$ is an $\mathcal{E}$-map then the spaces UP and UQ and the map $Uf: UP \to UQ$ will in future be denoted by P, Q and f respectively. This extends a familiar convention that applies to G-maps between G-spaces, where G is a topological group.

Let P be an $\mathcal{E}$-space. If $\{t\}$ is a singleton space (in practice t will belong to the unit interval I) there are canonical homeomorphisms:

$$P \times \{t\} \cong P \cong \{t\} \times P,$$

so $P\times\{t\}$ and $\{t\}\times P$ carry the structure of $\mathcal{E}$-spaces that are $\mathcal{E}$-homeomorphic to P. It will sometimes be convenient for us to identify, i.e. fail to distinguish between, such equivalent $\mathcal{E}$-spaces.

Let $f: P \to Q$ and $g: P \to Q$ be $\mathcal{E}$-maps. Then an *$\mathcal{E}$-homotopy* from f to g is a homotopy $F: P \times I \to Q$ such that $F_0 = f, F_1 = g$ and, for each $t \in I$, $F_t: P \to Q$ defined by $F_t(x) = F(x,t)$ for all $x \in P$, is an $\mathcal{E}$-map. There is an obvious associated concept of *$\mathcal{E}$-homotopy equivalence.*

Given a category $\mathcal{E}$ of enriched spaces, an *enriched overspace*, *$\mathcal{E}$-overspace* or *$\mathcal{E}$-space over B* is a map $p: X \to B$ together with, for each $b \in B$, an associated structure of an $\mathcal{E}$-space on the *fibre* $X|b = p^{-1}(b)$, for each $b \in B$. For example if P is a given $\mathcal{E}$-space and B is a space then the projection $\pi_B: P \times B \to B$ is an $\mathcal{E}$-space over B, we will refer to it as a *trivial $\mathcal{E}$-overspace.*

If $f: A \to B$ is a map and $q: Y \to B$ is an $\mathcal{E}$-overspace then $Y \times_B A$ will denote the pullback space $\{(y,a) \in Y \times A | q(y) = f(a)\}$, and $q_f: Y \times_B A \to A$ the projection map defined by $q_f(y,a) = a$, where $(y,a) \in Y \times_B A$. There is an obvious $\mathcal{E}$-overspace structure on q_f, which will be referred to as the *$\mathcal{E}$-overspace induced from q by f*. The projection $Y \times_B A \to Y$ will be denoted by f_q.

For example, if $\pi_B: C \times B \to B$ denotes the projection then the $\mathcal{E}$-overspace induced from q by π_B can be identified with the $\mathcal{E}$-overspace $1_C \times q: C \times Y \to C \times B$. If V is a subspace of B and $Y|V$ denotes $q^{-1}(V)$ then the restriction $q|(Y|V): Y|V \to V$ carries, in an obvious way, the structure of an $\mathcal{E}$-overspace and will be denoted by $q|V$. Clearly $q|V$ can be identified with the $\mathcal{E}$-overspace induced from q by the inclusion $V \subset B$. In particular taking $V = \{b\}$, where $b \in B$, the fibre over b will be denoted by $Y|b$.

If $p: X \to A$ and $q: Y \to B$ are $\mathcal{E}$-overspaces then an *$\mathcal{E}$-pairwise map* $< f,g >$ from p to q consists of maps $f: X \to Y$ and $g: A \to B$ such that $qf = gp$, and with the property that, for each $a \in A$, $f|(X|a): X|a \to Y|g(a)$ is an $\mathcal{E}$-map. An $\mathcal{E}$-pairwise map from the $\mathcal{E}$-overspace $p \times 1: X \times I \to A \times I$ to q will be called an *$\mathcal{E}$-pairwise homotopy.*

Taking $A = B$ and fixing g to be 1_B, the $\mathcal{E}$-pairwise map concept reduces to *$\mathcal{E}$-map over B*. If $f: X \to Y$ is an $\mathcal{E}$-map over B and $V \subset B$ then *the restriction of f over V*, $f|V = f|(X|V): X|V \to Y|V$, is clearly an $\mathcal{E}$-map over V. In particular taking $V = \{b\}$, where $b \in B, f|b = f|(X|b): X|b \to Y|b$ is an $\mathcal{E}$-map.

Taking $A = B \times I$ and fixing g to be the projection $B \times I \to B$, there is an obvious associated idea of *$\mathcal{E}$-homotopy over B*, and hence of *$\mathcal{E}$-homotopy equivalence over B*. The last concept will, however, be named *$\mathcal{E}$-fibre homotopy equivalence* or $\mathcal{E}$FHE.

Let $p: X \to B$ and $q: Y \to B$ be $\mathcal{E}$-spaces over B. If $f: X \to Y$ is an $\mathcal{E}$-map over B, and a homeomorphism with the property that $f^{-1}: Y \to X$

is also an $\mathcal{E}$-map over B, then f will be called an *$\mathcal{E}$-homeomorphism* over B.

We now give two fairly obvious technical results that will be used later.

(1.1) If $p: X \to A$ and $q: Y \to B$ are $\mathcal{E}$-overspaces and $< f, g >$ is an $\mathcal{E}$-pairwise map from p to q, then there is an $\mathcal{E}$-map $(f,p): X \to Y \times_B A$ over A defined by $(f,p)(x) = (f(x), p(x))$, where $x \in X$

(1.2) Let $r: Z \to C$ be an $\mathcal{E}$-overspace and $f: A \to B$ and $g: B \to C$ be maps. There is a homeomorphism $h: Z \times_C A \to (Z \times_C B) \times_B A$ defined by $h(z,a) = (z, f(a), a)$, where $(z,a) \in Z \times_C A$, that is also an $\mathcal{E}$-homeomorphism over A, i.e. from $r_{(gf)}$ to $(r_g)_f$.

Let us now consider the particular case where $\mathcal{E} = \mathcal{T}$. A *$\mathcal{T}$-space over B* is simply a space over B, i.e. a map into B. In the remainder of this paper *we adopt the convention of omitting the prefix $\mathcal{T}$* in order to conform to normal usage. For example a $\mathcal{T}$-homotopy equivalence becomes a *homotopy equivalence*, a $\mathcal{T}$-homotopy over B a *homotopy over B*, denoted by $\simeq_B$, and the term $\mathcal{T}$FHE is shortened to *FHE*.

2 Fibred mapping spaces.

If P and Q are $\mathcal{E}$-spaces then $\mathcal{E}(P,Q)$ will denote the set of all $\mathcal{E}$-maps from P to Q, topologized (when we so wish) by the (of course cg-ified) compact-open topology. If P, Q and R and $\mathcal{E}$-spaces and $f: Q \to R$ is an $\mathcal{E}$-map then the associated function

$$f_\# : \mathcal{E}(P,Q) \to \mathcal{E}(P,R), f_\#(g) = fg, \text{ where } g \in \mathcal{E}(P,Q),$$

is continuous.

Given that $q: Y \to B$ and $r: Z \to C$ are $\mathcal{E}$-overspaces, the *fibred mapping space* or *functional space* $Y \Box Z$ has underlying set

$$\bigcup_{b \in B, c \in C} \mathcal{E}(Y|b, Z|c),$$

and the *functional projection* $q\Box r: Y\Box Z \to B \times C$ is defined by $(q\Box r)(f) = (b,c)$, where $f \in \mathcal{E}(Y|b, Z|c)$.

We present an auxilliary construction before topologizing $Y\Box Z$. Let $Y^+ = Y \cup \{\infty\}$ be the disjoint union of the set Y with the point ∞, with the following topology: $C \subset Y^+$ is closed if either $C = Y^+$ or C is a closed subset of Y.

Let $b \in B$, $c \in C$, and $f: Y|b \to Z|c$ be a given map. We define a function $j = j_{Y\Box Z}: Y\Box Z \to \mathcal{T}(Y, Z^+)$ by taking $j(f)(y) = f(y)$ if $y \in Y|b$, and $j(f)(y) = \infty$ otherwise. We know that B is weak Hausdorff, so B is T_1, $Y|b$ is closed and hence $j(f) \in \mathcal{T}(Y, Z^+)$. Then $Y\Box Z$ will be given the initial (i.e. strong) topology relative to j and $q\Box r$. Clearly $q\Box r$ is continuous.

For example if B and C are one-point spaces then Y and Z are $\mathcal{E}$-spaces and $Y\Box Z = \mathcal{E}(Y, Z)$ carries the cg-ification of the compact-open topology.

Proposition 2.1 *Let $p: X \to B, q: Y \to C$ and $r: Z \to C$ be $\mathcal{E}$-overspaces.*

(i) If $f: Y \to Z$ is an $\mathcal{E}$-map over C then $f_\#: X\Box Y \to X\Box Z$, defined by $f_\#(g) = (f|c)g \in \mathcal{E}(X|b, Z|c)$, is a map over $B \times C$ where $g \in \mathcal{E}(X|b, Y|c)$ with $b \in B$ and $c \in C$.

(ii) The rule $f \to f_\#$ is functorial, i.e. $(1_Y)_\# = 1_{X\Box Y}$ and if $h: Z \to W$ is another $\mathcal{E}$-map over C then $(hf)_\# = h_\# f_\#$.

(iii) If $F: Y \times I \to Z$ is an $\mathcal{E}$-homotopy over C then $F_*: (X\Box Y) \times I \to X\Box Z$, defined by $F_*(g, t)(x) = F(g(x), t)$, where $g \in \mathcal{E}(X|b, Y|c)$ for some $b \in B$ and $c \in C$, $t \in I$, $x \in X$ and $p(x) = b$, is a homotopy over $B \times C$.

Proof:

(i) Clearly f induces a map $f^+: Y^+ \to Z^+$, $f^+(y) = f(y)$ is $y \in Y$ and $f^+(\infty) = \infty$, and hence a map $f': \mathcal{T}(X, Y^+) \to \mathcal{T}(X, Z^+), f'(g) = f^+ g$, where $g \in \mathcal{T}(X, Y^+)$. Then $j_{X\Box Z} f_\# = f' j_{X\Box Y}$ and $(p\Box r) f_\# = p\Box q$, so the expressions to the left of these two equalities are continuous, and hence $f_\#$ is continuous.

(ii) We prove only the second part. If $b \in B, c \in C$ and $g \in \mathcal{E}(X|b, Y|c)$ then $(hf)_\#(g) = ((hf)|c)g = (h|c)(f|c)g = h|c f_\#(g) = h_\# f_\#(g)$.

(iii) It is easily seen that F determines a homotopy $F^+: Y^+ \times I \to Z^+$ defined by $F^+(y, t) = F(y, t)$ if $y \in Y$ and $t \in I$, and $F^+(\infty, t) = \infty$, where $t \in I$. The continuity of the homotopy $F': \mathcal{T}(X, Y^+) \times I \to \mathcal{T}(X, Z^+)$, defined by $F'(g, t)(x) = F^+(g(x), t)$, where $g \in \mathcal{T}(X, Y^+), t \in I$ and $x \in X$, follows easily using (0.1) twice. F_* is clearly well defined as a function and it is easily checked that $(p\Box r)F_* = \pi((p\Box q) \times 1_I)$, where $\pi: B \times C \times I \to B \times C$ denotes the projection, and that $j_{X\Box Z} F_* = F'(j_{X\Box Y} \times 1_I)$. So $(p\Box r)F_*$ and $j_{X\Box Z} F_*$ are continuous and hence F_* is continuous.

Theorem 2.2 (Fibred exponential law.) *Let $f: A \to B$ and $g: A \to C$ be maps, and $q: Y \to B$ and $r: Z \to C$ be $\mathcal{E}$-overspaces. Then there is a bijective correspondence between:*

(i) *maps* $h: Y \times_B A \to Z$ *such that* $< h, g >$ *is an* $\mathcal{E}$*-pairwise map from* q_f *to* r*, and*

(ii) *maps* $h^0: A \to Y\Box Z$ *such that* $(q\Box r)h^0 = (f, g): A \to B \times C$*,*

defined by $h(y, a) = h^0(a)(y)$*, where* $q(y) = f(a)$*.*

Proof: Given h as described in (i), it is clear that $h^0(a) \in \mathcal{E}(Y|f(a) \times \{a\}, Z|g(a)) = \mathcal{E}(Y|f(a), Z|g(a))$, so h^0 is a well defined function and $(q\Box r)h^0$ (f, g). The continuity of h^0 follows from the continuity of $(q\Box r)h^0 = (f, g)$ and of jh^0, so we only need verify the latter condition. Let us define $\ell: Y \times A \to Z^+$ by $\ell(y, a) = h(y, a)$ if $(y, a) \in Y \times_B A$ and ∞ otherwise. The weak Hausdorffness of B ensures that $Y \times_B A = (q \times f)^{-1}(\Delta_B)$ is closed in $Y \times A$, and so the continuity of h ensures that of ℓ. Now ℓ corresponds by 0.1 to $jh^0: A \to \mathcal{T}(Y, Z^+)$ since $jh^0(a)(y) = \ell(y, a)$, where $a \in A$ and $y \in Y$, and so jh^0 is continuous.

Conversely let us assume that h^0 satisfies the condition described in (ii). Clearly h is well defined as a function and $rh = g(q_f)$. The continuity of jh^0 ensures that of the corresponding ℓ, and $h = \ell|Y \times_B A$ must also be continuous.

(2.2.1) If B and C are one-point spaces then Y and Z are $\mathcal{E}$-spaces and the maps $h: Y \times A \to Z$ and $h^0: A \to \mathcal{E}(Y, Z)$ are related by $h(y, a) = h^0(a)(y)$, where $y \in Y$ and $a \in A$.

(2.2.2) If $A = B = C$ and $f = g = 1_B$ then $Y \times_B A = Y \times_B B$ can be identified with Y and h can therefore be taken to be simply an $\mathcal{E}$-map of Y to Z over B. Further h^0 is a lifting of the diagonal map $\delta: B \to B \times B$, $\delta(b) = (b, b)$ where $b \in B$, over $q\Box r$. Hence we have determined a *bijective correspondence between* $\mathcal{E}$*-maps* $h: Y \to Z$ *over* B *and maps* h^0: $B \to Y\Box Z$ *over* $B \times B$*, defined by the rule* $h|b = h^0(b)$ for all $b \in B$.

(2.2.3) In particular, if in (2.2.2) we take $q = r$ and $h = 1_Y$, the associated map $(1_Y)^0: B \to Y\Box Y$ is the lift of δ over $q\Box q$ defined by $(1_Y)^0(b) = 1_{Y|b}$, for all $b \in B$.

Our next result is a homotopy version of 2.2.2.

Proposition 2.3 *Let us assume that* $q: Y \to B$ *and* $r: Z \to B$ *are* $\mathcal{E}$*-overspaces. Then there is a bijective correspondence between:*

(i) $\mathcal{E}$*-homotopies* $H: Y \times I \to Z$ *over* B*, and*

(ii) *homotopies* $H^0: B \times I \to Y\Box Z$ *over* $B \times B$*, i.e. that lift* $(\pi_B, \pi_B): B \times I \to B \times B$ *over* $q\Box r$*,* *defined by* $H(y,t) = H^0(b,t)(y)$ *where* $q(y) = b$ *and* $t \in I$.

Proof: This follows from theorem 2.2 with $A = B \times I$, $C = B$ and $f = g = \pi_B : B \times I \to B$, using the identification $Y \times_B (B \times I) = Y \times I$.

Lemma 2.4 *Given that* $p: X \to B, q: Y \to B$ *and* $r: Z \to B$ *are* $\mathcal{E}$*-overspaces, and that* $f: X \to Y$ *and* $g: Y \to Z$ *are* $\mathcal{E}$*-maps over* B. *Then* $g_\# f^0 = (gf)^0$.

Proof: We notice that $f^0: B \to X\Box Y$, $g_\#: X\Box Y \to X\Box Z$ and $(gf)^0: B \to X\Box Z$. Then if $b \in B: (gf)^0(b) = (gf|b) = (g|b)(f|b) = g_\#(f|b) = g_\# f^0(b)$.

(2.4.1) Taking B to be a one point space $*$ then $f^0: * \to \mathcal{E}(X,Y)$ has value $f, (gf)^0(*) = gf$ and 2.4 just states that $g_\#(f) = gf$, i.e. it is the defining equation for $g_\#$: $\mathcal{E}(X,Y) \to \mathcal{E}(X,Z)$.

Proposition 2.5 *Given that* $q: Y \to B$ *and* $r: Z \to C$ *are* $\mathcal{E}$*-overspaces, and that* U *and* V *are subspaces of* B *and* C *respectively. Then the functional projections* $q\Box r|U\times V: Y\Box Z|U\times V \to U\times V$ *and* $(q|U)\Box(r|V): (Y|U)\Box(Z|V) \to U \times V$ *coincide.*

Proof: It is immediate that the two constructions agree at the set-function level. Let $p: A \to U \times V$ be a map. It is a consequence of theorem 2.2 that a function $A \to Y\Box Z|U \times V$, over $U \times V$, is a map if and only if the same function $A \to (Y|U)\Box(Z|V)$ is a map. The result follows.

We now consider the special case where U and V consist of single points, i.e. $b \in B$ and $c \in C$, respectively.

(2.5.1) $Y\Box Z|\{(b,c)\}$, i.e. the fibre of $q\Box r$ over $(b,c) \in B \times C$, is $\mathcal{E}(Y|b, Z|c)$.

Proposition 2.6 *Let* $p: X \to B, q: Y \to C$ *and* $r: Z \to C$ *be* $\mathcal{E}$*-overspaces and* $f: Y \to Z$ *be an* $\mathcal{E}$*-map over* C. *If* U *and* V *are subspaces of* B *and* C*, respectively, there are maps* $(f|V)_\#: (X|U)\Box(Y|V) \to (X|U)\Box(Z|V)$ *and* $f_\#|U \times V: X\Box Y|U \times V \to X\Box Z|U \times V$. *Then, under the identification of proposition 2.5,* $(f|V)_\# = f_\#|U \times V$.

Proof: This is immediate from the definitions involved.

We now take $U = \{b\}$ and $V = \{c\}$, where $b \in B$ and $c \in C$.

(2.6.1) The maps $(f|\{c\})_\#$ and $f_\#|\{(b,c)\}$ coincide.

3 On the $\mathcal{E}$-covering homotopy property

Let $q{:}\,Y \to B$ be an $\mathcal{E}$-overspace. Let us assume that for all choices of a space A, an $\mathcal{E}$-overspace $p{:}\,X \to A$, a homotopy $F{:}\,A \times I \to B$ and a map $f{:}\,X \times \{0\} \to Y$ which satisfy the condition that $< f, F|A \times \{0\} >$ is an $\mathcal{E}$-pairwise map from $p \times 1_{\{0\}}$ to q, there always exists a homotopy $H{:}\,X \times I \to Y$ extending f and such that $< H, F >$ is an $\mathcal{E}$-pairwise homotopy from $p \times 1_I$ to q. Then q will be said to satisfy the *$\mathcal{E}$-covering homotopy property* or $\mathcal{E}$CHP.

For example if F is any $\mathcal{E}$-space and B is any space then the trivial $\mathcal{E}$-overspace $\pi_B{:}\,F \times B \to B$ satisfies the $\mathcal{E}$CHP.

It is easily seen that the $\mathcal{T}$CHP is equivalent to the apparently weaker version of the same condition, with X required to be A and $p = 1_A$. Hence, modulo our cg-space restriction, the $\mathcal{T}$CHP is the familiar CHP of [Do], i.e. the defining condition for a Hurewicz fibration.

The next lemma, which gives an alternative form of the "q satisfies the $\mathcal{E}$CHP" condition, will be used in section 7 as well as in proving the subsequent corollary.

Lemma 3.1 *The condition that the $\mathcal{E}$-overspace $q{:}\,Y \to B$ satisfies the $\mathcal{E}$CHP is equivalent to the apparently weaker condition that the $\mathcal{E}$CHP-definition is satisfied only for quintuples (A, X, p, F, f) such that $X = \overline{Y}_0, p = \overline{q}_0$ and $f = \overline{F}_0$.*

Proof: Given q, X, A, p, F and f as described in the $\mathcal{E}$CHP definition. It follows from the "weaker condition" that there is a homotopy $K{:}\,\overline{Y}_0 \times I \to Y$ such that $< K, F >$ is an $\mathcal{E}$-pairwise homotopy, from $\overline{q}_0 \times 1$ to q, which also extends $\overline{F}_0$. Then, taking $H = K((f,p) \times 1_I){:}\,X \times I \to Y$, we see that q satisfies the $\mathcal{E}$CHP.

Proposition 3.2 *(=[Mo1, prop. 2.1.2]) If the $\mathcal{E}$-overspace $q{:}\,Y \to B$ satisfies the $\mathcal{E}$CHP then, forgetting the $\mathcal{E}$-structure, the map q satisfies the CHP.*

Proof: This is immediate once we use lemma 3.1 to reinterpret the $\mathcal{E}$CHP and CHP (= $\mathcal{T}$CHP) definitions.

Proposition 3.3 *(=[Mo2, prop. 1]) If $q{:}\,Y \to B$ and $r{:}\,Z \to C$ are $\mathcal{E}$-overspaces that satisfy the $\mathcal{E}$CHP then $q \square r$ satisfies the CHP.*

Theorem 3.4 *(=[Mo2, prop. 2]) Let $q{:}\,Y \to B$ be an $\mathcal{E}$-overspace. Then q has the $\mathcal{E}$CHP if and only if $q \square q$ has the CHP.*

The two results of [Mo2] just quoted are stated as holding in a more restricted context, i.e. extra assumptions are made about the category $\mathcal{E}$, as compared with our present argument. However the proofs of [Mo2] depend only on certain formal properties of fibred mapping spaces and therefore extend, without alteration, to the broader context of this paper. Also [Mo2] uses an argument that depends on lemma 1.2 of reference [1] of that paper. Now this lemma is incompletely stated in that reference; the reader may prefer to replace that result with our complete and more general theorem 2.2.

A locally finite cover $\mathcal{V}$ of a space B will be said to be *numerable* if there is a partition of unity for B, consisting of maps $\{\lambda_V: B \to I\}_{V \in \mathcal{V}}$, such that $V = \lambda_V^{-1}(0,1]$ for each $V \in \mathcal{V}$.

Lemma 3.5 *If $\mathcal{U}$ and $\mathcal{V}$ are numerable covers of B and C, respectively, then $\mathcal{U} \times \mathcal{V} = \{U \times V | U \in \mathcal{V}$ and $V \in \mathcal{V}\}$ is a numerable cover of $B \times C$.*

Proof: Let $\{\lambda_U: B \to I\}_{U \in \mathcal{U}}$ and $\{\lambda_V: B \to I\}_{V \in \mathcal{V}}$ be partitions of unity for B and C respectively. If $U \in \mathcal{U}$ and $V \in \mathcal{V}$ we define $\lambda_{U \times V}: B \times C \to I$ by pointwise multiplication of λ_U and λ_V, i.e. $\lambda_{U \times V}(b,c) = \lambda_U(b)\lambda_V(c)$, where $b \in U$ and $c \in V$. It is easily seen that $\{\lambda_{U \times V} | U \in \mathcal{U}$ and $V \in \mathcal{V}\}$ is a partition of unity for $B \times C$.

Theorem 3.6 *(see [Ma, thm. 3.8], and also [Do, thm. 4.8] for the $\mathcal{T}$-case). Let $q: Y \to B$ be an $\mathcal{E}$-overspace and $\mathcal{V}$ be a numerable cover of B. If for each $V \in \mathcal{V}$ the $\mathcal{E}$-overspace $q|V: Y|V \to V$ satisfies the $\mathcal{E}$CHP then so also does q.*

Proof: For each choice of $V_1, V_2 \in \mathcal{V}$ we know by proposition 3.3 that $(q|V_1)\Box(q|V_2)$ has the CHP, and so we see by proposition 2.5 that $(q\Box q)|V_1 \times V_2$ has the CHP. Now $\mathcal{V} \times \mathcal{V}$ is a numerable cover of $B \times B$ (lemma 3.5) so it follows from [Do, thm. 4.8] that $q\Box q$ has the CHP. We see via theorem 3.4 that q has the $\mathcal{E}$CHP.

4 On the $\mathcal{E}$-weak covering homotopy property

The homotopy $F: A \times I \to B$ will be said to be *stationary on* $A \times [0,1/2]$ if for all $t \in [0,1/2]$ and all $a \in A$, $F(a,t) = F(a,0)$.

If the $\mathcal{E}$-overspace $q: Y \to B$ satisfies the weakened version of the defining condition for the $\mathcal{E}$CHP, in which the homotopics $F: A \times I \to B$ that are used are required to be stationary on $A \times [0,1/2]$, then q will be said to possess the *$\mathcal{E}$-weak covering homotopy property* or $\mathcal{E}$WCHP.

Examples obviously include all $\mathcal{E}$-overspaces satisfying the $\mathcal{E}$CHP. It is easily seen that the $\mathcal{T}$WCHP is equivalent to the apparently weaker version of the same condition in which $X = A$ and $p = 1_A$. Hence, modulo our cg-space requirement, the $\mathcal{T}$WCHP is the familiar WCHP of [Do], i.e. the defining condition for what are often known as *Dold fibrations* or *h-fibrations.*

It will sometimes be convenient for us to use the following abbreviated notations.

Given an $\mathcal{E}$-overspace $q: Y \to B$ and a homotopy $F: A \times I \to B$, the $\mathcal{E}$-overspaces induced from q by F and $F_t: A \to B$ will be denoted by $\overline{q}: \overline{Y} \to A \times I$ and $\overline{q}_t: \overline{Y}_t \to A$ respectively, where $t \in I$. The projections $\overline{Y} \to Y$ and $\overline{Y}_t \to Y$ will be denoted by $\overline{F}$ and $\overline{F}_t$ respectively.

The following lemma will be useful both in this chapter and in chapter 6.

Lemma 4.1 *Let $q: Y \to B$ be an $\mathcal{E}$-overspace satisfying the $\mathcal{E}$WCHP, and $F: A \times I \to B$ be a homotopy. Then there are $\mathcal{E}$-maps and $\mathcal{E}$-homotopies:*

(i) $\theta: \overline{Y}_1 \to \overline{Y}_0$ *over* A, *(ii)* $\phi: \overline{Y}_1 \to \overline{Y}_1$ *over* A

(iii) $\alpha: \overline{Y} \to \overline{Y}_0 \times I$ *over* $A \times I$, *(iv)* $\beta: \overline{Y}_1 \times I \to \overline{Y}$ *over* $A \times I$,

(v) $\gamma: \overline{Y}_1 \times I \to \overline{Y}_1 \times I$ *over* $A \times I$.

These maps and homotopies satisfy:

$$\alpha|A \times \{0\} = 1: \overline{Y}_0 \to \overline{Y}_0 \qquad \alpha|A \times \{1\} = \theta$$
$$\beta|A \times \{0\} = \theta \qquad \beta|A \times \{1\} = \phi$$
$$\gamma|A \times \{0\} = \phi \quad \gamma|A \times \{1\} = 1: \overline{Y}_1 \to \overline{Y}_1 .$$

Proof: In this argument we assume that F is stationary on $A \times [0, 1/2]$. If it does not already possess this property then we can make it apply by redefining F as $F': A \times I \to B$ by

$$F'(a,t) = \begin{cases} F(a,0) & 0 \le t \le \frac{1}{2} \\ F(a, 2t-1) & \frac{1}{2} \le t \le 1 \end{cases},$$

where $a \in A$ and $t \in I$.

Let $G: A \times I \times I \to B$ be defined by

$$G(a,s,t) = \begin{cases} F(a,0) & s \le t \\ F(a,s) & t \le \frac{1}{2} \\ F(a, 1+s-2t) & \frac{1}{2} \le t \le s \end{cases},$$

where $a \in A$ and $s, t \in I$. We notice that G is stationary on $A \times I \times [0, 1/2]$, i.e. if $s \in I$ and $t \in [0, 1/2]$ then $G(a, s, t)$ is independent of the value of t chosen. Identifying $\overline{Y}$ with $\overline{Y} \times \{0\}$ we see that $q\overline{F} = G(\overline{q} \times 1_I)|\overline{Y} \times \{0\}$.

Applying the $\mathcal{E}$WCHP we obtain a homotopy $H: \overline{Y} \times I \to Y$ extending $\overline{F}$ and such that $< H, G >$ is an $\mathcal{E}$-pairwise map from $\overline{q} \times 1_I$ to q.

Viewing $\overline{Y} \times I$ as a space over $A \times I \times I$, we will consider the subspaces of $\overline{Y} \times I$ over $A \times \{(1,1)\}, A \times \{(1,1/2)\}$, $\{(a,t,t) | a \in A, t \in I\}$, $A \times \{1\} \times [1/2,1]$ and $A \times \{1\} \times [0,1/2]$. In each case 1.2 can be applied, the f of 1.2 always being the inclusion of the given subspace in $A \times I \times I$ and the g being the composite map $F\pi_{A\times I}: A \times I \times I \to B$, where $\pi_{A\times I}: A \times I \times I \to A \times I$ projects on to the first two factors. Hence the five subspaces of $\overline{Y} \times I$ under consideration can be taken to be $\overline{Y}_1 = \overline{Y}_1 \times \{(1,1)\}, \overline{Y}_1 = \overline{Y}_1 \times \{(1,1/2)\}, \overline{Y}$, $\overline{Y}_1 \times [1/2,1] = \overline{Y}_1 \times \{1\} \times [1/2,1]$ and $\overline{Y}_1 \times [0,1/2] = \overline{Y}_1 \times \{1\} \times [0,1/2]$, respectively.

Restricting H to the first three of these spaces, G to the corresponding subspaces of $A \times I \times I$, and in each case applying 1.1, we obtain the required maps θ, ϕ and α. Identifying $[1/2,1]$ and $[0,1/2]$ with I by the homeomorphisms $s \to 2-2s$ and $t \to 1-2t$, respectively, where $s \in [1/2,1]$ and $t \in [0,1/2]$, we can redefine the last two of the five subspaces of $\overline{Y} \times I$. Applying 1.1 in both cases we obtain the required homotopies β and γ respectively.

The next two lemmas give alternative forms of the "q satisfies the $\mathcal{E}$WCHP" and the "$q\Box r$ satisfies the WCHP" conditions, respectively, thereby facilitating the remaining proofs of this section. The proposition, a consequence of the first lemma and of some interest in itself, is required in the proof of theorem 5.5.

Lemma 4.2 *The condition that the $\mathcal{E}$-overspace $q: Y \to B$ satisfies the $\mathcal{E}$ WCHP is equivalent to the apparently weaker condition that the $\mathcal{E}$ WCHP-definition is satisfied only for quintuples (A, X, p, F, f) such that $X = \overline{Y}_0, p = \overline{q}_0$ and $f = \overline{F}_0$.*

Proof: This is the same as the proof of lemma 3.1, except that the $\mathcal{E}$WCHP now replaces the $\mathcal{E}$CHP and the homotopy F must be stationary on $A \times I$.

Proposition 4.3 *If the $\mathcal{E}$-overspace $q: Y \to B$ satisfies the $\mathcal{E}$ WCHP then, forgetting the $\mathcal{E}$-structure, the map q satisfies the WCHP.*

Proof: This is immediate once we use lemma 4.2 to reinterpret the $\mathcal{E}$WCHP and WCHP ($= \mathcal{T}$WCHP) definitions.

Lemma 4.4 *Let $q: Y \to B$ and $r: Z \to C$ be $\mathcal{E}$-overspaces. Then $q\Box r$ satisfies the WCHP if and only if the following condition is satisified:*

for all choices of a pair of homotopies $F: A \times I \to B$ and $G: A \times I \to C$ that are stationary on $A \times [0,1/2]$, and of a map $h: \overline{Y}_0 \to Z$ such that

$< h, G_0 >$ is an $\mathcal{E}$-pairwise map from $\overline{q}_0$ to r, there exists a map $H:\overline{Y} \to Z$ extending h and such that $< H, G >$ is an $\mathcal{E}$-pairwise map from $\overline{q}$ to r.

Proof: Applying theorem 2.2 we see that there is a map $H:\overline{Y} \to Z$ such that $< H, G >$ is an $\mathcal{E}$-map from $\overline{q}$ to r if and only if there is a homotopy $K: A \times I \to Y\Box Z$ such that $(q\Box r)K = (F, G)$, and these are related by $H(y, a, t) = K(a, t)(y)$ where $q(y) = F(a, t)$. Again, by theorem 2.2, there is a map $h:\overline{Y}_0 \to Z$ such that $< h, G_0 >$ is an $\mathcal{E}$-map from $\overline{q}_0$ to r if and only if there is a map $k: A \to Y\Box Z$ such that $(q\Box r)k = (F_0, G_0)$, related by $h(y, a) = k(a)(y)$ where $q(y) = F(a, 0)$. Identifying A with $A \times \{0\}$, we see that H extends h if and only if K extends k. The result follows easily.

Proposition 4.5 *If the $\mathcal{E}$-overspaces $q: Y \to B$ and $r: Z \to C$ satisfy the $\mathcal{E}$WCHP then $q\Box r$ satisfies the WCHP.*

Proof: Let us assume that we are given homotopies $F: A \times I \to B$ and $G: A \times I \to C$ that are stationary on $A \times [0, 1/2]$ and a map $h:\overline{Y}_0 \to Z$ such that $< h, G_0 >$ is an $\mathcal{E}$-pairwise map from $\overline{q}_0$ to r. Identifying $\overline{Y}_0 \times \{0\}$ with $\overline{Y}_0$ we have $rh = G(\overline{q}_0 \times 1_I)|\overline{Y}_0 \times \{0\}$, so it follows by the $\mathcal{E}$WCHP for r that there is a homotopy $K:\overline{Y}_0 \times I \to Z$ extending h and such that $< K, G >$ is an $\mathcal{E}$-pairwise homotopy from $\overline{q}_0 \times 1_I$ to r. Now q also satisfies the $\mathcal{E}$WCHP so there exists an $\alpha:\overline{Y} \to \overline{Y}_0 \times I$ over $A \times I$ as described in lemma 4.1. Then, defining $H = K\alpha$, we have $rH = rK\alpha = G(\overline{q}_0 \times 1)\alpha = G\overline{q}$ and $H|\overline{Y}_0 = K\alpha|\overline{Y}_0 = (K|\overline{Y}_0)(\alpha|\overline{Y}_0) = h1 = h$. The result then follows by lemma 4.4.

Theorem 4.6 *Let $q: Y \to B$ be an $\mathcal{E}$-overspace. Then q has the $\mathcal{E}$WCHP if and only if $q\Box q$ has the WCHP.*

Proof: The "only if" part is immediate from proposition 4.4.

Given a homotopy $F: A \times I \to B$, we notice that if $\pi: A \times I \to A$ denotes the projection then $F_0\pi: A \times I \to B$ is a homotopy. Replacing F, G, h and $r: Z \to C$ of lemma 4.4 by our $F_0\pi, F, \overline{F}_0$ and $q: Y \to B$, respectively, we see by 1.2 that $\overline{q}:\overline{Y} \to A \times I$ of lemma 4.4 must be replaced by $(\overline{q}_0)_\pi = \overline{q}_0 \times 1:\overline{Y}_0 \times I \to A \times I$. It follows from lemma 4.4 that there is a homotopy $H:\overline{Y}_0 \times I \to Y$ satisfying precisely the conditions specified in lemma 4.2 to ensure that q possesses the $\mathcal{E}$WCHP.

Theorem 4.7 *(see [Do, thm. 5.12] for the $\mathcal{T}$-case) Let $q: Y \to B$ be an $\mathcal{E}$-overspace and $\mathcal{V}$ be a numerable cover of B. If for each $V \in \mathcal{V}$ the $\mathcal{E}$-overspace $q|V: Y|V \to V$ satisfies the $\mathcal{E}$WCHP then so also does q.*

Proof: This is as the proof of theorem 3.6, except that the terms CHP and $\mathcal{E}$CHP, [Do, thm. 4.8] and our results 3.3 and 3.4 must be replaced by WCHP and $\mathcal{E}$WCHP, [Do, thm. 5.12], and our results 4.5 and 4.6, respectively.

5 On $\mathcal{E}$-fibre homotopy equivalence.

If P, Q and R are $\mathcal{E}$-spaces and $f: Q \to R$ is an $\mathcal{E}$-homotopy equivalence then it is easily seen that $f_{\#}: \mathcal{E}(P,Q) \to \mathcal{E}(P,R)$ is a homotopy equivalance; our next result generalizes this statement to the fibred mapping space level.

Lemma 5.1 *Given that $p: X \to A, q: Y \to B$ and $r: Z \to B$ are $\mathcal{E}$-overspaces. If $f: Y \to Z$ is an $\mathcal{E}FHE$ then $f_{\#}: X\Box Y \to X \Box Z$ is an FHE.*

Proof: Let $g: Z \to Y$ be an $\mathcal{E}$-fibre homotopy inverse of f, then it follows by 2.1 that $g_{\#}$ is a fibre homotopy inverse to $f_{\#}$.

We saw in proposition 2.1 that $X\Box Y$ is covariantly functorial in the second variable Y. It can be shown, in a similar way that $X\Box Y$ is contravariantly functorial in the first variable X, and a contravarient analogue of lemma 5.1 established. It follows that if the $\mathcal{E}$-overspaces $p_0: X_0 \to A$ and $p_1: X_1 \to A$ are $\mathcal{E}$FHE and the $\mathcal{E}$-overspaces $q_0: Y_0 \to A$ and $q_1: Y_1 \to A$ are $\mathcal{E}$FHE, then $p_0 \Box q_0$ and $p_1 \Box q_1$ are FHE.

Theorem 5.2 *Let $p: X \to B$ and $q: Y \to B$ be $\mathcal{E}$-overspaces and $f: X \to Y$ be an $\mathcal{E}$-map over B. Then f is an $\mathcal{E}FHE$ if and only if $f_{\#}: Z\Box X \to Z\Box Y$ is an FHE for both $Z = X$ and $Z = Y$.*

Proof: The "only if" part is given by lemma 5.1.

Let $\lambda_Z: Z\Box Y \to Z\Box X$ denote a fibre homotopy inverse for $f_{\#}$, in the cases $Z = X$ and $Z = Y$. We recall (2.2.3) the existence of a map $(1_Y)^0: B \to Y\Box Y$ over $B \times B$, then $g^0 = \lambda_Y(1_Y)^0: B \to Y\Box X$ is also a map over $B \times B$ and the associated $g: Y \to X$ (2.2.2) is an $\mathcal{E}$-map over B. Now $f_{\#}\lambda_Y \simeq_{B\times B} 1_{Y\Box Y}$ so, by lemma 2.4, $(fg)^0 = f_{\#}g^0 = f_{\#}\lambda_Y(1_Y)^0 \simeq_{B\times B} (1_Y)^0$. Hence by proposition 2.3: $fg \simeq_B 1_Y$.

From this point on we take $Z = X$. Then $f_{\#}g_{\#} = (fg)_{\#}$ and, by proposition 2.1 (ii), $(fg)_{\#} \simeq_{B\times B} (1_Y)_{\#} = 1_{X\Box Y}$. Hence $g_{\#} \simeq_{B\times B} (\lambda_X f_{\#})g_{\#} = \lambda_X(f_{\#}g_{\#}) \simeq_{B\times B} \lambda_X 1_{X\Box Y} = \lambda_X$. So $g_{\#}f_{\#} \simeq_{B\times B} \lambda_X f_{\#} \simeq_{B\times B} 1_{X\Box X}$. Recalling that $(1_X)^0: B \to X\Box X$ is a map over $B \times B$ (2.2.3), we see that:

$$\begin{aligned} (gf)^0 &= (gf1_X)^0 \\ &= (gf)_{\#}(1_X)^0 \quad \text{(see lemma 2.4)} \\ &= g_{\#}f_{\#}(1_X)^0 \quad \text{(see proposition 2.1(ii))} \\ &\simeq_{B\times B} (1_X)^0 \quad \text{(see previous sentence).} \end{aligned}$$

It follows by proposition 2.3 that $gf \simeq_B 1_X$, and so g is an $\mathcal{E}$-fibre homotopy inverse for f.

Theorem 5.3 *(see [Ma, thm. 1.5], also [Do, thm. 3.3]) for the $\mathcal{E} = \mathcal{T}$ version). Let $p: X \to B$ and $q: Y \to B$ be $\mathcal{E}$-overspaces and $f: X \to Y$ be and $\mathcal{E}$-map over B. If $\mathcal{V}$ is a numberable cover of B, then f is an $\mathcal{E}FHE$ if and only if, for each choice of $V \in \mathcal{V}$, $f|V: X|V \to Y|V$ is an $\mathcal{E}FHE$.*

Proof: The "only if" part is immediate by restriction.

It follows from lemma 5.1 that

$$(f|V)_{\#}: (Z|U)\Box(X|V) \to (Z|U)\Box(Y|V)$$

is an FHE, both for $Z = X$ and $Z = Y$, where $U, V \in \mathcal{V}$. Then, by propositions 2.5 and 2.6,

$$f_{\#}|U \times V: Z\Box X|U \times V \to Z\Box Y|U \times V$$

is an FHE and, by [Do, thm. 3.3] $f_{\#}: Z\Box X \to Z\Box Y$ is an FHE. The result follows by theorem 5.2.

A space B will be said to be *numerably contractible* if it admits a numerable cover $\mathcal{V}$ such that, for each $V \in \mathcal{V}$, the inclusion $V \to B$ is null homotopic. Such spaces include CW-complexes, classifying spaces, locally contractable paracompact spaces and spaces having the homotopy types of other numerably contractible spaces [Do, p. 243/4].

Theorem 5.4 *(see [Ma, thm. 2.6] for the $\mathcal{E}$CHP case, and [Do, thm. 6.3] for the $\mathcal{E} = \mathcal{T}$ case).*

Let B be a numerably contractible space, $p: X \to B$ and $q: Y \to B$ be $\mathcal{E}$-overspaces satisfying the $\mathcal{E}$WCHP and $f: X \to Y$ be an $\mathcal{E}$-map over B. Then f is an $\mathcal{E}$FHE if and only if, for at least one choice of $b \in B$ in each path component of B, $f|b: X|b \to Y|b$ is an $\mathcal{E}$-homotopy equivalence.

Proof: The "only if" part is immediate by restriction over $\{b\}$. Conversely, for each $a \in B$, and each b of the specified type in B, $(f|b)_{\#}: \mathcal{E}(Z|a, X|b) \to \mathcal{E}(Z|a, Y|b)$ is a homotopy equivalence for both $Z = X$ and $Z = Y$ (see the sentence preceding lemma 5.1), i.e. $f_{\#}|Z\Box X|\{(a, b)\}$ is a homotopy equivalence for both $Z = X$ and $Z = Y$ (see 2.5.1 and 2.6.1). Now $p\Box p$, $p\Box q$, $q\Box p$ and $q\Box q$ satisfy the WCHP (see proposition 4.5) and it follows via [Do, thm. 6.3] that $f_{\#}$ is an FHE for both $Z = X$ and $Z = Y$. The result then follows from theorem 5.2.

Theorem 5.5 *(compare with [Do, thm. 6.1]). Let $\mathcal{E}$ be a category of enriched spaces, and C be a class of $\mathcal{E}$-spaces with the property that any $\mathcal{E}$-map f, which is between spaces in C and is also a homotopy equivalence, will necessarily be an $\mathcal{E}$-homotopy equivalence. Let B be a numerably contractible space and $p: X \to B$ and $q: Y \to B$ be $\mathcal{E}$-overspaces, satisfying the $\mathcal{E}$WCHP, and such that each path-component of B contains at least one point b with the property that both $X|b$ and $Y|b$ are in C.*

If $g: X \to Y$ is an $\mathcal{E}$-map over B, then g is a an $\mathcal{E}$FHE if and only if it is an ordinary homotopy equivalence between the underlying spaces of X and Y.

Proof: The "if" part if immediate. It follows from proposition 4.3 and [Do, thm. 6.1] that g is an FHE and hence that, for each $b \in B$, $g|b$ is a homotopy equivalence. Our data ensures that, for those b with $X|b$ in $\mathcal{C}$, $g|b$ is an $\mathcal{E}$-homotopy equivalence and so, by theorem 5.4, g is an $\mathcal{E}$FHE.

6 Alternative definitions of $\mathcal{E}$-fibrations.

An $\mathcal{E}$-overspace $q: Y \to B$ will be said to *$\mathcal{E}$-homotopy trivial* if it is $\mathcal{E}$FHE to a trivial $\mathcal{E}$-space over B, i.e. $\pi_B: P \times B \to B$ for some choice of an $\mathcal{E}$-space P.

An $\mathcal{E}$-overspace $q: Y \to B$ will be said to be *$\mathcal{E}$-locally homotopy trivial* or $\mathcal{E}$LHT if there is a numerable cover $\mathcal{V}$ of B such that, for each $V \in \mathcal{V}$, $q|V: Y|V \to V$ is $\mathcal{E}$-homotopy trivial.

We do not assume in the $\mathcal{E}$LHT case that P is fixed: it may vary with V. It is clear that if B is not path-corrected we can find examples where different $\mathcal{E}$-spaces P of different $\mathcal{E}$-homotopy types correspond to different path-components of B.

Let $q: Y \to B$ be an $\mathcal{E}$-overspace. If, for all choices of a space A and a pair of maps f and g from A to B such that $f \simeq g$, the induced $\mathcal{E}$-overspaces q_f and q_g are $\mathcal{E}$FHE, then we will say that q satisfies the *$\mathcal{E}$-homotopy induced property* or $\mathcal{E}$HIP. If this condition is satisfied then it is easily seen, taking A to be a point, that all fibres of q over a given path component of B have the same $\mathcal{E}$-homotopy type.

Let (B, b_0) be a pointed space and $q: Y \to B$ be a given $\mathcal{E}$-overspace. Then if (A, a_0) is a pointed space and $f: A \to B$ is a pointed map the distinguished fibre $Y \times_B A|a_0 = q_f^{-1}(a_0)$ can, of course, be identified with $Y|b_0$. Two $\mathcal{E}$-overspaces q_f and q_g induced in this way by pointed maps f and g from A and B will be said to be *identity $\mathcal{E}$-fibre homotopy equivalent* (1$\mathcal{E}$FHE) if there is an $\mathcal{E}$FHE h from g_f to q_g such that $h|a_0: Y|b_0 \to Y|b_0$ is $\mathcal{E}$-homotopic to the identity on $Y|b_0$. We notice that 1$\mathcal{E}$FHE is an equivalence relation on the class of $\mathcal{E}$-overspaces induced from q by pointed maps from a space A.

We now specify a property of $\mathcal{E}$-overspaces that is useful in cases where the Brown representability theorem is applied to the classification of $\mathcal{E}$-overspaces, such as in [A] and [Bo2].

Let $q: Y \to B$ be an $\mathcal{E}$-overspace. If there is a point $b_0 \in B$ such that for all choices of a pointed space (A, a_0) and a of pair of pointed maps f and g from A to B, such that f is pointed homotopic to g, the induced $\mathcal{E}$-overspaces q_f and q_g are 1$\mathcal{E}$FHE, then we will say that q satisfies the *$\mathcal{E}$-based homotopy induced property* or $\mathcal{E}$BHIP.

Proposition 6.1 *Let $q: Y \to B$ be an $\mathcal{E}$-overspace satisfying the $\mathcal{E}$WCHP* *and b_0 be an arbitrarily chosen point of B. Then q satisfies the $\mathcal{E}BHIP$.*

Proof: Let (A, a_0) be a based space and f and g be based maps from A to B that are homotopic via a based homotopy $F: A \times I \to B$. Our proof consists of showing that $\theta: \overline{Y}_1 \to \overline{Y}_0$, as defined in lemma 4.1, is a 1$\mathcal{E}$FHE.

Applying lemma 4.1 we obtain $\theta, \phi, \alpha, \beta$ and γ as described there. Defining a homotopy $G: A \times I \to B$ by $G(a,t) = F(a, 1-t)$, where $a \in A$ and $t \in I$, and again applying lemma 4.1, but with F replaced by G, we obtain analogous maps and homotopies $\theta': \overline{Y}_0 \to \overline{Y}_1, \phi': \overline{Y}_0 \to \overline{Y}_0, \alpha': \overline{Z} \to \overline{Y}_1 \times I$, $\beta': \overline{Y}_0 \times I \to \overline{Z}$ and $\gamma': \overline{Y}_0 \times I \to \overline{Y}_0 \times I$. The space $\overline{Z}$ has underlying set $\{(y,a,t) | q(y) = G(a,t)\}$, i.e. $\{(y,a,t) | q(y) = F(a, 1-t)\}$, in contrast to $\overline{Y}$ which has underlying set $\{(y,a,t) | q(y) = F(a,t)\}$. However there is a homeomorphism $\lambda : \overline{Y} \to \overline{Z}, \lambda(y,a,t) = (y,a,1-t)$, which preserves $\mathcal{E}$-structure, and is such that $\lambda | (\overline{Y} | A \times 0) = \lambda | \overline{Y}_0$ and $\lambda | (\overline{Y} | A \times 1) = \lambda | \overline{Y}_1$ are the identities on $\overline{Y}_0$ and $\overline{Y}_1$ respectively.

Then there is an $\mathcal{E}$-homotopy over A, i.e. $K: \overline{Y}_1 \times I \to \overline{Y}_1$, running from $\theta'\theta$ to $1: \overline{Y}_1 \to \overline{Y}_1$ and defined by:

$$K(y,a,t) = \begin{cases} \pi_1 \alpha'(\lambda\beta(y,a,2t), 1-2t) & 0 \le t \le \frac{1}{2} \\ \pi_1 \gamma(y,a,2t-1) & \frac{1}{2} \le t \le 1, \end{cases}$$

where $\pi_1 : \overline{Y}_1 \times I \to \overline{Y}_1$ denotes the projection and $(y,a) \in \overline{Y}_1$. Similarly, taking $\mu : \overline{Z} \to \overline{Y}$ to be the inverse of λ, there is an $\mathcal{E}$-homotopy over $A, i.e. L: \overline{Y}_0 \times I \to \overline{Y}_0$, running from $\theta\theta'$ to $1: \overline{Y}_0 \to \overline{Y}_0$ and defined by:

$$L(y,a,t) = \begin{cases} \pi_0 \alpha(\mu\beta'(y,a,2t), 1-2t) & 0 \le t \le \frac{1}{2} \\ \pi_0 \gamma'(y,a,2t-1) & \frac{1}{2} \le t \le 1, \end{cases}$$

where $\pi_0 : \overline{Y}_0 \times I \to \overline{Y}_0$ denotes the projection and $(y,a) \in \overline{Y}_0$. Hence θ is an $\mathcal{E}$FHE.

We notice that $\overline{Y}_0 | a_0, \overline{Y}_1 | a_0, \overline{Y} | a_0 \times I$ and $\overline{Y}_0 \times I | a_0 \times I = (\overline{Y}_0 | a_0) \times I$ can be identified with $Y | b_0, Y | b_0, (Y | b_0) \times I$ and $(Y | b_0) \times I$ respectively. Thus $\theta | a_0$ and $\alpha | (a_0 \times I)$ can be viewed as an $\mathcal{E}$-map $Y | b_0 \to Y | b_0$ and an $\mathcal{E}$-homotopy $(Y | b_0) \times I \to (Y | b_0) \times I$ respectively. Composing the latter with the projection $(Y | b_0) \times I \to Y | b_0$ we obtain an $\mathcal{E}$-homotopy between $1: Y | b_0 \to Y | b_0$ and $\theta | b_0$. Hence θ is a 1$\mathcal{E}$FHE.

Proposition 6.2 *Let $\mathcal{E}$ be a category of enriched spaces and $q: Y \to B$ be an $\mathcal{E}$-overspace. Then the conditions that q satisfies:*

(i) the $\mathcal{E}LHT$ property, *(ii) the $\mathcal{E}WCHP$,*

(iii) the $\mathcal{E}BHIP$, and *(iv) the $\mathcal{E}HIP$*

are related according to the scheme $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$.

Proof: $(i) \Rightarrow (ii)$. Let $\mathcal{V}$ be a numerable cover associated an the $\mathcal{E}$LHT structure on q. Now the $\mathcal{E}$WCHP is easily seen to be preserved by $\mathcal{E}$FHEs of $\mathcal{E}$-overspaces, so for each $V \in \mathcal{V}, q|V$ satisfies the $\mathcal{E}$WCHP. Then q has that same property by theorem 4.7.

$(ii) \Rightarrow (iii)$. This is proposition 6.1

$(iii) \Rightarrow (iv)$. If $\sqcup$ is used to denote disjoint topological sum, then any homotopy $F: A \times I \to B$ determines a based homotopy $G: (A \sqcup \{a_0\}) \times I \to B$ by $G(a,t) = F(a,t)$ where $a \in A$ and $G(a_0,t) = b_0$. Applying the $\mathcal{E}$BHIP for q with G we obtain the $\mathcal{E}$HIP for q with F.

Theorem 6.3 *(compare with [Do, thm. 6.4], which asserts that the $\mathcal{T}$LHT and $\mathcal{T}$WCHP conditions are equivalent). Let B be a numerably contractible space, $\mathcal{E}$ be a category of enriched spaces and $q: Y \to B$ be an $\mathcal{E}$-overspace. Then the $\mathcal{E}$LHT property, the $\mathcal{E}$WCHP, the $\mathcal{E}$BHIP and the $\mathcal{E}$HIP for q are all equivalent.*

Proof: As a consequence of proposition 6.2 we just have to show that the $\mathcal{E}$LHT property is a consequence of the $\mathcal{E}$HIP. If V is a subspace of B such that the inclusion $V \to B$ is homotopic to a constant map to a point, say b, of B then it follows via the $\mathcal{E}$HIP that $q|V$ is $\mathcal{E}$FHE to $\pi_B: (Y|b) \times V \to V$. If we select a cover of B that satisfies the numerably contractible condition, and apply the above argument to each member V of that cover, we see that q satisfies the $\mathcal{E}$LHT property.

7 First example: sectioned fibrations.

In this section we will take $\mathcal{E}$ to be $\mathcal{T}^0$ and $U: \mathcal{T}^0 \to \mathcal{T}$ to be the functor that forgets distinguished points.

A map $q: Y \to B$ together with a section, i.e. a *not necessarily continuous* function $t: B \to Y$ such that $qt = 1_B$, will be called a *sectioned space over* B. Now such a t selects a distinguished point in each fibre $Y|b$ of q over B, where $b \in B$: hence sectioned spaces can be identified with $\mathcal{T}^0$-overspaces and vice-versa.

For example, if B is a space and (P, x_0) is a pointed space, $\pi_B: P \times B \to B$ will denote the projection on B and $\sigma^B: B \to P \times B$ the section to π_B given by $\sigma^B(b) = (x_0, b)$, for all $b \in B$. Then (π_B, σ^B) is the *trivial sectioned space* over B with fibre (P, x_0).

The symbols $(p,s) = (p: X \to A, s: A \to X)$ and $(q,t) = (q: Y \to B, t: B \to Y)$ will be used below, sometimes without any further explanation, to denote sectioned spaces.

We now reformulate various $\mathcal{T}^0$-concepts in sectioned space terminology.

Given a map $g: A \to B$ then $t^g: A \to Y \times_B A, t^g(a) = (tg(a), a), a \in A$, is a section to q_g. Then (q_g, t^g) is the *sectioned space induced from* (q,t) *by* g. In particular if V is a subspace of B then $(q|V: Y|V \to V, t|V: V \to Y|V)$ is a sectioned space.

If $f: X \to Y$ and $g: A \to B$ are maps such that $qf = gp$ and $fs = tg$ then $< f, g >$ will be a *sectioned pairwise map* from (p, s), to (q, t).

There is a sectioned space $(p \times 1: X \times I \to A \times I, s \times 1: A \times I \to X \times I)$, and sectioned pairwise maps $< F, G >: (p \times 1, s \times 1) \to (q, t)$ are *sectioned pairwise homotopies.* Taking $A = B$, constraining g to be 1_B and G to be the projection, there are associated concepts of *sectioned map over* B, *sectioned homotopy over* B and *sectioned fibre homotopy equivalence (sectioned* FHE).

We will say that (q,t) is *sectioned homotopy trivial* if it is sectioned FHE to a trivial sectioned space (π_B, σ^B). If there is a numerable cover $\mathcal{V}$ of B and, for each $V \in \mathcal{V}, (q|V, t|V)$ is sectioned homotopy trivial then (q,t) will be said to be *sectioned locally homotopy trivial* or *sectioned* LHT.

Let us assume that for all choices of a sectioned space $(p: X \to A, s: A \to X)$, a homotopy $F: A \times I \to B$ and a map $f: X \times \{0\} \to Y$ which satisfy the condition that $< f, F|A \times \{0\} >$ is a sectioned pairwise map from $p \times 1_{\{0\}}$ to q, there always exists a homotopy $H: X \times I \to Y$ such that $< H, F >$ is a sectioned pairwise homotopy from $(p \times 1, s \times 1)$ to (q,t). Then (q,t) will be said to possess the *sectioned covering homotopy property* or *sectioned* CHP.

We notice that the various sectioned concepts just defined agree with the analogous $\mathcal{T}^0$-concepts, i.e. the sectioned definitions are just rephrased versions of the $\mathcal{T}^0$-definitions.

Proposition 7.1 *If* (q,t) *is sectioned LHT then* t *is continuous.*

Proof: The sections σ^B to trivial sectioned spaces are continuous and so also, by composition, are the sections $t|V$ for all V in a numerable cover $\mathcal{V}$ of the base space. It follows that t is continuous.

A *well sectioned fibration* is a sectioned space $(q: Y \to B, t: B \to Y)$, where q is a Hurewicz fibration and t is a closed cofibration.

Proposition 7.2 *If* (q,t) *is a well sectioned fibration and* $f: A \to B$ *is a map then* (q_f, t^f) *is also a well sectioned fibration.*

The proof is removed to section 10; the next corollary refers to the special case where f is the inclusion of a point in B.

A pointed space will be said to be *well pointed* if the inclusion of the point in the space is a closed cofibration.

Corollary 7.3 *If* (q,t) *is a well sectioned fibration over* B *then the fibres* $(Y|b, t(b))$ *are well pointed spaces, for all* $b \in B$.

Theorem 7.4 *If (q,t) is a well sectioned fibration then it satisfies the sectioned CHP.*

Proof: Given a homotopy $F: A \times I \to B$ and taking $f = F_0: A \to B$, we notice that $\overline{F}_0: \overline{Y}_0 \times \{0\} = \overline{Y}_0 \to Y$ and $tF(\overline{q}_0 \times 1_I)|t^f(A) \times I: t^f(A) \times I \to Y$ agree on $t^f(A) \times \{0\}$ so they combine to give a map

$$h = \overline{F}_0 \cup (tF(\overline{q}_0 \times 1_I)|t^f(A) \times I): (\overline{Y}_0 \times \{0\}) \cup (t^f(A) \times I) \to Y.$$

Then $qh = F(\overline{q}_0 \times 1_I)|(\overline{Y}_0 \times \{0\}) \cup (t^f(A) \times I)$ and, according to the $\mathcal{E} = \mathcal{T}^0$ version of lemma 3.1, we just have to construct a homotopy $H: \overline{Y}_0 \times I \to Y$ extending h and such that $qH = F(\overline{q}_0 \times 1_I)$. Now t^f is a closed cofibration (proposition 7.2) hence so also is $t^f \times 1: A \times I \to \overline{Y}_0 \times I$ (see [Br, 7.3.3]). The result follows by [St 1, thm.4].

The following corollaries 7.5, 7.6 and 7.8 are direct consequencies of theorem 7.4 in conjunction with theorems 5.4, 6.2 (((*ii*) $\Rightarrow$ (*iv*)) and 6.3 respectively; corollary 7.7 is the particular case of corollary 7.6 where f is the inclusion of a point in B.

Corollary 7.5 *(compare with [E, thm. 3.9]). Let (p,s) and (q,t) be well sectioned fibrations over a numerably contractible space B, and f be a sectioned map over B from (p,s) to (q,t). We will assume that there is at least one point in each path-component of B with the property that $f|b$ is a pointed homotopy equivalence between pointed fibres. Then f is a sectioned FHE.*

Corollary 7.6 *(=HIP for well sectioned fibrations). If (q,t) is a well sectioned fibration, A is a space and f and g are homotopic maps from A to B then (q_f, t^f) and (q_g, t^g) are sectioned FHE.*

Corollary 7.7 *(= [E, prop. 3.5]). If (q,t) is a well sectioned fibration and b_0 and b_1 are a pair of points in the same path-component of B, then the pointed spaces $(Y|b_0, t(b_0))$ and $(Y|b_1, t(b_1))$ are pointed homotopy equivalent.*

Corollary 7.8 *(compare with [E, thm. 3.6]). Any well sectioned fibration over a numerably contractible space is sectioned LHT.*

It is well known, and a particular case of [Br, 7.4.2], that if (X, x_0) and (Y, y_0) are well pointed spaces then any pointed map from X to Y which is a homotopy equivalence, must also be a pointed homotopy equivalence. Hence if we take $\mathcal{E} = \mathcal{T}^0$, then the class of all *weakly well pointed spaces*, i.e. pointed spaces that are pointed homotopy equivalent to well pointed spaces, is an example of a class $\mathcal{C}$, as described in theorem 5.5.

Proposition 7.9 *Let (p,s) and (q,t) be well sectioned fibrations over a numerably contractible space B and $f: X \to Y$ be a sectioned map over B. Then f is a sectioned FHE if and only if it is an ordinary homotopy equivalence between the spaces X and Y.*

Proof: This follows from the above discussion, theorem 5.5, corollary 7.3 and theorem 7.4.

8 Further examples.

(i) Let $\mathcal{E}$ denote $\mathcal{T}$ and $U : \mathcal{T} \to \mathcal{T}$ be the identity functor on $\mathcal{T}$. Then our results 3.6, 4.7, 5.4, 5.5, 6.2 and 6.3 repeat information already known from [Do], in addition 6.2 and 6.3 contain some further information.

(ii) Let G be a topological monoid. If $\mathcal{E} = G\mathcal{T}^{he}$ and $U: G\mathcal{T}^{he} \to \mathcal{T}$ forgets the G-action, then a *principal G-fibration* may be defined to be a $G\mathcal{T}^{he}$-overspace satisfying the $G\mathcal{T}^{he}$LHT condition.

The next two examples are different generalizations of the sectioned space example.

(iii) Let $\mathcal{E}$ denote the category of pairs of spaces and maps of pairs [Sp, p. 22/23]. If $f:(P,P_0) \to (Q_1Q_0)$ is a map of pairs then we define U by $U(P,P_0) = P$ and $Uf = f$.

Then a *pair-overspace* consists of a pair (Y, Y_0) and a map $q: Y \to B$, the fibre of q over b being the pair $(Y|b, Y_0|b)$. This provides an appropriate framework for the study of *pair-fibrations*, such as appear in, for example, the Leray-Hirsch theorem [H, ch. 16, thm. 1.1 and remark 1.2] and in the last (spectral sequence) chapter of [Sp].

(iv) We now take $\mathcal{E}$ to be the category $\mathcal{T}^A$ of spaces under a fixed space A. The objects are maps such as $i: A \to P$ and $j: A \to Q$, and morphisms from i to j are maps $f: P \to Q$ such that $fi = j$. The functor $U: \mathcal{T}^A \to \mathcal{T}$ forgets A and forgets maps out of A, in the obvious way.

A $\mathcal{T}^A$-space over B can clearly be taken as consisting of a pair (q,t), where $q : Y \to B$ is a map and $t : A \times B \to Y$ is a function such that qt is the projection $\pi_B : A \times B \to B$ and, for each $b \in B$, $t|A \times \{b\} : A \times \{b\} \to Y|b$ is continous. In general t itself may not be continuous. However there is an easy direct generalization of proposition 7.1 and its proof, which tells us that: if (q,t) corresponds to a $\mathcal{T}^A$-space that is $\mathcal{T}^A LHT$, then t is continuous.

If B is a given space, Y contains $A \times B$ as a subspace, and $q: Y \to B$ extends π_B, then the fibre $Y|b$ contains a subspace $A \times \{b\}$ that may be identified with A. On this basis q can be viewed as a $\mathcal{T}^A$-overspace. There is a corresponding theory of *fibrations which extend trivial fibrations.*

(v) Let $\mathcal{E}$ be $\mathcal{T}_A$, the category of spaces over a fixed space A. Thus the objects are maps such as $d: P \to A$ and $e: Q \to A$, and a morphism from d to e is a map $f: P \to Q$ such that $d = ef$. The functor $U: \mathcal{T}_A \to \mathcal{T}$ forgets A, and forgets maps into A, in the obvious fashion.

A $\mathcal{T}_A$-space over B determines a corresponding function $q : Y \to A \times B$ in an obvious fashion; in fact a $\mathcal{T}_A$-space over B can be taken to be a function $q : Y \to A \times B$, where $\pi_B q : Y \to B$ is continuous, and for each $b \in B$ $\pi_A q|(Y|A \times \{b\}) : Y|(A \times \{b\}) \to A$ is continuous. Taking $\pi_A : A \times B \to A$ to denote the projection, we see that in general $\pi_A q : Y \to A$, and hence q, may not be continuous. However, by an argument similar to that used in proposition 7.1, it can be shown that: if $q : Y \to A \times B$ corresponds to a $\mathcal{T}_A$-space that is $\mathcal{T}_A LHT$ then $\pi_A q$, and hence q, are continuous.

If $q: Y \to A \times B$ is a map we notice that the fibre of $\pi_B q: Y \to B$ over $b \in B$, i.e. $q|A \times \{b\}: Y|A \times \{b\} \to A \times \{b\}$, can be viewed as an object of $\mathcal{T}_A$. So q can be identified with the $\mathcal{T}_A$-space over B determined by $\pi_B q: Y \to B$, and we have the foundation for a theory of *fibrations over product spaces.*

(vi) We will now take $\mathcal{E}$ to be the category $\mathcal{F}ib$ with objects that are at the same time both Dold fibrations and identification maps, and morphisms that are $\mathcal{T}$-pairwise maps. The functor U: $\mathcal{F}ib \to \mathcal{T}$ remembers only the total spaces (= domains) of the objects of $\mathcal{F}ib$, and only the f of morphisms $< f, g >$.

If $q: Y \to M$ and $m: M \to B$ are Dold fibrations and q is also an (of course surjective) identification then there is an associated $\mathcal{F}ib$-overspace $(mq: Y \to B, \{q|b: Y|b \to M|b\}_{b \in B})$. In fact a modification of the proof of the result that the composite of two Dold fibrations is a Dold fibration, shows that such pairs of Dold fibrations determine $\mathcal{F}ib$-overspaces that satisfy the $\mathcal{F}ib$WCHP. Thus this example provides a suitable context for the study of *composites of Dold fibrations.* It may be generalized to cover *n-stage fibrations* and *infinite towers of fibrations.*

9 Proof of proposition 7.2

It is well known and easily seen that q_f is a Hurewicz fibration. We will show **(i)** $t^f(A)$ is closed in $Y \times_B A$, and that t^f is a cofibration when **(ii)** f is a cofibration and also when **(iii)** f is a Hurewicz fibration. Now every map f is the composite of a cofibration and a Hurewicz fibration [St3, prop. 2], so it follows via 1.2 that t^f is a closed cofibration in the general case.

Proof of (i): $t^f(A) = \{(tf(a),a)|a \in A\} = \{(y,a) \in Y \times_B A|y \in t(B)\} = (f_q)^{-1}t(B)$. Now $t(B)$ is closed (data), so $t^f(A)$ is closed.

Proof of (ii): (given by P.Heath) We will assume that f is a cofibration. The composite of two cofibrations is a cofibration, so tf is a cofibration. Now $(f_q)(t^f) = tf$ and f_q is a cofibration ([St2, thm. 12] or [Bo1, cor. 3]), so t^f is a cofibration [St3, lemma 5].

The next three lemmas are fibred mapping space results, we require that $\mathcal{E} = \mathcal{T}$ and that C is a single point. Then lemma 9.1 is a particular case of theorem 2.2, lemma 9.2 is a homotopy version of lemma 9.1 that can be derived from lemma 9.1 by replacing the $q: Y \to B$ of lemma 9.1 by the composite of q with $\pi_Y: Y \times I \to Y$, and lemma 9.3 is the particular case of lemma 9.2 where $Y = B$ and $q = 1_B$.

We assume that $q: Y \to B$ and $f: A \to B$ are given maps and Z is a given space. Regarding Z as a space over a point via a map $r: Z \to *$, we see that $A \Box Z$ has underlying set $\bigcup_{b \in B} \mathcal{T}(A|b, Z)$ and that $f \Box r$ can be taken to be the obvious projection of $A \Box Z$ to B.

Lemma 9.1 *There is a bijective correspondence between maps $g: Y \times_B A \to Z$ and maps $g^0: Y \to A \Box Z$ over B defined by $g(y,a) = g^0(y)(a)$, where $q(y) = f(a)$.*

Lemma 9.2 *There is a bijective correspondence between homotopies $G: (Y \times_B A) \times I \to Z$ and homotopies over B, $G^0: Y \times I \to A \Box Z$, defined by $G(y,a,u) = G^0(y,u)(a)$, where $q(y) = f(a)$ and $u \in I$.*

Lemma 9.3 *There is a bijective correspondence between homotopies $H: A \times I \to Z$ and homotopies over B, $H^0: B \times I \to A \Box Z$, defined by $H(a,u) = H^0(b,u)(a)$, where $f(a) = b$ and $u \in I$.*

Proof of (iii):. We now assume that f is a Hurewicz fibration.

Let Z be a given space, $g: (Y \times_B A) \times \{0\} \to Z$ be a map and $H: A \times I \to Z$ be a homotopy such that $g(t^f \times 1_{\{0\}}) = H|A \times \{0\}$.

Applying lemmas 9.1 and 9.3 to g and H, respectively, we obtain a map $g^0: Y \times \{0\} \to A \Box Z$ over B with $g(y,a,0) = g^0(y,0)(a)$, where $q(y) = f(a)$, and a homotopy $H^0: B \times I \to A \Box Z$ over B with $H(a,u) = H^0(b,u)(a)$, where $f(a) = b$ and $u \in I$. Then $g^0(t \times 1_{\{0\}})(b,0)(a) = g^0(t(b),0)(a) = g(t(b),a,0) = H(a,0) = H^0(b,0)(a)$, where $f(a) = b$, so $g^0(t \times 1_{\{0\}}) = H^0|B \times \{0\}$.

Hence there is a well defined map $k: (Y \times \{0\}) \cup (t(B) \times I) \to A \Box Z$ defined by $k(y,0) = g^0(y,0)$ when $y \in Y$, and $k(t(b),u) = H^0(b,u)$, where $b \in B$ and $u \in I$ (see [St1, thm. 1]). Now g^0 and H^0 are over B so it follows that $(f\Box r)k = q\pi_Y|(Y \times \{0\}) \cup (t(B) \times I)$, where $\pi_Y: Y \times I \to Y$ denotes the projection.

Taking $\mathcal{E} = \mathcal{T}$ in proposition 3.3, we see that $f\Box r$ is a Hurewicz fibration. Applying the relative covering homotopy property theorem [St1, thm. 4] we see that there is a homotopy over B, $G^0: Y \times I \to A\Box Z$, such that $(f\Box r)G^0 = q\pi_Y$ and $G^0|(Y \times \{0\}) \cup (t(B) \times I) = k$. It follows via lemma 9.2 that there is a homotopy $G: (Y \times_B A) \times I \to Z$ with $G(y,a,u) = G^0(y,u)(a)$, where $q(y) = f(a)$ and $u \in I$. Then $G(y,a,0) = G^0(y,0)(a) = k(y,0)(a) = g^0(y,0)(a) = g(y,a,0)$, so $G|(Y \times_B A) \times \{0\} = g$. Also, if $f(a) = b$ and $u \in I, G(t^f \times 1_I)(a,u) = G(t(b),a,u) = G^0(t(b),u)(a) = k(t(b),u)(a) = H^0(b,u)(a) = H(a,0)$ and thus $G(t^f \times 1_I) = H$, and the proof is complete.

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