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THE CARTESIAN CLOSED TOPOLOGICAL HULL OF THE CATEGORY OF COMPLETELY REGULAR FILTERSPACES

by ***H.L. BENTLEY*** *and* ***E. LOWEN-COLEBUNDERS***

RESUME. Dans cet article, on construit l'enveloppe topologique cartésienne fermée de la catégorie de tous les espaces fibrés complètement réguliers, et des applications uniformément continues. Les objets de cette nouvelle catégorie sont caractérisés: ce sont les espaces filtrés pseudotopologiques μ-réguliers, à domaine μ-fermé.

1. INTRODUCTION.

It is a well known fact that **Creg**, the category of completely regular topological spaces and continuous maps, is not cartesian closed and hence is inconvenient for many purposes in homotopy theory, topological algebra and functional analysis. Fortunately **Creg** can be fully embedded in a cartesian closed topological hull. This hull is known to be the category **Cemb** of c-embedded convergence spaces and continuous maps. The objects of **Cemb** have been internally characterized as those convergence spaces that are ω-regular, ω-closed domained and pseudotopological [19]. The category **Cemb** has proved to be extremely useful in the study of function spaces of realvalued maps [6]. For a discussion of all the nice aspects of a cartesian closed category we refer to the original paper of H. Herrlich [13] and to a survey paper of L. Nel [21]. The role of cartesian closedness in functional analysis is clearly demonstrated in [11].

Complete regularity is one of the most interesting notions in topology, since it is closely related to the real number system. In order to develop a theory about completely regular extensions of topological spaces, Bentley, Herrlich & Ori introduced a notion of complete regularity for merotopic spaces [4]. In one of their main results it is stated that a merotopic space is a subspace in **Mer** (the category of merotopic spaces and uniformly continuous maps) of a completely regular topological space if and only if it is a completely regular filterspace.

From this theorem, it follows that the category **CregFil** of completely regular filterspaces plays an important role when studying extensions or completions. However, like its topological counterpoint, **CregFil** is not cartesian closed. Therefore, with regard to some particular problems related to the areas mentioned above, it has some deficiencies. It is well known that the larger category **Fil** of all filtermerotopic spaces is a cartesian closed supercategory of **CregFil**. In some respects, it is too big to retain enough structure of **CregFil**. Hence it seems desirable to find a cartesian closed topological hull of **CregFil**. The construction of this hull and the internal characterization of its objects are the main subjects of this paper. The construction of this cartesian closed topological hull is a new example in the list of "improvement by enlargement". Other examples are **Cemb**, the cartesian closed topological hull of **Creg**, the category **Pstop** of pseudotopological spaces which is the cartesian closed topological hull of the category **Prtop** of pretopological spaces, and the category **Ant** of Antoine spaces which is the cartesian closed topological hull of **Top**. A survey on the situation of **Top** and **Prtop** can be found in [15].

2. PRELIMINARIES.

In order to keep the exposition as brief as possible we assume familiarity with merotopic spaces (see e.g. [14, 17, 2, 20]). We recall the basic definitions.

A *merotopic space* is a set X together with a subset γ of P^2X such that:

(M1) If $\mathcal{A}$ corefines[1]) $\mathcal{B}$ and $\mathcal{A} \in \gamma$, then $\mathcal{B} \in \gamma$.
(M2) $\forall\, x \in X$, on a[2]) $\dot{x} \in \gamma$.
(M3) $\{\emptyset\} \in \gamma$ and $\emptyset \not\in \gamma$.
(M4) If $\mathcal{A} \cup \mathcal{B} \in \gamma$ then $\mathcal{A} \in \gamma$ or $\mathcal{B} \in \gamma$.

The members of γ are said to be *micromeric.* γ is called the *merotopy*. The explicit mention of the merotopy is often suppressed.

A map $f: X \to Y$ between merotopic spaces X and Y is called *uniformly continuous* iff whenever $\mathcal{A}$ is micromeric in X, then

$$f(\mathcal{A}) = \{f(A) \mid A \in \mathcal{A}\}$$

is micromeric in Y.

Mer is the concrete category of merotopic spaces and uniformly continuous maps. **Mer** is a topological construct. A

1) $\mathcal{A}$ corefines $\mathcal{B}$ iff $\forall\, A \in \mathcal{A}\ \exists B \in \mathcal{B}$ with $B \subset A$. We denote $\mathcal{A} < \mathcal{B}$ or $\mathcal{B} > \mathcal{A}$.
2) $\dot{x} = \{F \subset X \mid x \in F\}$.

collection $\boldsymbol{B}$ of subsets of a merotopic space X is called a *uniform cover* of X if whenever $\boldsymbol{A}$ is micromeric in X there is a $B \in \boldsymbol{B}$ and $A \in \boldsymbol{A}$ such that $A \subset B$.

X is called a *filter(merotopic) space* if it is a merotopic space and every micromeric collection is corefined by a micromeric filter.

Micromeric filters are also called Cauchy filters. It is clear that a filterspace is completely determined by its Cauchy filters. The full subcategory of **Mer** whose objects are the filterspaces is denoted by **Fil**.

Fil is bicoreflective in **Mer** and **Fil** is cartesian closed.

A *Cauchy space* is a filterspace which satisfies: If $\boldsymbol{A}$ is micromeric and $\boldsymbol{B}$ is micromeric with $\emptyset \not\in \boldsymbol{A} \wedge \boldsymbol{B}$, then $\boldsymbol{A} \vee \boldsymbol{B}$ is micromeric, where:

$$\boldsymbol{A} \wedge \boldsymbol{B} = \{A \cap B \mid A \in \boldsymbol{A},\ B \in \boldsymbol{B}\},\ \boldsymbol{A} \vee \boldsymbol{B} = \{A \in \boldsymbol{A},\ B \in \boldsymbol{B}\}.$$

A merotopic space is said to be *completely regular* [4] if whenever $\boldsymbol{A}$ is a micromeric collection in X, then so is the collection

$$\{B \subset X \mid A \text{ is completely within } B \text{ for some } A \in \boldsymbol{A}\}.$$

That A is *completely within* B means that there exists a uniformly map $f: X \to [0,1]$ with $f(A) \subset \{0\}$ and $f(X|B) \subset \{1\}$. Here $[0,1]$ is understood to carry its usual topological structure.

The category of all symmetric ($= R_0$) convergence spaces and then a fortiori also the category of all symmetric topological spaces can be fully embedded in ***Fil***. The embedding functor F maps a symmetric convergence space X to the filter space FX where $\boldsymbol{A}$ is micromeric in FX iff there is a filter $\boldsymbol{F}$, convergent in X and corefining $\boldsymbol{A}$. We will identify the category of symmetric convergence spaces (symmetric topological spaces) with its isomorphic image through F and call this category $\mathbf{Conv_S}$ ($\mathbf{Top_S}$ respectively). The objects in $\mathbf{Conv_S}$ (or $\mathbf{Top_S}$) will simply be called convergence spaces (or topological spaces).

In particular the real line with its usual topology will be considered as a topological space and also as a filterspace. For both the notation $\mathbf{R}$ will be used.

Thus X is a convergence space iff it is a filterspace satisfying the following additional properties:

– If $\boldsymbol{A}$ and $\boldsymbol{B}$ are micromeric in X and if for some point $x \in X$ we have $\boldsymbol{A} \vee \dot{x}$ micromeric and $\boldsymbol{B} \vee \dot{x}$ micromeric, then $\boldsymbol{A} \vee \boldsymbol{B}$ is micromeric.

– For every micromeric collection $\boldsymbol{A}$ in X there exists $x \in X$ such that $\boldsymbol{A} \vee \dot{x}$ is micromeric.

In any merotopic space X we will use the short notation $\boldsymbol{A} \xrightarrow{X} x$ or $\boldsymbol{A} \to x$ for the expression "$\boldsymbol{A} \vee \dot{x}$ is micromeric on X". We say that $\boldsymbol{A}$ converges to x. Note that the convergence spaces we are dealing with satisfy Fisher's axioms [10].

Also, in our context, a convergence space is automatically symmetric. It means that a filter $\boldsymbol{F}$ converges to x whenever $\boldsymbol{F} \vee \dot{x}$ is convergent. This will cause no problems when applying results from the literature on convergence spaces (without implicit symmetry) since we will mainly be interested in c-embedded convergence spaces, and they are always symmetric.

In [8] Bourdaud described the cartesian closed topological hull of the category of completely regular topological spaces. In order to give an internal characterization of the objects of the hull, he used the following notions:

A convergence space X is *pseudotopological* [9] if a filter $\boldsymbol{F}$ converges to x whenever all ultrafilters finer than $\boldsymbol{F}$ converge to x. A convergence space X is ω-*regular* [22] if $\mathrm{cl}_{\omega X}\boldsymbol{A}$ converges to x whenever $\boldsymbol{A}$ converges to x. Here

$$\mathrm{cl}_{\omega X}\boldsymbol{A} = \{\mathrm{cl}_{\omega X}A \mid A \in \boldsymbol{A}\}$$

where $\mathrm{cl}_{\omega X}$ is the closure operator of the initial topology ωX determined by the source $(f: X \to \mathbf{R})_{f \in \mathrm{Hom}(X,\mathbf{R})}$.

In view of the isomorphism of the category of symmetric convergence spaces and $\mathbf{Conv_S}$ we may use both terminologies for the functions in Hom(X,$\mathbf{R}$): we may call them continuous (between the convergence spaces X and $\mathbf{R}$) or uniformly continuous (between X and $\mathbf{R}$, considered as filterspaces).

A convergence space X is ω-*closed domained* [8] if for every filter $\boldsymbol{F}$ on X the set

$$\lim \boldsymbol{F} = \{x \in X \mid \boldsymbol{F} \to x\}$$

is ωX-closed. Note that, even without implicit assumption, this property implies symmetry.

Bourdaud has shown that the category of ω-regular, ω-closed domained pseudotopological convergence spaces is a cartesian closed hull of **Creg**. This hull is also called the category of c-embedded spaces. Other characterizations of c-embedded spaces can be found in the literature, see e.g. [6]. In most papers on this subject however the Hausdorff property is implicitly assumed.

A convergence space is *reciprocal* (satisfies R) [2, 20] if it satisfies: If $x,y \in X$ and if there exists a filter $\boldsymbol{F}$ on X converging to x and y, then for every filter $\boldsymbol{G}$ on X we have $\boldsymbol{G} \to x$ iff $\boldsymbol{G} \to y$. This axiom is also known as axiom P in [18] and is stronger than the symmetry axiom. A convergence space is reciprocal iff it is a Cauchy space.

3. PENDANTS OF CONVERGENCE SUBCATEGORIES IN THE CATEGORY Fil.

Following the terminology used in [4] if $\boldsymbol{B}$ is a subcategory of $\mathbf{Conv_S}$ and $\boldsymbol{B}'$ is a subcategory of **Fil** we say that $\boldsymbol{B}'$ is a *pendant of* $\boldsymbol{B}$ iff $\boldsymbol{B}' \cap \mathbf{Conv_S} = \boldsymbol{B}$. For each of the subcategories described in the previous section, we will define pendants in **Fil**.

In this section, X is a filterspace, $\mathbf{R}$ is the real line with the usual topology (considered as a filterspace) and Hom(X, $\mathbf{R}$) is the collection of all uniformly continuous functions on X to $\mathbf{R}$.

Let **CregFil** be the full subcategory of **Fil** whose objects are the completely regular filterspaces. This category is a pendant of the category of completely regular topological spaces. This result follows from Proposition 6 in [4] and is formulated in terms of filterspaces in the next proposition.

PROPOSITION 3.1. *A topological space X is completely regular (as a topological space) iff it is a completely regular filterspace.*

PROPOSITION 3.2. *The category* **CregFil** *is bireflective in* **Fil**.

PROOF. Let $(f_i: X \to X_i)_{i \in I}$ be an initial source in **Fil** and suppose all spaces X_i are completely regular filterspaces. Let $\boldsymbol{A}$ be a Cauchy filter in X and let

$$\boldsymbol{B} = \{B \subset X \mid A \text{ is completely within } B \text{ for some } A \in \boldsymbol{A}\}.$$

For $i \in I$ the collection $f_i(\boldsymbol{A})$ is micromeric in X_i. Therefore so is

$$\boldsymbol{B}_i = \{B_i \subset X_i \mid A_i \text{ is completely within } B_i \text{ for some } A_i \in f_i(\boldsymbol{A})\}.$$

For $B_i \in \boldsymbol{B}_i$, let $A_i \in f_i(\boldsymbol{A})$ be such that A_i is completely within B_i. Let $A \in \boldsymbol{A}$ be such that $A_i = f_i(A)$. Let $B = f_i^{-1}(B_i)$. It is clear that A is completely within B. Therefore $B \in \boldsymbol{B}$ and $f_i(B) \subset B_i$. It follows that $\boldsymbol{B}_i$ corefines $f_i(\boldsymbol{B})$ and therefore $f_i(\boldsymbol{B})$ is micromeric in X.

Next we introduce a pendant for the category of ω-regular convergence spaces. Recall that when X is any filterspace we denote $t\mathrm{Hom}(X,\mathbf{R})$ the initial topology on the underlying set of X determined by the source $(f: X \to \mathbf{R})_{f \in \mathrm{Hom}(X,\mathbf{R})}$.

DEFINITION 3.3. X is *μ-regular* if $\mathrm{cl}_{t\mathrm{Hom}(X,\mathbf{R})}\boldsymbol{A}$ is micromeric in X whenever $\boldsymbol{A}$ is micromeric in X. Here

$$\mathrm{cl}_{t\mathrm{Hom}(X,\mathbf{R})}\boldsymbol{A} = \{ \mathrm{cl}_{t\mathrm{Hom}(X,\mathbf{R})}A \mid A \in \boldsymbol{A}\}.$$

Remark that a μ-regular filterspace was called a "completely regular filterspace" by Katetov in [17]. μ-regular Cauchy spaces were studied in [12].

In [7] the authors have shown that the category of μ-regular filterspaces contains the category of completely regular filterspaces and is bireflective in **Fil**. The explicit description of the bireflector can also be found in [7]. The next proposition states that the category of μ-regular filterspaces is a pendant of the category of ω-regular convergence spaces.

PROPOSITION 3.4. *A convergence space* X *is ω-regular iff it is μ-regular.*

PROOF. Suppose X is an ω-regular convergence space. Then X is a filterspace. Let $\boldsymbol{A}$ be a Cauchy filter. Then for some $x \in X$ we have $\boldsymbol{A} \to x$. Moreover

$$\mathrm{cl}_{\omega X}\boldsymbol{A} = \mathrm{cl}_{t\mathrm{Hom}(X,\mathbf{R})}\boldsymbol{A} \to x.$$

Hence $\mathrm{cl}_{t\mathrm{Hom}(X,\mathbf{R})}\boldsymbol{A}$ is micromeric. In order to show the other implication, suppose X is μ-regular and $\boldsymbol{F} \to x$. Then we have $\mathrm{cl}_{\omega X}\boldsymbol{F} = \mathrm{cl}_{t\mathrm{Hom}(X,\mathbf{R})}\boldsymbol{F}$ is micromeric. Moreover since $\mathrm{cl}_{\omega X}\boldsymbol{F} < \dot{x}$ we can conclude that $\mathrm{cl}_{\omega X}\boldsymbol{F} \to x$.

Next we introduce a pendant for the category of ω-closed domained convergence spaces.

DEFINITION 3.5. X is *μ-closed domained* if: whenever $\boldsymbol{F}$ and $\boldsymbol{G}$ are Cauchy filters such that $\boldsymbol{F} \vee \boldsymbol{G}$ is not Cauchy, then there exists $h \in \mathrm{Hom}(X, \mathbf{R})$ with $\lim h(\boldsymbol{F}) \not\ni \lim h(\boldsymbol{G})$.

Remark that a μ-closed domained filterspace satisfying T_1 ($\dot{x} \vee \dot{y}$ micromeric $\Rightarrow x = y$) is called "Cauchy separated Cauchy space" by Gazik & Kent [12]. In this paper, T_1 separation is not implicitly assumed.

PROPOSITION 3.6. *A μ-closed domained filterspace is a Cauchy space.*

PROOF. If F and G are Cauchy filters, $\emptyset \not\in F \wedge G$, then $F \wedge G$ is a Cauchy filter. So for every $h \in \mathrm{Hom}(X, \mathbf{R})$ we have $\lim h(F) = \lim h(G)$. Hence $F \vee G$ is a Cauchy filter too.

PROPOSITION 3.7. *The category of μ-closed domained filterspaces is bireflective in* **Fil.**

PROOF. If X is any filterspace then its μ-closed domained reflection is given by $\mathrm{id}: X \to \mu X$, the identity map on the underlying sets where we define: F is a Cauchy filter in μX if there exist Cauchy filters G and H in X such that $F > G \vee H$ and $\lim f(G) = \lim f(H)$ whenever $f \in \mathrm{Hom}(X, \mathbf{R})$. μX clearly is a filterspace and $\mathrm{id}: X \to \mu X$ is uniformly continuous. Further if h: $X \to Z$ is uniformly continuous and Z is a μ-closed domained filterspace, let F be a μX-Cauchy filter. Take G and H Cauchy filters in X such that

$$F > G \vee H \text{ and } \lim f(G) = \lim f(H)$$

whenever $f \in \mathrm{Hom}(X, \mathbf{R})$. $h(G)$ and $h(H)$ are Cauchy filters on Z and whenever $g \in \mathrm{Hom}(Z, \mathbf{R})$ we have $g \circ h \in \mathrm{Hom}(X, \mathbf{R})$. Hence $\lim g(h(G)) = \lim g(h(H))$. It follows that $h(G) \vee h(H)$ is a Cauchy filter on Z. Finally $h(F)$ is a Cauchy filter on Z.

PROPOSITION 3.8. *For a convergence space* X *the following are equivalent (even without any symmetry assumption):*

(1) X *is ω-closed domained.*
(2) For each filter F on X: $\lim_X F = \emptyset$ *or* $\lim_X F = \lim_{\omega X} F$.
(3) X *is a μ-closed domained filterspace.*

PROOF. (1) $\Rightarrow$ (2) is Lemma 4.5 in Bourdaud [8].
(2) $\Rightarrow$ (3): Suppose X is a convergence space satisfying (2). If

$$\dot{y} \xrightarrow{X} x \text{ and } G \xrightarrow{\omega X} x,$$

also $G \xrightarrow{\omega X} x$. Since ωX is symmetric this implies $G \xrightarrow{\omega X} y$. Finally, in view of (2) we can conclude $G \xrightarrow{X} y$. It follows that X is symmetric and therefore a filter space.
Next we show that it is μ-closed domained. Suppose that F converges to x and G to y in X and $F \vee G$ is not convergent. Then $y \not\in \lim_X F$ and hence by (2) $y \not\in \lim_{\omega X} F$. So there exists a (uniformly) continuous function f on X to $\mathbf{R}$ satisfying $f(F) \not\to f(y)$. Since $f(G) \to f(y)$ we can conclude that $\lim f(F) \neq f(G)$.

(3)$\Rightarrow$(1): Suppose X is a convergence space which is also a μ-closed domained filterspace. Further let $\boldsymbol{F}$ be a filter on X. Either $\lim_X \boldsymbol{F} = \emptyset$ and then $\lim_X \boldsymbol{F}$ is ωX-closed. Or $\lim_X \boldsymbol{F} \neq \emptyset$. We clearly have $\lim_X \boldsymbol{F} \subset \lim_{\omega X} \boldsymbol{F}$. Suppose

$$\boldsymbol{F} \xrightarrow{X} x \quad \text{and } \boldsymbol{F} \xrightarrow{\omega X} x .$$

For any (uniformly) continuous function h from X to $\mathbf{R}$ we have $h(\boldsymbol{F}) \to h(x)$ and $h(\boldsymbol{F}) \to h(y)$, hence $h(x) = h(y)$. In particular we have $\lim\ h(\dot{x}) = \lim h(\dot{y})$. Since X is μ-closed domained we can conclude that $\dot{x} \vee \dot{y}$ is convergent on X. In view of Proposition 3.6, we can conclude that x and y have the same convergent filters. Hence $\boldsymbol{F}$ converges to y in X. So finally $\lim_X \boldsymbol{F} = \lim_{\omega X} \boldsymbol{F}$. Therefore $\lim_X \boldsymbol{F}$ is ωX-closed.

Next we introduce a pendant for the category of pseudotopological convergence spaces.

DEFINITION 3.9. A filterspace is *pseudotopological* if a filter $\boldsymbol{F}$ is a Cauchy filter iff for all ultrafilters $\boldsymbol{V}$, $\boldsymbol{W}$ finer than $\boldsymbol{F}$ we have that $\boldsymbol{V} \vee \boldsymbol{W}$ is a Cauchy filter.

Remark that, when Definition 3.9 is applied to Cauchy spaces, it coincides with the definition given by Gazik & Kent in [12].

PROPOSITION 3.10. *The category of pseudotopological filterspaces is bireflective in* **Fil**.

PROOF. If X is any filterspace then its pseudotopological reflection is given by id: X$\to$ pX, the identity map on the underlying sets, where we define $\boldsymbol{F}$ Cauchy filter in pX if $\boldsymbol{V} \vee \boldsymbol{W}$ is Cauchy in X whenever $\boldsymbol{V}$ and $\boldsymbol{W}$ are ultrafilters finer than $\boldsymbol{F}$. It is clear that pX is a filterspace, and X$\to$ pX is uniformly continuous. Whenever $\boldsymbol{V}$ and $\boldsymbol{W}$ are ultrafilters, $\boldsymbol{V} \vee \boldsymbol{W}$ is X-Cauchy iff it is pX-Cauchy. It follows that pX is pseudotopological. Further if h: X$\to$Z is uniformly continuous and Z is a pseudotopological filterspace, let $\boldsymbol{F}$ be a pX-Cauchy filter and let $\boldsymbol{V}$ and $\boldsymbol{W}$ be ultrafilters on Z both finer than $h(\boldsymbol{F})$. Take ultrafilters $\boldsymbol{V}'$ and $\boldsymbol{W}'$ on X both finer than $\boldsymbol{F}$ such that we have $h(\boldsymbol{V}') < \boldsymbol{V}$ and $h(\boldsymbol{W}') < \boldsymbol{W}$. By definition of pX, the filter $\boldsymbol{V}' \vee \boldsymbol{W}'$ is Cauchy on X. It follows that $\boldsymbol{V} \vee \boldsymbol{W}$ is Cauchy on Z. Finally, since Z is pseudotopological, we can conclude that $h(\boldsymbol{F})$ is micromeric.

The following proposition is well known; see e.g. [20]. It shows that the category of pseudotopological Cauchy spaces is a pendant for the category of reciprocal pseudotopological convergence spaces.

PROPOSITION 3.11. *For a convergence space* X *the following properties are equivalent:*

(1) X *is pseudotopological and reciprocal.*

(2) X *is a pseudotopological Cauchy space.*

From Propositions 3.3, 3.4, 3.7, 3.8, 3.10 and 3.11 we can now state the following concluding result.

THEOREM 3.12. *The full subcategory of* **Fil** *whose objects are the μ-regular, μ-closed domained pseudotopological filterspaces is bireflective in* **Fil** *and contains the category of completely regular filterspaces.*

In case the T_1 property is assumed the category of μ-regular, μ-closed domained, pseudotopological filterspaces coincides with the category of c^-embedded Cauchy spaces introduced by Gazik & Kent in [12]. In Proposition 3.6, we already mentioned that a μ-closed domained filterspace is a Cauchy space. A μ-closed domained T_1 Cauchy space has been called Cauchy separated in [12], and Gazik & Kent showed that the μ-regular Cauchy separated pseudotopological Cauchy spaces coincide with the c^-embedded spaces. Gazik & Kent also developed a nice completion theory for c^-embedded spaces.

4. CARTESIAN CLOSEDNESS.

A category **C** with finite products is cartesian closed if for each **C**-object X, the functor $X\times - : \mathbf{C}\to\mathbf{C}$ has a right adjoint, denoted by $[X,-]$.

For topological constructs, cartesian closedness is characterized by the existence of canonical function spaces, i.e., the power $[X,Y]$ is given by the set Hom(X,Y), endowed with a **C**-structure ξ satisfying the following conditions:

a) the evaluation map

$$ev: X\times(\mathrm{Hom}(X,Y),\xi)\to Y,\ ev(x,f) = f(x),$$

is a morphism;

b) for each **C**-object W and each morphism $h: X\times W=Y$ the map

$h^*: W\to(\mathrm{Hom}(X,Y),\xi)$ defined by $h^*(w)(x) = h(x,w)$

is a morphism.

The map h^* is called the transpose of h. We refer to [1] for further details on this notion. It is well known that the topological construct **Fil** is cartesian closed. This was shown in [5] via categorical methods (and using grills instead of filters). The existence of canonical function spaces in **Fil** was demonstrated by Katetov in [17]. In [3] it was shown that the full subcategory **Chy** of Cauchy filterspaces is also Cartesian closed. The canonical function spaces in **Chy** are formed as in **Fil**.

As we already mentioned in Proposition 3.6, the category of μ-regular, μ-closed domained pseudotopological filterspaces is contained in **Chy**. In this section we prove that it is cartesian closed by showing that the canonical function spaces in **Fil** of μ-regular, μ-closed domained pseudotopological filterspaces are μ-regular, μ-closed domained and pseudotopological.

We recall the description of the canonical function spaces in **Fil**.

DEFINITION 4.1 [17]. For filterspaces X, Y a filter structure ξ on Hom(X,Y) is defined by: Ψ is a Cauchy filter iff $\Psi(\boldsymbol{F})$ is micromeric on Y whenever $\boldsymbol{F}$ is a Cauchy filter on X. Here Ψ is a filter on Hom(X,Y), $\boldsymbol{F} \times \Psi$ is the collection

$$\{F \times \Psi \mid F \in \boldsymbol{A},\ \boldsymbol{A} \in \Psi\}$$

and $\Psi(\boldsymbol{F}) = \mathrm{ev}(\boldsymbol{F} \times \Psi)$. The space (Hom(X,Y),$\xi$) is denoted by [X,Y].

PROPOSITION 4.2. *If* X *and* Y *are filterspaces and* Y *is* μ*-regular, then* [X,Y] *is* μ*-regular.*

PROOF. Let $\boldsymbol{A} \subset \mathrm{Hom}(X,Y)$ and $x \in X$. Suppose

$$g \in \mathrm{cl}_{t\mathrm{Hom}([X,Y],\mathbf{R})} \boldsymbol{A}.$$

We first show that this implies that

$$g(x) \in \mathrm{cl}_{t\mathrm{Hom}(X,\mathbf{R})} \boldsymbol{A}(x) \text{ where } \boldsymbol{A}(x) = \mathrm{ev}(\{x\} \times \boldsymbol{A}).$$

Let $h_1, \ldots, h_n$ be uniformly continuous maps $h_i: Y \to \mathbf{R}$ and let $V_1, \ldots, V_n$ be open subsets of $\mathbf{R}$, and assume that

$$g(x) \in \bigcap_{i=1}^{n} h_i^{-1}(V_i).$$

For $i \in \{1, \ldots, n\}$ take the uniformly continuous function

$$\varphi_i: [X,Y] \to \mathbf{R}: f \mapsto h_i \circ f(x).$$

The set

$$V = \bigcap_{i=1}^{n} \varphi_i^{-1}(V_i)$$

is open in $t\mathrm{Hom}([X,Y],\mathbf{R})$, so it intersects $\boldsymbol{A}$. Clearly every

$$k \in \boldsymbol{A} \cap (\bigcap_{i=1}^{n} \varphi_i^{-1}(V_i))$$

satisfies

$$k(x) \in \boldsymbol{A}(x) \cap (\bigcap_{i=1}^{n} h_i^{-1}(V_i)).$$

It follows that whenever Ψ is a Cauchy filter on $[X,Y]$ and $\boldsymbol{F}$ is a Cauchy filter on X, the collection $\mathrm{cl}_{t\mathrm{Hom}(Y,\mathbf{R})}(\Psi(\boldsymbol{F}))$ corefines $(\mathrm{cl}_{t\mathrm{Hom}([X,Y],\mathbf{R})}\Psi)(\boldsymbol{F})$. Finally, if Y is μ-regular, so is $[X,Y]$.

PROPOSITION 4.3. *If* X *and* Y *are filterspaces and* Y *is μ-closed domained, then* $[X,Y]$ *is μ-closed domained.*

PROOF. Let Ψ and Φ be Cauchy filters on $[X,Y]$ and suppose $\Psi \vee \Phi$ is not Cauchy. Let $\boldsymbol{F}$ be a Cauchy filter on X such that the filter $\Psi(\boldsymbol{F}) \vee \Phi(\boldsymbol{F})$ is not micromeric on Y. Since Y is μ-closed domained, there is a function $h \in \mathrm{Hom}(X,\mathbf{R})$ such that we have $\lim h(\Psi(\boldsymbol{F})) \neq \lim h(\Phi(\boldsymbol{F}))$. Let

$$\varphi : [X,Y] \to \mathbf{R} : f \mapsto \lim h(f(\boldsymbol{F})).$$

Note that this definition is meaningful since $h \circ f : X \to \mathbf{R}$ is uniformly continuous. For any Cauchy filter ϑ on $[X,Y]$ and $\boldsymbol{A} \in \vartheta$, $F \in \boldsymbol{F}$, $f \in \boldsymbol{A}$ we have

$$\lim h(f(\boldsymbol{F})) \in \mathrm{cl}_{\mathbf{R}} h(\boldsymbol{A}(F)) \text{ where } \boldsymbol{A}(F) = \mathrm{ev}(F \times \boldsymbol{A}).$$

Hence $\varphi(\boldsymbol{A}) \subset \mathrm{cl}_{\mathbf{R}} h(\boldsymbol{A}(F))$. So we can conclude that $\mathrm{cl}_{\mathbf{R}} h(\vartheta(\boldsymbol{F}))$ corefines $\varphi(\vartheta)$. It follows that $\varphi \in \mathrm{Hom}([X,Y],\mathbf{R})$. Moreover

$$\lim \varphi(\Psi) = \lim \mathrm{cl}_{\mathbf{R}} h(\Psi(\boldsymbol{F})) = \lim h(\Psi(\boldsymbol{F})).$$

On the other hand

$$\lim \varphi(\Phi) = \lim \mathrm{cl}_{\mathbf{R}} h(\Phi(\boldsymbol{F})) = \lim h(\Phi(\boldsymbol{F})).$$

Therefore $[X,Y]$ is μ-closed domained.

PROPOSITION 4.4. *If* X *and* Y *are filterspaces and* Y *is pseudotopological, then* $[X,Y]$ *is pseudotopological.*

PROOF. Let Ψ be a filter on $\mathrm{Hom}(X,Y)$ and suppose that $\vartheta \vee \Phi$ is Cauchy on $[X,Y]$ whenever ϑ and Φ are ultrafilters finer than Ψ. Let $\boldsymbol{F}$ be a Cauchy filter on X. First we prove that whenever $\boldsymbol{V}$ is an ultrafilter finer than $\Psi(\boldsymbol{F})$ there exists an ultrafilter ϑ finer than Ψ such that $\boldsymbol{V} > \vartheta(\boldsymbol{F})$. Suppose on the contrary that for every $\vartheta > \Psi$ we have $\boldsymbol{V} \not> \vartheta(\boldsymbol{F})$. For every $\vartheta > \Psi$ choose $\boldsymbol{A} \in \vartheta$ and $F \in \boldsymbol{F}$ such that $\boldsymbol{A}(F) \not\in \boldsymbol{V}$. We can find $\vartheta_1, \dots, \vartheta_n$ and corresponding sets $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n$ in $\vartheta_1, \ldots, \vartheta_n$ respectively, $F_1, \ldots, F_n$ in $\boldsymbol{F}$ with

$$\boldsymbol{A}_i(F_i) \not\in \Psi \text{ and } \bigcup_{i=1}^{n} \boldsymbol{A}_i \in \Psi.$$

If not, a filter finer than Ψ would exist, containing all complements of finite unions of the choosen sets $\boldsymbol{A}_i \in \vartheta_i$, and this is impossible. Let

$$\boldsymbol{A} = \bigcup_{i=1}^{n} \boldsymbol{A}_i \quad \text{and} \quad F = \bigcap_{i=1}^{n} F_i,$$

then

$$\boldsymbol{A}(F) \in \boldsymbol{V} \text{ and } \boldsymbol{A}(F) \subset \bigcup_{i=1}^{n} \boldsymbol{A}_i(F_i),$$

which is impossible. Next we show that $\Psi(\boldsymbol{F})$ is micromeric on Y. Let $\boldsymbol{V}$ and $\boldsymbol{W}$ be ultrafilters on Y, both finer than $\Psi(\boldsymbol{F})$. Let ϑ and Φ be ultrafilters on Hom(X,Y) such that

$$\vartheta > \Psi, \ \Phi > \Psi, \ \boldsymbol{V} > \vartheta(\boldsymbol{F}), \ \boldsymbol{W} > \Phi(\boldsymbol{F}).$$

It follows that $\vartheta \vee \Psi$ is Cauchy on [X,Y] and therefore $\vartheta \vee \Psi(\boldsymbol{F})$ is micromeric on Y. Finally we can conclude that $\boldsymbol{V} \vee \boldsymbol{W}$ is Cauchy on Y and since $\boldsymbol{V}$ and $\boldsymbol{W}$ are arbitrary, it follows that $\Psi(\boldsymbol{F})$ is micromeric on Y.

THEOREM 4.5. *The full subcategory of* **Fil** *whose objects are the μ-regular, μ-closed domained pseudotopological filterspaces is Cartesian closed.*

5. A CARTESIAN CLOSED TOPOLOGICAL HULL OF THE CATEGORY OF COMPLETELY REGULAR FILTERSPACES.

In the previous section we have proved that the category of μ-regular μ-closed domained pseudotopological filterspaces is a cartesian closed topological extension of **CregFil**. Is it the smallest one? First we have to define what is exactly meant by this [1].

A subclass $\boldsymbol{D}$ of a topological construct $\boldsymbol{B}$ is called *finally dense* in $\boldsymbol{B}$ if for each $\boldsymbol{B}$-object X, there is a final sink $(f_i : D_i \to X)_{i \in I}$ with all D_i in $\boldsymbol{D}$. If $E : \boldsymbol{D} \to \boldsymbol{B}$ is an embedding of a construct $\boldsymbol{D}$ into $\boldsymbol{B}$ then the embedding E and the extension $\boldsymbol{B}$ of $\boldsymbol{D}$ are said to be finally dense if the class

$$\{E(X) \mid X \in \text{Ob}\, \boldsymbol{D}\}$$

is finally dense in $\boldsymbol{B}$. A subclass $\boldsymbol{D}$ of a topological construct $\boldsymbol{B}$ is called *initially dense* in $\boldsymbol{B}$ if for each $\boldsymbol{B}$-object X there is an initial source $(f_i : X \to D_i)_{i \in I}$ with all D_i in $\boldsymbol{D}$.

A cartesian closed topological construct $\boldsymbol{B}$ is called *carte-*

sian closed topological hull of a construct $\boldsymbol{A}$ if $\boldsymbol{B}$ is a finally dense extension of $\boldsymbol{A}$ with the property that any finally dense embedding of $\boldsymbol{A}$ into a cartesian closed topological construct can be uniquely extended to $\boldsymbol{B}$. The cartesian closed topological hull of a construct is unique up to isomorphism, if it exists. It can be characterized in terms of function spaces in the following way.

PROPOSITION 5.1 [16]. *The cartesian closed topological hull* $\boldsymbol{B}$ *of a construct* $\boldsymbol{A}$ *is characterized by the following properties:*

(1) $\boldsymbol{B}$ *is a cartesian closed topological construct and* $\boldsymbol{A}$ *is a full subconstruct of* $\boldsymbol{B}$.

(2) $\boldsymbol{A}$ *is finally dense in* $\boldsymbol{B}$.

(3) $\{[X,Y] \mid X$ *and* Y $\boldsymbol{A}$*-objects*$\}$ *is initially dense in* $\boldsymbol{B}$.

We refer to [1] for further details on these notions.

In this section we prove that the category of μ-regular, μ-closed domained pseudotopological filterspaces is the cartesian closed topological hull of **CregFil**.

PROPOSITION 5.2. *The category* **CregFil** *of all completely regular filterspaces is finally dense in* **Fil**.

PROOF. Let X be an arbitrary filterspace. For a filter $\boldsymbol{F}$ on X let $X_{\boldsymbol{F}}$ be the filterspace on the underlying set $|X|$ of X defined as follows. A filter $\boldsymbol{G}$ on $|X|$ is a Cauchy filter for $X_{\boldsymbol{F}}$ iff $\boldsymbol{G} = \dot{x}$ for some $x \in X$ or $\boldsymbol{G} > \boldsymbol{F}$. Each $F \in \boldsymbol{F}$ is completely within F and if $x \in X$ with $\boldsymbol{F} \not< \dot{x}$ then $\{x\}$ is completely within $\{x\}$. This implies that

$$\{B \subset X \mid F \text{ is completely within } B \text{ for some } F \in \boldsymbol{F}\} = \boldsymbol{F}$$

and

$$\{B \subset X \mid \{x\} \text{ is completely within } B\} = \dot{x}$$

if $\boldsymbol{F} \not< \dot{x}$, and is corefined by $\boldsymbol{F}$ if $\boldsymbol{F} < \dot{x}$. Hence it is a completely regular filterspace. Next consider the sink

$$(\mathrm{id}: X_{\boldsymbol{F}} \to X)_{\boldsymbol{F} \text{ Cauchy on } X}.$$

Then clearly this is a final epi sink in **Fil**.

In view of Theorem 3.12 we also have the following result:

PROPOSITION 5.3. *The category* **CregFil** *is finally dense in the category of all* μ*-regular,* μ*-closed domained pseudotopological* *filterspaces.*

The proof of the next proposition is essentially the same as the proof of Theorem 2.3 in [12] for T_1 Cauchy spaces.

PROPOSITION 5.4. *If* X *is a* μ*-regular,* μ*-closed domained pseudotopological filterspace, then the source* $j: X \to [[X,\mathbf{R}],\mathbf{R}]$ *is initial in* **Fil**. *Here* j *is the map where for* $x \in X$, *the function* $j(x): \mathrm{Hom}(X,\mathbf{R}) \to \mathbf{R}$ *maps* f *to* $f(x)$.

PROOF. Suppose X is a μ-regular, μ-closed domained pseudotopological filterspace. The map $j: X \to [[X,\mathbf{R}],\mathbf{R}]$ is uniformly continuous, so the initial filter structure of this source is coarser than the given structure on X. First we prove that the initial filter structure is also finer than the given X on ultrafilters. Suppose $\boldsymbol{W}$ is an ultrafilter on X such that $j(\boldsymbol{W})$ is Cauchy on $[[X,\mathbf{R}],\mathbf{R}]$ but $\boldsymbol{W}$ is not Cauchy on X. It follows that $\mathrm{cl}_{t\mathrm{Hom}([X,Y],\mathbf{R})}\boldsymbol{H}$ does not corefine $\boldsymbol{W}$ for every Cauchy filter $\boldsymbol{H}$ on X. For a Cauchy filter $\boldsymbol{H}$ on X we choose $H_{\boldsymbol{H}} \in \boldsymbol{H}$ such that $H_{\boldsymbol{H}}$ is $t\mathrm{Hom}(X,\mathbf{R})$-closed and $H \notin \boldsymbol{W}$.

$$\boldsymbol{A} = \{H_{\boldsymbol{H}} \mid \boldsymbol{H} \text{ Cauchy on } X\}$$

is a uniform cover of X, and so is the collection $\boldsymbol{B}$ consisting of all finite unions of elements of $\boldsymbol{A}$. For $\varepsilon > 0$ and $B \in \boldsymbol{B}$ put

$$\boldsymbol{N}_{\varepsilon,B} = \{f \in \mathrm{Hom}(X,\mathbf{R}) \mid |f(x)| < \varepsilon,\ \forall x \in B\}$$

and

$$\Psi = \{\boldsymbol{M} \subset \mathrm{Hom}(X,\mathbf{R}) \mid \boldsymbol{N}_{\varepsilon,B} \subset \boldsymbol{M} \text{ for some } \varepsilon > 0,\ B \in \boldsymbol{B}\}.$$

Clearly $\Psi \vee \dot{0}$ is Cauchy on $[X,\mathbf{R}]$. It follows that $j(\boldsymbol{W})(\Psi)$ converges to 0 on $\mathbf{R}$. Choose

$$W \in \boldsymbol{W} \text{ and } \boldsymbol{N}_{\varepsilon,B} \in \Psi \text{ such that } j(W)(\boldsymbol{N}_{\varepsilon,B}) \subset]-1/2,1/2[.$$

Clearly W is not contained in B. Choose $x \in W$, $x \notin B$. Since B is closed in $t\mathrm{Hom}(X,\mathbf{R})$ we can find $f \in \mathrm{Hom}(X,\mathbf{R})$ such that we have $f(B) \subset \{0\}$ and $f(x) = 1$. This function belongs to $\boldsymbol{N}_{\varepsilon,B}$ and since $x \in W$ we should have $f(x) \in]-1/2,1/2[$; this is a contradiction.

Next we prove that the initial structure is finer than the given structure on X. Suppose $\boldsymbol{G}$ is a Cauchy filter for the initial structure. In view of the pseudotopological property of X it suffices to show that $\boldsymbol{W} \vee \boldsymbol{V}$ is Cauchy on X whenever $\boldsymbol{W} > \boldsymbol{G}$, $\boldsymbol{V} > \boldsymbol{G}$ are ultrafilters. From the first part of the proof we know that $\boldsymbol{W}$ and $\boldsymbol{V}$ are Cauchy filters on X. If $\boldsymbol{W} \vee \boldsymbol{V}$ is not Cauchy on X, since X is μ-closed domained, there is an $f \in \mathrm{Hom}(X,\mathbf{R})$

such that $\lim f(\boldsymbol{W}) \neq \lim f(\boldsymbol{V})$. On the other hand $j(\boldsymbol{W} \vee \boldsymbol{V})$ is micromeric on $[[X,\mathbf{R}],\mathbf{R}]$ and therefore $j(\boldsymbol{W} \vee \boldsymbol{V})(\dot{f})$ converges on $\mathbf{R}$. It follows that $f(\boldsymbol{W}) \vee f(\boldsymbol{V})$ converges on $\mathbf{R}$. This is a contradiction. It follows that $\boldsymbol{W} \vee \boldsymbol{V}$ is Cauchy on X, and finally $\boldsymbol{G}$ too is Cauchy on X.

PROPOSITION 5.5. *The powers of completely regular filterspaces are initially dense in the category of all μ-regular, μ-closed domained pseudotopological filterspaces.*

PROOF. Let X be a μ-regular, μ-closed domained pseudotopological filterspace. By Propositions 4.2, 4.3, 4.4, $[X,\mathbf{R}]$ is μ-regular, μ-closed domained and pseudotopological. By Proposition 5.2, there is a final sink (in the category of μ-regular, μ-closed domained pseudotopological spaces) $(f_i : Y_i \to [X,\mathbf{R}])_{i \in I}$ where all Y_i are completely regular filterspaces. By Lemma 6 [16] the Hom functor transforms this final epi sink into an initial source

$$([[X,\mathbf{R}],\mathbf{R}] \to [Y_i,\mathbf{R}])_{i \in I}.$$

Since by Proposition 5.4 the source $j : X \to [[X,\mathbf{R}],\mathbf{R}]$ is also initial, the two initial sources can be composed and we are done.

Combining Theorems 3.12, 4.5 and Propositions 5.3, 5.5, all conditions for a cartesian closed topological hull as formulated in Proposition 5.1 are satisfied. So we can state our final result.

THEOREM 5.6. *The category of all μ-regular, μ-closed domained pseudotopological filterspaces is the cartesian closed topological hull of the category* **CregFil**.

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