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General associativity and general composition for double categories

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GENERAL ASSOCIATIVITY AND GENERAL COMPOSITION FOR DOUBLE CATEGORIES

by Robert DAWSON and Robert PARE

Résumé. Nous démontrons que la composition dans une catégorie double vérifie une loi d'associativité générale. Nous étudions aussi une classe de catégories doubles qui admettent une composition générale associative.

Introduction

The question we wish to consider is whether a compatible arrangement of double cells in a double category has a unique composite, independent of the order in which the operations of horizontal and vertical composition are performed. The following example will serve to illustrate what we mean. Consider the following arrangement of double cells:

<table>
<tr><td>α</td><td>β</td><td rowspan="2">γ</td></tr>
<tr><td rowspan="2">δ</td><td>ϵ</td></tr>
<tr><td>ϕ</td><td>ψ</td></tr>
</table>

We can evaluate this either as

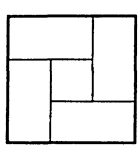
$$((\alpha\beta)\cdot(\delta(\epsilon\cdot\phi)))(\gamma\cdot\psi)$$

or

$$(\alpha\cdot\delta)(((\beta\cdot\epsilon)\gamma)\cdot(\phi\psi)).$$

The question is whether these are equal.

The answer will be yes, in general, but the proof, while easy, is not totally straightforward. The problem is that there are compatible arrangements of double cells which cannot be evaluated at all. The simplest example is the *pinwheel*



Although the whole arrangement fits together nicely, no two double cells are compatible so there is no place to start composing. As we shall see below, the "obvious proof" of general associativity snags on this fact. Note, for example, that if we first compute $\alpha\beta$ and then $\phi\psi$ in our original arrangement, we cannot proceed; we have arrived at the pinwheel! If there are arrangements with no composite, might there not be arrangements with several?

Why call this law general associativity when the order of the various factors seems to change, not only the parenthesis? In fact, the order of the factors does not change in the two-dimensional arrangement, just the order in which the operations are performed. It is merely a by-product of trying to reduce what is essentially a two-dimensional composite to linear notation, which causes the confusion.

To fix notation, we give a brief overview of the theory of double categories. They were introduced by Ehresmann [5] in 1963 (see also [6]). Since then much has been done, much by the Ehresmann group (see [1], [7]). They have also been used extensively in the context of homotopical algebra (see [2], [14], [15]). However, they are mentioned only in passing in the work of the Australian school [10] and Gray [8] on 2-categories and bicategories. There the emphasis is on the foundational aspect of bicategories. They are universes in which to do mathematics (i.e. category theory). We believe that double categories play an important rôle in this context as well, notably in the two dimensional theory of limits. That side of double categories is developed in [11].

A double category $\mathbb{D}$ is a category object, $\mathbf{D}_2 \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathbf{D}_1 \substack{\rightarrow\\\leftarrow\\\rightarrow} \mathbf{D}_0$, in **Cat**. It is a structure with objects (the objects of $\mathbf{D}_0$), vertical morphisms (the morphisms of $\mathbf{D}_0$), horizontal morphisms (the objects of $\mathbf{D}_1$), and double morphisms (or cells) (the morphisms of $\mathbf{D}_1$). The vertical morphisms form a category ($\mathbf{D}_0$) whose composition is denoted $\cdot$ and identities id_A. The horizontal structure also forms a category with composition denoted by juxtaposition and identities by 1_A. In fact the whole structure can be described by the pasting of double morphisms. A double morphism has a horizontal domain and codomain, and a vertical domain and codomain. It can be pictured

$$\begin{tikzcd} A \arrow[r, "h'"] \arrow[d, dotted, "v'"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, dotted, "v"] \\ C \arrow[r, "h"'] & D \end{tikzcd}$$

or when the domains and codomains are not important simply as

$$\boxed{\alpha}$$

Horizontal pasting is denoted

$$\boxed{\alpha \mid \beta} \longmapsto \boxed{\alpha\beta}$$

and vertical pasting by

$$\begin{array}{|c|}\hline \alpha \\ \hline \gamma \\ \hline \end{array} \longmapsto \begin{array}{|c|}\hline \alpha\cdot\gamma \\ \hline \end{array}$$

Each of these operations gives a category structure with identities

$$v \begin{array}{|c|}\hline\hline 1_v \\ \hline\hline \end{array} v \qquad \text{and} \qquad \begin{array}{c} h \\ \begin{array}{||c||}\hline id_h \\ \hline \end{array} \\ h \end{array}$$

respectively. The two category structures commute in the sense that we have the middle four interchange for

$$\begin{array}{|c|c|}\hline \alpha & \beta \\ \hline \gamma & \delta \\ \hline \end{array}$$

$(\alpha\beta) \cdot (\gamma\delta) = (\alpha \cdot \gamma)(\beta \cdot \delta)$.

The question which we address is whether the associativities plus interchange give uniqueness of all possible ways of evaluating compatible arrangements of double cells. Thus we may view our results as coherence theorems for pasting of double cells.

A 2-category may be viewed [10] as a double category in which all vertical morphisms are identities. A 2-category also gives rise [13] to a double category in which the double morphisms are squares with a 2-cell in them:

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

Thus our results yield coherence for pasting in 2-categories.

Mike Johnson has a general pasting theorem for n-categories [9]. When specialized to 2-categories, his pasting schemes take into account any arrangement in the double categories constructed from a 2-category as above. It is not clear whether the converse holds, but we believe our tilings are better behaved as everything is rectangular. Nor do his results seem to apply to double categories in general. John Power [12] also has a general pasting scheme for 2-categories. His approach is more geometric than combinatorial, relying on planar graphs. Our tilings have a combinatorial aspect and a planar component as well. The precise relationship between these various notions is not yet completely clear.

There are various dualities for double categories. There is *op* which switches domain and codomain for the horizontal structure, *co* which does the same for the vertical structure, and there is transpose, *tr*, which interchanges horizontal and vertical. There are also combinations of these, eight in all counting the identity, corresponding to symmetries of the square:

$\mathbb{D}$
$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, dotted, "v"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, dotted, "w"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

$\mathbb{D}^{op}$
$$\begin{tikzcd} B \arrow[r, "h"] \arrow[d, dotted, "w"'] \arrow[dr, phantom, "\alpha"] & A \arrow[d, dotted, "v"] \\ D \arrow[r, "k"'] & C \end{tikzcd}$$

$\mathbb{D}^{co}$
$$\begin{tikzcd} C \arrow[r, "k"] \arrow[d, dotted, "v"'] \arrow[dr, phantom, "\alpha"] & D \arrow[d, dotted, "w"] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

$\mathbb{D}^{coop}$
$$\begin{tikzcd} D \arrow[r, "k"] \arrow[d, dotted, "w"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, dotted, "v"] \\ B \arrow[r, "h"'] & A \end{tikzcd}$$

$\mathbb{D}^{tr}$
$$\begin{tikzcd} A \arrow[r, "v"] \arrow[d, dotted, "h"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, dotted, "k"] \\ B \arrow[r, "w"'] & D \end{tikzcd}$$

$\mathbb{D}^{optr}$
$$\begin{tikzcd} B \arrow[r, "w"] \arrow[d, dotted, "h"'] \arrow[dr, phantom, "\alpha"] & D \arrow[d, dotted, "k"] \\ A \arrow[r, "v"'] & C \end{tikzcd}$$

$\mathbb{D}^{cotr}$
$$\begin{tikzcd} C \arrow[r, "v"] \arrow[d, dotted, "k"'] \arrow[dr, phantom, "\alpha"] & A \arrow[d, dotted, "h"] \\ D \arrow[r, "w"'] & B \end{tikzcd}$$

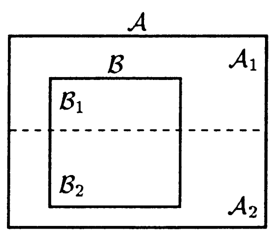
$\mathbb{D}^{cooptr}$
$$\begin{tikzcd} D \arrow[r, "w"] \arrow[d, dotted, "k"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, dotted, "h"] \\ C \arrow[r, "v"'] & A \end{tikzcd}$$

1 General associativity:

For our proof, we need the following lemma.

Lemma 1.1 *If a compatible arrangement of double cells is composable then any subset of them which forms a compatible arrangement is also composable.*

Proof: The proof is by induction on the total number of double cells. If there is only one double cell the result is obvious. Let $\mathcal{A}$ be a compatible arrangement with n double cells and $\mathcal{B} \subseteq \mathcal{A}$ a subarrangement. Assume that $\mathcal{A}$ is composable, so there is some order in which the operations can be performed so as to get a value for $\mathcal{A}$. Let $\gamma(\mathcal{A})$ denote any such value. Let us denote by $*$ the last operation performed in the evaluation of $\gamma(\mathcal{A})$. We can write $\gamma(\mathcal{A}) = \gamma(\mathcal{A}_1) * \gamma(\mathcal{A}_1)$. Let $\mathcal{B}_i = \mathcal{B} \cap \mathcal{A}_i (i = 1, 2)$.

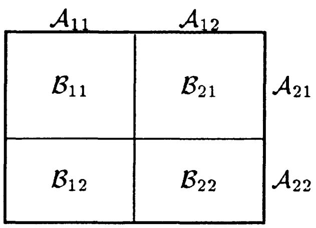


Since $\mathcal{A}_i$ has less than n cells and is composable, unless $\mathcal{B}_i$ is empty, it too is composable by induction. If $\mathcal{B}_i$ is empty, then $\mathcal{B} = \mathcal{B}_{2-i}$ is composable. Otherwise, $\gamma(\mathcal{B}) = \gamma(\mathcal{B}_1) * \gamma(\mathcal{B}_2)$. □

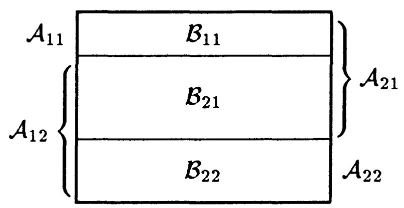
We are now in the position to prove associativity.

Theorem 1.2 *If a compatible arrangement of double cells is composable in two different ways, the results are equal.*

Proof: Let $\mathcal{A}$ be a compatible arrangement of n double cells, composable in two different ways, giving $\gamma_1(\mathcal{A})$ and $\gamma_2(\mathcal{A})$. We shall prove that $\gamma_1(\mathcal{A}) = \gamma_2(\mathcal{A})$ by induction on n. If $n = 1$, the result is trivial. As before $\gamma_i(\mathcal{A}) = \gamma_i(\mathcal{A}_{i1}) *_i \gamma_i(\mathcal{A}_{i2}), (i = 1, 2)$. Let $\mathcal{B}_{ij} = \mathcal{A}_{1i} \cap \mathcal{A}_{2j}$. Some of these may be empty. There are a number of possibilities (8 in all) for the relative positions of the $\mathcal{A}_{ij}$ of which the following two are representative:



(i)



(ii)

By the lemma, each of the $\mathcal{B}_{ij}$ is composable, and by the induction hypothesis the composite, $\gamma(\mathcal{B}_{ij})$, is uniquely determined.

Thus, in case (i) we have

$$\begin{aligned}
\gamma_1(\mathcal{A}) &= \gamma_1(\mathcal{A}_{11})\gamma_1(\mathcal{A}_{12}) \\
&= (\gamma(\mathcal{B}_{11}) \cdot \gamma(\mathcal{B}_{12}))(\gamma(\mathcal{B}_{21}) \cdot \gamma(\mathcal{B}_{22})) \\
&= (\gamma(\mathcal{B}_{11})\gamma(\mathcal{B}_{21})) \cdot (\gamma(\mathcal{B}_{12})\gamma(\mathcal{B}_{22})) \\
&= \gamma_2(\mathcal{A}_{21}) \cdot \gamma_2(\mathcal{A}_{22}) \\
&= \gamma_2(\mathcal{A}).
\end{aligned}$$

In case (ii),

$$\begin{aligned}
\gamma_1(\mathcal{A}) &= \gamma_1(\mathcal{A}_{11}) \cdot \gamma_1(\mathcal{A}_{12}) \\
&= \gamma(\mathcal{B}_{11}) \cdot (\gamma(\mathcal{B}_{21}) \cdot \gamma(\mathcal{B}_{22})) \\
&= (\gamma(\mathcal{B}_{11}) \cdot \gamma(\mathcal{B}_{21})) \cdot \gamma(\mathcal{B}_{22}) \\
&= \gamma_2(\mathcal{A}_{21}) \cdot \gamma_2(\mathcal{A}_{22}) \\
&= \gamma_2(\mathcal{A}).
\end{aligned}$$

The other cases are similar.

□

2 General composition:

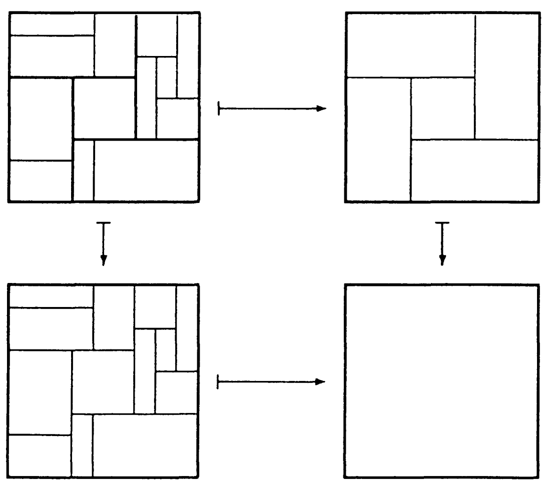
Why is it that the problem of composability does not arise in the case of 2-categories? In most cases occurring naturally, there is a way of composing the pinwheel and, in fact, any other compatible arrangement. For example, if the double morphisms of the pinwheel are commutative squares (or pullbacks) in a category then the large one is also. The pasting theorems of Johnson and Power also show that the same holds for the double categories constructed from 2-categories as discussed above. Thus double categories often have operations of higher arity, not derivable from the two basic binary ones.

In this section we study simple conditions, satisfied by most of the usual double categories, which produce a composite for all compatible arrangements.

Let $\mathrm{Arr}(\mathbb{D})$ denote the set of compatible arrangements in $\mathbb{D}$. We want a multiplication, $\mu : \mathrm{Arr}(\mathbb{D}) \to \mathbb{D}$, associating to each arrangement a uniquely determined double cell. Presumably, if this composite is to be of any use it should satisfy general associativity

$$\begin{tikzcd}
\mathrm{Arr}\,\mathrm{Arr}(\mathbb{D}) \arrow[r, "\mathrm{Arr}(\mu)"] \arrow[d, "\mu\,\mathrm{Arr}(\mathbb{D})"'] & \mathrm{Arr}(\mathbb{D}) \arrow[d, "\mu"] \\
\mathrm{Arr}(\mathbb{D}) \arrow[r, "\mu"'] & \mathbb{D}
\end{tikzcd}$$

An example will clarify what we mean:



The top left corner is an arrangement of arrangements, whereas the bottom left corner is simply an arrangement.

In its simplest form, if $\mathcal{B}$ is obtained from $\mathcal{A}$ by a single composition (Hor or Vert), then associativity would give $\mu(\mathcal{A}) = \mu(\mathcal{B})$. But we can also view this in the reverse direction, i.e. $\mathcal{A}$ is obtained from $\mathcal{B}$ by factoring a double cell, and their composite is still the same. So the idea is to reduce an arbitrary arrangement to a single cell using composites and factorizations. In order to do this we shall assume that we have certain factorizations.

Definition 2.1 *We say that a double category has* RL-factorizations *if for every double cell*

$$
\begin{tikzcd}
\cdot \arrow[r] \arrow[d, dotted] \arrow[dr, phantom, "\alpha"] & \cdot \arrow[d, dotted, "a"] \\
\cdot \arrow[r] & \cdot
\end{tikzcd}
$$

and every factorization of its horizontal codomain, $a = a_1 \cdot a_2$, there exists a factorization of α as $\alpha_1 \cdot \alpha_2$ where the horizontal codomain of α_i is a_i

$$
\begin{tikzcd}
\cdot \arrow[r] \arrow[d, dotted] \arrow[dr, phantom, "\alpha"] & \cdot \arrow[d, dotted, "a"] \\
\cdot \arrow[r] & \cdot
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\cdot \arrow[r] \arrow[d, dotted] \arrow[dr, phantom, "\alpha_1"] & \cdot \arrow[d, dotted, "a_1"] \\
\cdot \arrow[r] \arrow[d, dotted] \arrow[dr, phantom, "\alpha_2"] & \cdot \arrow[d, dotted, "a_2"] \\
\cdot \arrow[r] & \cdot
\end{tikzcd}
$$

Note: RL stands for "right-to-left". There are three "dual" notions: LR-, TB-, and BT- factorizations.

Examples 2.2 Given a 2-category the double category whose double morphisms are 2-cells in a square

$$
\begin{tikzcd}
\cdot \arrow[r, "F"] \arrow[d, "K"'] \arrow[dr, phantom, "\overset{t}{\Rightarrow}"] & \cdot \arrow[d, "G"] \\
\cdot \arrow[r, "H"'] & \cdot
\end{tikzcd}
$$

has all four factorizations. For example, for the RL-factorization, assume $G = G_1 G_2$. Then

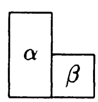
$$
\begin{tikzcd}
\cdot \arrow[r, "F"] \arrow[d, "K"'] \arrow[dr, phantom, "\overset{t}{\Rightarrow}"] & \cdot \arrow[d, "G"] \\
\cdot \arrow[r, "H"'] & \cdot
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\cdot \arrow[r, "F"] \arrow[d, "id"'] \arrow[dr, phantom, "="] & \cdot \arrow[d, "G_1"] \\
\cdot \arrow[r, "FG_1"] \arrow[d, "K"'] \arrow[dr, phantom, "\overset{t}{\Rightarrow}"] & \cdot \arrow[d, "G_2"] \\
\cdot \arrow[r, "H"'] & \cdot
\end{tikzcd}
$$

If a 2-category is considered as a double category whose vertical arrows are identities, it will have RL- and LR-factorizations but not TB- and BT- ones in general.

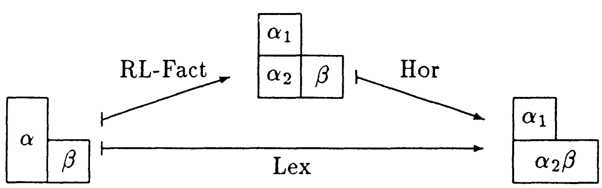
On the other hand, the double category of pullback squares constructed from a category with pullbacks has RL- and BT- factorizations but rarely the other two.

There are two problems with the use of factorization in the reduction of arrangements. The first has to do with our method of proof, which is induction. Factoring increases the number of cells in an arrangement and so takes us further from the composite. In fact, any rank function which composition decreases will be increased by factorization. We solve this by combining a factorization with a composite in certain circumstances.

Suppose that we have double cells in the following position

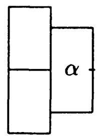


and suppose that α can be RL- factored compatibility with β, then the transformation

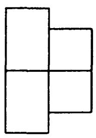
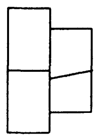
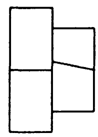


is called *left exchange*. We shall use this and dual notions in our proofs.

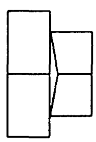
The second problem has to do with the very notion of compatible arrangement. Suppose that α has several left neighbours



and we wish to perform an RL-factorization on α (to use Lex for example). One might wonder whether, after factorization the situation would look like

 ,  or 

In fact it would more likely look like



with a little wedge appearing to the left of α, and this is not allowed in what we consider to be a compatible arrangement.

This seems like a good point to give a clearer account of what we do consider to be a compatible arrangement of double cells in a double category.

3 Tilings:

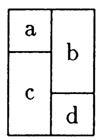
If a rectangle is decomposed into a finite number of smaller rectangles, there is induced on the set T of subrectangles two binary relations: the horizontal neighbour and the vertical neighbour relations H and V. rHs holds if and only if the right side of r intersects the left side of s in more than one point. The definition of rVs expresses that r is an upper neighbour of s in a similar fashion. A *tiling* $\mathcal{T} = (T, H, V)$ is a set T whose elements are called tiles, equipped with two binary relations H and V, which can be represented as the neighbour relations on the decomposition of a rectangle. For example, if we take $T = \{a, b, c, d, e\}$, $H = \{(a, b),\ (c, d),\ (c, e), (d, b)\}$, and $V = \{(a, c),\ (a, d),\ (b, e),\ (d, e)\}$, then (T, H, V) can be represented as a pinwheel.

An intrinsic characterization of tilings, independent of a geometric representation is given in [3]. One of the important properties for us which is studied there is the *neighbour chain property* (NCP). In a tiling the set of right neighbours of a given tile forms a chain with respect to V, i.e. they can be listed as $a_1, \ldots, a_n$ such that $a_i V a_{i+1}$ for all $i = 1, 2, \ldots, n-1$. The full NCP is that all four such conditions hold, i.e, for right, left, top, and bottom.

Tilings are what index our rectangular arrangements. They are the arities for our general composition. The intersections used in theorem 1.2 should be taken at the level of tilings and not at the level of cells where coincidences might occur.

A *compatible arrangement* of double cells in a double category $\mathbb{D}$ consists of a tiling $\mathcal{T} = (T, H, V)$ together with a function $\mathcal{A}$ taking tiles to double cells of $\mathbb{D}$, elements of H to vertical morphisms of $\mathbb{D}$, and elements of V to horizontal morphisms, in such a way that if a has any right neighbour at all, then the horizontal codomain of $\mathcal{A}(a)$ is the composite of the chain $\{\mathcal{A}(a,b) | (a,b) \in H\}$ (which is hereby required to be composable), and the same for left, top, and bottom neighbours. We may denote this by $\mathcal{A} : \mathcal{T} \to \mathbb{D}$.

This definition excludes wedges of the sort mentioned at the end of the previous section. For example, in any arrangement $\mathcal{A}$ indexed by the tiling



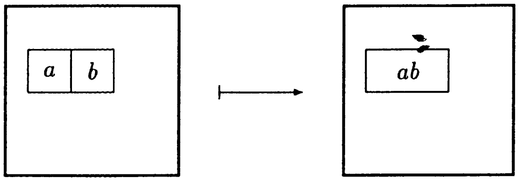
the presence of $(c, b) \in H$, precludes a potential wedge as it must be represented by a vertical morphism which goes into the horizontal domain of $\mathcal{A}(b)$ as well as the horizontal codomain of $\mathcal{A}(c)$.

Remark: A tiling $\mathcal{T}$ generates a free double category $\mathbb{T}$ in the obvious way. Rectangular arrangements $\mathcal{A} : \mathcal{T} \to \mathbb{D}$ are in natural bijection with double functors $\mathbb{A}$: $\mathbb{T} \to \mathbb{D}$.

Representing a tiling $\mathcal{T}$ as the neighbour relations on a decomposition of a rectangle is the same as a compatible arrangement $\mathcal{R} : \mathcal{T} \to \mathbb{R}^2$ where $\mathbb{R}^2$ is double category whose objects are pairs of real numbers (x, y) and whose double morphisms are quadruples (x, y, x', y') with $x \leq x'$ and $y' \leq y$ (with identities corresponding to one or other of the equalities).

It will make things easier later if our rectangular representation $\mathcal{R} : \mathcal{T} \to \mathbb{R}^2$ has integer values, i.e. $\mathcal{R} : \mathcal{T} \to \mathbb{Z}^2 \subseteq \mathbb{R}^2$. There is no loss in generality if we assume this. For example let ϕ: $\mathbb{R} \to \mathbb{R}$ be any strictly increasing function taking all coordinates of corners of rectangles $\mathcal{R}(a)$ to integers. Then $\mathcal{R} : \mathcal{T} \to \mathbb{R}^2 \xrightarrow{\phi^2} \mathbb{R}^2$ has that property.

Horizontal and vertical composition in a compatible arrangement can be described independently of the geometric picture. For example, let $\mathcal{A} : \mathcal{T} \to \mathbb{D}$ be an arrangement, with a and b tiles such that b is the only right neighbour of a and a is the only left neighbour of b. I.e. for $(x, y) \in H$, $x = a \iff y = b$. Create a new tiling $\mathcal{T}' = (T', H', V')$ by identifying a and b, i.e. $T' = T/a \sim b$. We denote the equivalence class $\{a, b\}$ by ab. The right neighbours of ab are the right neighbours of b : $abH'x \iff bHx$. Similarly $xH'ab \iff xHa$, $xV'ab \iff xVa$ or xVb, $abV'x \iff aV'x$ or $bV'x$. A geometric representation of $\mathcal{T}$ gives rise to a canonical one for $\mathcal{T}'$ by erasing the edge between a and b



$\mathcal{T}$ $\qquad\qquad\qquad$ $\mathcal{T}'$

The arrangement $\mathcal{A}$ is descended to $\mathcal{A}' : \mathcal{T}' \to \mathbb{D}$ by defining $\mathcal{A}'(ab) = \mathcal{A}(a)\mathcal{A}(b)$ which is defined because the horizontal codomain of $\mathcal{A}(a)$ is the composite of all $\mathcal{A}(a,x)$ for $(a,x) \in H$, i.e. $\mathcal{A}(a,b)$, and the horizontal domain of $\mathcal{A}(b)$ is $\mathcal{A}(a,b)$ for the same reason. For all other $x \in T'$, $\mathcal{A}'(x) = \mathcal{A}(x)$.

$\mathcal{A}'(x,y) = \mathcal{A}(x,y)$ as long as neither x nor y is ab. If y is a right neighbour of ab, then $\mathcal{A}'(ab,y) = \mathcal{A}(b,y)$, and similarly for left neighbours. If a and b share a lower neighbour c in $\mathcal{T}$, then $\mathcal{A}'(ab,c) = \mathcal{A}(a,c)\mathcal{A}(b,c)$ which exists because a and b are horizontal neighbours in the chain of upper neighbours of c. On the other hand, if c is a lower neighbour of a or b but not both, then $\mathcal{A}'(ab,c) = \mathcal{A}(a,c)$ or $\mathcal{A}(b,c)$ whichever makes sense.

Checking the domain and codomain conditions is straightforward, except possibly the vertical ones for ab. The lower neighbours of a form a chain $c_1, c_2, \ldots, c_k$ and the vertical codomain of $\mathcal{A}(a)$ is $\partial\mathcal{A}(a) = \mathcal{A}(a,c_1)\mathcal{A}(a,c_2)\ldots\mathcal{A}(a_1c_k)$ which is assumed composable. Similarly $\partial_1\mathcal{A}(b) = \mathcal{A}(b,d_1)\mathcal{A}(b,d_2)\ldots\mathcal{A}(b,d_l)$. Thus

$$\begin{aligned}\partial_1\mathcal{A}'(ab) &= \partial_1(\mathcal{A}(a)\mathcal{A}(b)) = (\partial_1\mathcal{A}(a))(\partial_1\mathcal{A}(b)) \\ &= \mathcal{A}(a,c_1)\ldots\mathcal{A}(a,c_k)\mathcal{A}(b,d_1)\ldots\mathcal{A}(b,d_l)\end{aligned}$$

which is the composite of the $\mathcal{A}'(ab,e_i)$ where the e_i are the lower neighbours of ab.

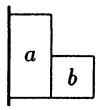
If $\mathcal{T}'$ comes for $\mathcal{T}$ by horizontal composition, then a rectangular representation for $\mathcal{T}$, $\mathcal{R} : \mathcal{T} \to \mathbb{R}^2$ gives, by this method, a canonically associated one for $\mathcal{T}'$, $\mathcal{R}' : \mathcal{T}' \to \mathbb{R}^2$. If we start with $\mathcal{R} : \mathcal{T} \to \mathbb{Z}^2$ then all successive ones have the same property.

The main contribution to our understanding of composition in a rectangular arrangement that the above admittedly abstract presentation gives is that it can in fact be done without pictures. It is of course much less understandable but it must be resorted to when there is danger of the diagrams suggesting things that are not there.

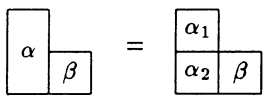
When we talk of *factorization* in theorems 4.1 and 5.1 we mean at the level of the whole arrangement. Thus when we say that $\mathcal{B}$ is obtained from $\mathcal{A} : \mathcal{T} \to \mathbb{D}$ by factorization, we mean that there exists a tiling $\mathcal{T}'$ and an arrangement $\mathcal{B} : \mathcal{T}' \to \mathbb{D}$ such that $\mathcal{A}$ is obtained from $\mathcal{B}$ by composition. So the existence of RL-factorizations does not imply that we can factor a cell in an arrangement if it lies inside.

A rectangular representation for $\mathcal{T}'$ gives one for $\mathcal{T}$ as described above, but when we use factorization we do not assume that this is the same one that we started with on $\mathcal{T}$. To say Fact: $\mathcal{A} \mapsto \mathcal{B}$ means exactly that there exists a $\mathcal{B}$ and Comp: $\mathcal{B} \longmapsto \mathcal{A}$.

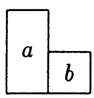
Left exchange is treated similarly. It is performed on a compatible arrangement $\mathcal{A} : \mathcal{T} \longrightarrow \mathbb{D}$. Let a and b be tiles in the following position



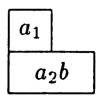
with *a on the left border* of $\mathcal{T}$. Abstractly, a has no left neighbours, several right neighbours of which b is the furthest down, and b has only a as left neighbour. Assume that $\mathcal{A}(a) = \alpha$ can be RL-factored compatibility with $\mathcal{A}(b) = \beta$:



Construct a new tiling $\mathcal{T}'$ by replacing



by



formally, and a new arrangement $\mathcal{A}' : \mathcal{T}' \longrightarrow \mathbb{D}$ by $\mathcal{A}'(a_1) = \alpha_1$ and $\mathcal{A}'(a_2b) = \alpha_2\beta$.

If $\mathcal{R} : \mathcal{T} \longrightarrow \mathbb{Z}^2$ is a rectangular representation of $\mathcal{T}$ then, because $\mathbb{Z}^2$ has RL-factorizations, we get a rectangular representation $\mathcal{R}' : \mathcal{T} \longrightarrow \mathbb{Z}^2$. Here the factorizations cause no problems because we are assuming that a lies on the left border. When we write

$$\mathrm{Lex} : \mathcal{A} \longmapsto \mathcal{A}'$$

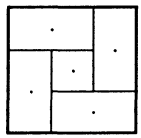
it is understood that if $\mathcal{A}$ has a rectangular representation, then $\mathcal{A}'$ inherits one in the above manner.

4 Key-based composition:

We show in this section, that in the presence of any of the four factorizations mentioned above, any rectangular arrangement can be reduced to a single double cell using a finite number of compositions and factorizations. We give a canonical reduction somewhat similar to placing all brackets to the left for one binary operation. Uniqueness of the result will be discussed in the next section.

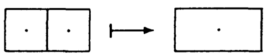
We shall assume that our double category has RL-factorizations, and so we take as our operations horizontal and vertical composition and left exchange (which involves a factorization and a composition). The other cases are treated dually.

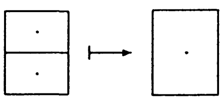
We define the *rank*, $\rho(\mathcal{R})$, of a rectangular representation $\mathcal{R} : \mathcal{T} \longrightarrow \mathbb{Z}^2$ to be the sum over all tiles of the distance from the centres of the corresponding rectangles to the top of the tiling. Thus, if $\mathcal{R}(a) = [l_a, r_a] \times [b_a, t_a]$, then $\rho(\mathcal{R}) = \sum_{a \in T} [\tau - (t_a + b_a)/2]$ where $\tau = y$ coordinate of the top. E.g. if the pinwheel is represented as a 3×3 square in the obvious way

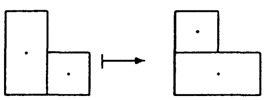


its rank is $\frac{1}{2} + 1 + 1\frac{1}{2} + 2 + 2\frac{1}{2} = 7\frac{1}{2}$. Note that, the higher a rectangle lies, the less it contributes to the rank.

Each of our operations decreases the rank:

Hor:  (one disappears)

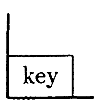
Vert:  (bottom one rises and the top disappears)

Lex:  (the right one stays at the same height, the other rises)

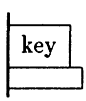
Theorem 4.1 *If the double category $\mathbb{D}$ admits any one of the above factorizations, then any rectangular arrangement of double cells can be reduced to a single double cell using a finite number of compositions and factorizations.*

Proof: We may assume, without loss of generality, that $\mathbb{D}$ has RL-factorizations. Let $\mathcal{A} : \mathcal{T} \to \mathbb{D}$ be a rectangular arrangement of double cells, and choose a representation of $\mathcal{T}, \mathcal{R} : \mathcal{T} \to \mathbb{Z}^2$. Its rank, $\rho(\mathcal{R})$, is a half integer and as (Hor), (Vert), and (Lex) each decrease it, any sequence of these operations performed on $\mathcal{A}$ must stop in $\leq 2\rho(\mathcal{R})$ steps.

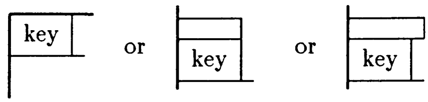
Define the *key tile* of $\mathcal{T}$ to be the furthest down of those tiles on the left border that have no more than one upper neighbour. The upper left corner tile has no upper neighbours so there is always at least one tile satisfying the condition, so the key tile always exists. If there is a tile below the key it is necessarily wider (has several upper neighbours) otherwise the key would not be the lowest. So the situation is either



or

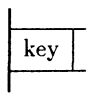


Either the key has no upper neighbours or the one above it is the same width or wider, so for upper neighbours the situation is



(a) (b) (c)

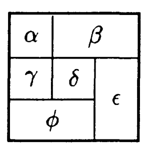
The *key operation* on $\mathcal{A}$ is the one involving the key tile defined as follows. If we are in situation (b) above, then it is vertical composition. Otherwise the situation looks like



If the key has a unique right neighbour, then a horizontal composite is possible and that is the key operation. If there are several right neighbours a left exchange is possible with the furthest down of them, and we take this as the key operation. Unless there is only one tile, the key operation can always be performed. This completes our proof as there are no infinite sequences of these operations. □

Now make a choice of RL-factorizations. Then left exchange is an actual (partially defined) operation. The procedure described in Theorem 4.1 will yield a well defined tile $\mu(\mathcal{A})$ for any rectangular arrangement $\mathcal{A} : \mathcal{T} \to \mathbb{D}$. We shall call this the *key-based composite* of $\mathcal{A}$.

As left exchange involves arbitrary choices of RL-factorizations, one would be surprised if μ satisfied general associativity, and in fact it doesn't. Consider, for example,



with two other cells α', γ' with the same domains and codomains as α and γ respectively and such that $\alpha' \cdot \gamma' = \alpha \cdot \gamma$ but no other relation. Also assume that the chosen RL-factorization of $\alpha \cdot \gamma$ is into α' and γ'. Then μ of the above is $(\alpha'\beta) \cdot (((\gamma'\delta) \cdot \phi)\,\epsilon)$, whereas if we first compose α with β and then take μ we get $(\alpha\beta) \cdot (((\gamma\delta) \cdot \phi)\,\epsilon)$. In the free situation, there is no way to transform either of these composites into anything.

Admittedly this is an artificial example. In the next section we see why it is difficult to find a natural one.

5 The main theorem.

Theorem 5.1 *If the double category $\mathbb{D}$ admits any two of the factorizations RL, LR, TB, and BT, then any two sequences of compositions and factorizations which reduce a compatible arrangement to single cells yield the same result. If one of these factorizations is used to define the key-based composite, then it satisfies general associativity.*

Proof: The second statement is a consequence of the first as general associativity is merely a question of the order in which various operations are performed.

Without loss of generality we may assume that $\mathbb{D}$ has RL- and LR-factorizations or RL- and BT-factorizations. The other cases follow by duality. Make a choice of RL-factorizations and use it to define the key-based composite, $\mu(\mathcal{A})$, of any compatible arrangement $\mathcal{A} : \mathcal{T} \to \mathbb{D}$. We wish to show that if $\lambda : \mathcal{A} \longmapsto \mathcal{A}'$ is the result of applying one instance of Hor or Vert, then $\mu(\mathcal{A}) = \mu(\mathcal{A}')$. This will prove our theorem as factorizations are just the reverse of compositions, so μ is also invariant under these.

Choose a rectangular representation of $\mathcal{T}$, $\mathcal{R} : \mathcal{T} \to \mathbb{Z}^2$. Our plan is to show that for each of the three cases, Vert (1), Hor (2), Lex (3) of the key operation $\kappa : \mathcal{A} \longmapsto \mathcal{B}$, there is path

$$\mathcal{B} = \mathcal{B}_0,\ \mathcal{B}_1, \ldots, \mathcal{B}_k = \mathcal{A}'$$

of arrangements of rank less than $\mathcal{A}$, such that for every consecutive pair, $\mathcal{B}_{i-1}$, $\mathcal{B}_i$, one is obtained from the other by one of our operations λ. It would then follow by induction that

$$\mu(\mathcal{B}) = \mu(\mathcal{B}_0) = \mu(\mathcal{B}_1) = \ldots = \mu(\mathcal{B}_k) = \mu(\mathcal{A}')$$

and as κ is the key operation, $\mu(\mathcal{A}) = \mu(\mathcal{B})$. So our proof would be complete.

There are six cases to consider for the various possibilities for κ and λ and five go through without problem. In order to deal with the sixth we must allow λ to be left exchange as well, and not just with the choice of RL-factorizations but arbitrary Lex. Now we have nine cases and $8\frac{1}{2}$ work without problem. To deal with the remaining one, we now allow λ to Rex and Tex as well. So there are five possibilities for λ : Vert (a), Hor (b), Lex(c), Rex (d), and Tex (e). There are now 15 cases (1a -3e) to consider, and $14\frac{1}{2}$ go through in a straightforward way without needing the extra factorization. It is only in one subcase of 3c that we use it.

First note that if the tiles which κ changes are distinct from those which λ changes, the operations commute, i.e. performing κ on $\mathcal{A}'$ gives the same as performing λ on $\mathcal{B}$:

$$\begin{tikzcd} \mathcal{A} \arrow[r, mapsto, "\lambda"] \arrow[d, mapsto, "\kappa"'] & \mathcal{A}' \arrow[d, mapsto, "\kappa"] \\ \mathcal{B} \arrow[r, mapsto, "\lambda"'] & \mathcal{B}' \end{tikzcd}$$

So in this case we are done.

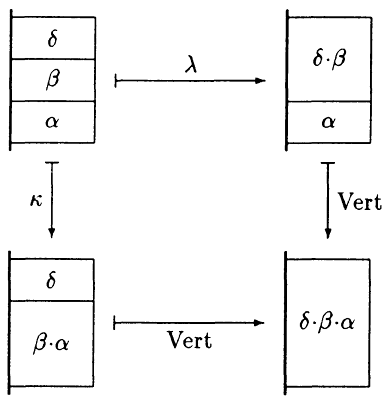
Thus we need only consider the situation when the operations "overlap". As each operation involves two tiles, there are at most four ways they can overlap, so the possibilities are well contained. We shall only do a few representative cases in detail. The others being similar are left to the reader.

In the following diagrams, α is the key tile so lies on the left border (shown as a thick line). κ operates on α and β, and λ on γ and δ. The cells not pictured do not change.

Case 1a:

$\kappa = \text{Vert} \quad ; \quad \lambda = \text{Vert}$

(i) $\gamma = \beta$.



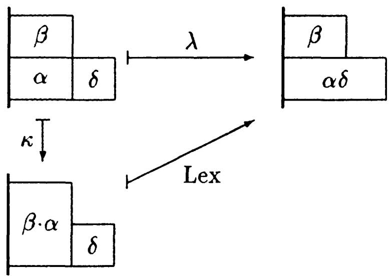
(ii) $\gamma = \alpha$ and $\beta = \delta$.
Then $\kappa = \lambda$ so $\mu(\mathcal{A}) = \mu(\mathcal{A}')$ trivially.

(iii) $\delta = \alpha$.
This is impossible as the tile below the key α must be wider.

Case 1b:

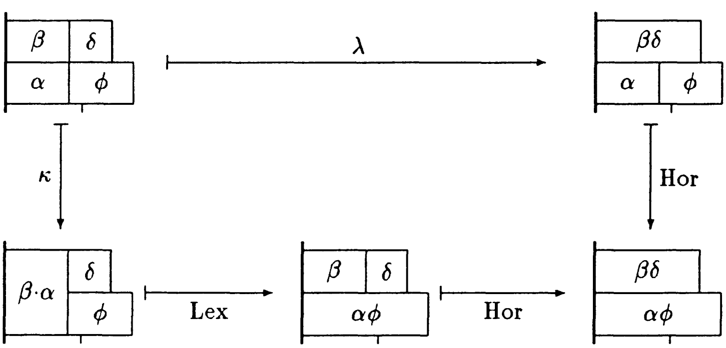
$\kappa = \text{Vert} \quad ; \quad \lambda = \text{Hor}$

(i) $\gamma = \alpha$

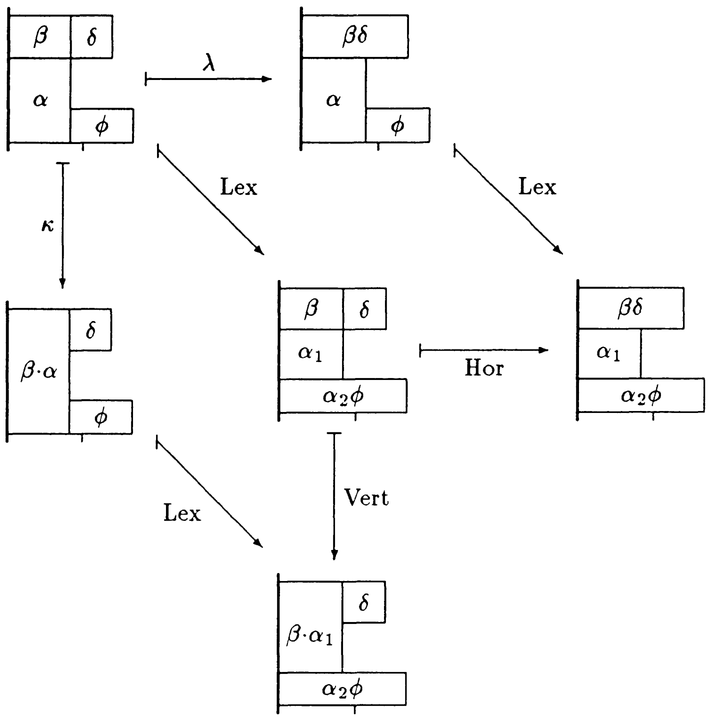


Note that Lex can be performed with any appropriate factorization, not just the chosen one which must be used with the key operations.

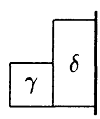
(ii) $\gamma = \beta$ and α has one right neighbour ϕ. Since α is the key tile, the one below it (if there is one) is necessarily wider. Thus not only is ϕ the unique right neighbour of α but α is the unique left neighbour of ϕ, i.e, α and ϕ are composable.



(iii) $\gamma = \beta$, α has several right neighbours. As in (ii) above, because α is the key tile, the lowest right neigbour ϕ of α matches with α so as to allow a left exchange.

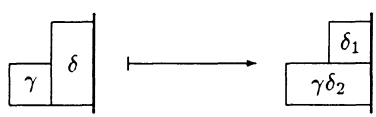


Before proceeding we should spell out what Rex and Tex are. *Right exchange* is applied to tiles in the position



with δ along the right border. If δ can be factored as $\delta_1 \cdot \delta_2$ with δ_2 composable with γ then

Rex :



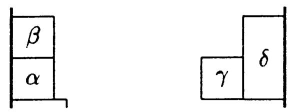
Top exchange is similar. If γ is along the top border and can be factored as $\gamma_1\gamma_2$ with γ_2 composable with δ then

Tex :

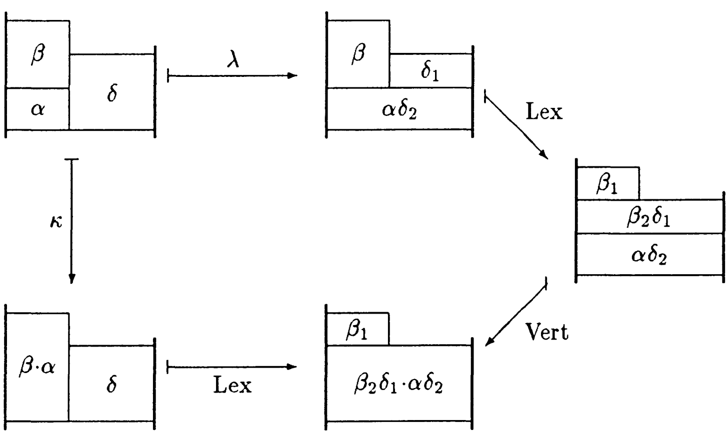
It is important that Rex and Tex both decrease the rank.

Case 1d is the most complicated as there are subcases which depend on the relative sizes of the tiles.

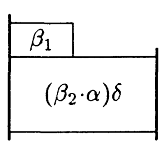
$\kappa = \text{Vert} \quad ; \quad \lambda = \text{Rex}$



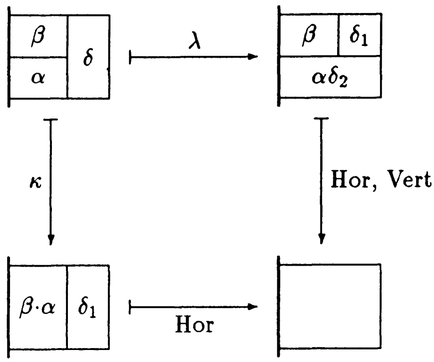
(i) $\gamma = \alpha$ and δ is shorter that $\beta \cdot \alpha$.



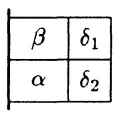
Here again we use the fact that Lex can be performed using any compatible RL-factorization. Thus we first factor β into β_1 and β_2 compatibly with δ_1, then we use the factorization of $\beta \cdot \alpha$ into β_1 and $\beta_2 \cdot \alpha$ to perform the exchange with δ. This gives



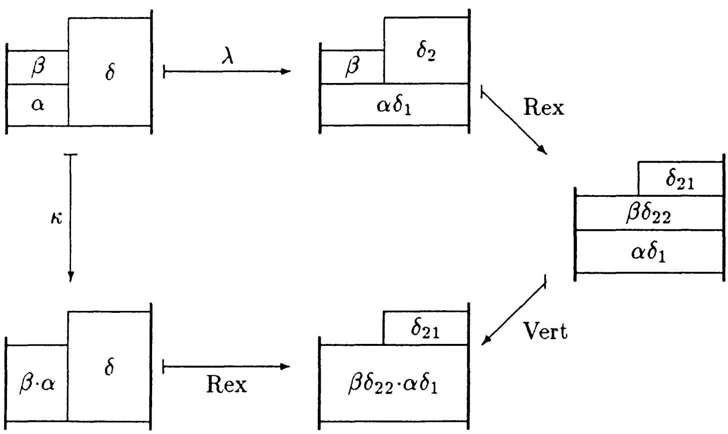
But $(\beta_2 \cdot \alpha)\delta = (\beta_2 \cdot \alpha)(\delta_1 \cdot \delta_2) = (\beta_2\delta_1) \cdot (\alpha\delta_2)$.
(ii) $\gamma = \delta$ and δ is the same height as $\alpha \cdot \beta$.



the right bottom corner being the unique composite of



(iii) $\gamma = \delta$ and δ is taller than $\alpha \cdot \beta$



(iv) $\gamma = \beta$ and α has one right neighbour ϕ. This subcase is similar to 1b subcase (ii) and left to the reader.

(v) $\gamma = \beta$ and α has several right neighbours. This is similar to 1b subcase (iii).

Case 1e:

$\kappa = \text{Vert} \quad ; \quad \lambda = \text{Tex}$

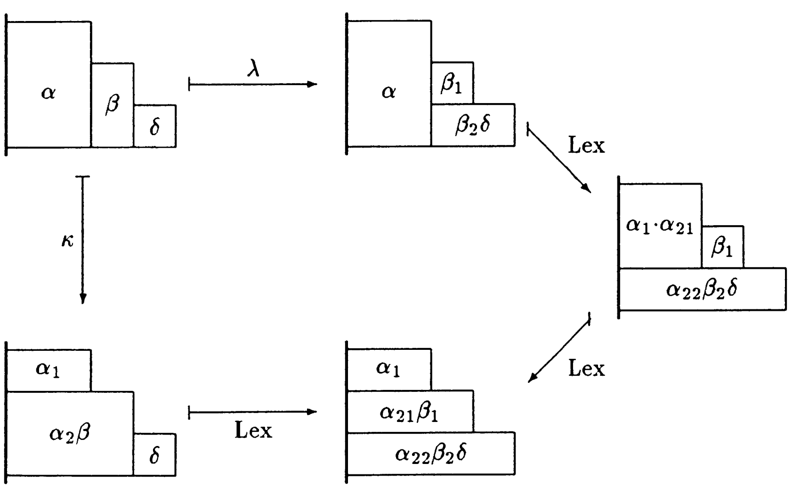
This case is easy as there is no possibility for overlap.

Fourteen of the fifteen cases go through in exactly the same manner as the above examples, the existence of the second factorization never being needed except in one subcase of 3c.

Case 3c:

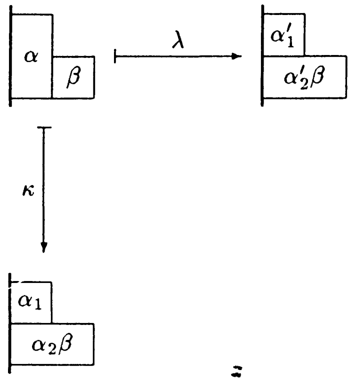
$\kappa = \text{Lex} \quad ; \quad \lambda = \text{Lex}$

(i) $\gamma = \beta$



The factorizations must be performed in the following order: α is factored into α_1 and α_2, the chosen factorization for the key exchange; β is factored into β_1 and β_2 compatibility with $\delta; \alpha_2$ is factored into α_{21} and α_{22} compatibility with β_2.

(ii) $\gamma = \delta$. Then $\beta = \delta$. κ is left exchange with the chosen factorization for α and κ is left exchange with another factorization. This case must be considered as in our previous cases we used left exchange with factorizations other than the chosen ones, e.g. 1bi and 1di.



$$\begin{tikzcd} \mathcal{A} \arrow[r, mapsto, "\lambda"] \arrow[d, mapsto, "\kappa"'] & \mathcal{A}' \\ \mathcal{B} & \end{tikzcd}$$

As our double category $\mathbb{D}$ has either LR- or BT -factorizations, there is an operation which can be performed on $\mathcal{A}$ similar to the key operation but on the right border or top border respectively. It might be Hor, Vert, Tex, or Rex but definitely not Lex. Call it $\nu : \mathcal{A} \longmapsto \mathcal{C}$. By applying the appropriate one of 3a, 3b, 3d, 3e, we have a path of lesser rank tilings joining $\mathcal{B}$ to $\mathcal{C}$. Since we assumed nothing about our choice of RL-factorization in defining the key operation, we could equally well have chosen α'_1 and α'_2 as our factorization of α, compatible with β. So the same argument as above gives a path of lesser rank tilings joining $\mathcal{A}'$ to $\mathcal{C}$. Thus by induction, $\mu(\mathcal{A}) = \mu(\mathcal{C}) = \mu(\mathcal{A}')$. This completes the proof. □

In future work [4] we shall show how canonical factorizations give the same result.

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