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SOME SPECTRAL SEQUENCES IN BREDON COHOMOLOGY

by J. SŁOMIŃSKA

Résumé. Soit G un groupe fini. Nous considérons certaines suites spectrales dont la limite et les $E_2^{p,q}$-groupes sont les groupes de cohomologie de Bredon de G-CW-complexes adéquats. Les exemples le plus intéressants sont relatifs au cas d'un G-CW-complexe BW qui est l'espace classifiant d'un certain G-ensemble partiellement ordonné ("G-poset") W. On étudie en particulier le cas oú W est le G-poset des sous-groupes de G .

Introduction.

Let G be a finite group. In this paper we present some examples of spectral sequences whose $E_2^{p,q}$-groups are Bredon cohomology groups. The limit groups of these spectral sequences can be also expressed in the terms of Bredon cohomology. In fact we will consider spectral sequences of the form

$$H_G^p(K, L^q) \Rightarrow H_G^{p+q}(K'', L)$$

where K, K'' are G-CW-complexes and L is a local G-coefficient system on K'' determined by L and by an appropriate G-cellular map $f : K'' \longrightarrow K$. All of these spectral sequences are derived from the spectral sequence defined in 1.1 which describes a more general case. We show that the spectral sequences 7.2 and 7.7 of [3] which describe Borel cohomology groups and the spectral sequence 1.18 of [11] are also consequences of 1.1.

In Section 1 we assume that K is an arbitrary G-CW-complex. Let SK be the poset of all cells of K and let $IK : SK \longrightarrow CW$ be the functor such that, for every open cell s of K, $IK(s)$ is the smallest subcomplex of K which contains s. We will consider a G-CW-complex K'' of the form $K'' = IK \times_{SK} U$ where

$$U : SK^{op} \longrightarrow CW$$

is a functor which satisfies additional conditions concerning the action of G on SK. In particular, for $s \in SK$, $U(s)$ is a G_s-CW-complex. Let *pt* be the constant one-point functor. Then $K = IK \times_{SK} pt$ and f is induced by the natural projection

$U \longrightarrow pt$. In 1.5 we describe a spectral sequence such that

$$L^q(s) = H^q_{G_s}(U(s), L(s))$$

where $L(s)$ is the restriction of L to $U(s)$. In 1.8 and 1.9 we investigate this spectral sequence in the case where K'' is a G-subcomplex of the product $K \times K'$ of G-CW-complexes and, for every $s \in SK$, $U(s)$ is a G_s-subcomplex of K'. We will also investigate homotopy properties of the functor $IK \times_{SK} (-)$ because it is a homotopy colimit in the case where, for every $s \in SK$, $IK(s)$ is contractible.

In Section 2 we assume that K is a classifying space BW of a certain poset W on which the group G acts preserving order. Then SK is the simplicial subdivision of W and will be denoted by S. We show that in this case

$$H^p_G(K, L^q) = \lim_{S/G}{}^p \ \mathcal{L}^q$$

where $\mathcal{L}^q([s]) = L^q(s)$ and that there is a functor $N_U : S/G \longrightarrow G-CW$ such that

$$K'' \cong \operatorname*{hocolim}_{S} U \cong \operatorname*{hocolim}_{S/G} N_U$$

as G-spaces.

In 2.5 we define an "equivariant join" and discuss a spectral sequence associated to this construction. In this case W is the poset $P(D)$ of all nonempty subsets of a finite G-set D. As an example, we consider a sphere of an induced representation of G.

From 2.7 to the end of Section 2 we assume that U is a functor from S^{op} to the category $\mathcal{P}(\mathcal{K}')$ of all subcomplexes of of a given G-CW-complex K'.

Example 2.7 generalizes the spectral sequence described in [3], Ch.VII.4 and can be treated as the equivariant Mayer-Vietoris spectral sequence. In this case $W = P(D)$. We assume that if $s = (D_0, \ldots, D_n)$, where $D_0 \subset \ldots \subset D_n \subseteq D$, is an element of S, then

$$U(s) = \bigcap_{d \in D_n} U(\{d\}).$$

Suppose that $K' = \bigcup_{d \in D} U(\{d\})$ and that M is a generic G-coefficient system. Then there exists a spectral sequence

$$\lim_{[s] \in S/G}{}^p \quad H^q_G(G \times_{G_s} U(s), M) \Rightarrow H^{p+q}_G(K', M)$$

because in this case there exists a G-homotopy equivalence $K' \cong K''$. We obtain the spectral sequence from [3], Ch.VII.4 if we assume that G is a trivial group.

Example 2.9 generalizes the Proposition 12.1 from [3], Ch.IX. In this case, for every $s = (w_0, \ldots, w_n) \in S$, $U(s) = U(w_n)$ and we have a G-homotopy equivalence

$$K'' \cong \operatorname*{hocolim}_{[s] \in S/G} (G \times_{G_s} U(w_n)).$$

We will use the notation

$$W(k') = \{w \in W; k' \in U(w)\}$$

whenever $k' \in K'$. If $U(w)$ is G_w-contractible whenever $w \in W$, then $K'' \cong BW$ as G-spaces and there exists a spectral sequence

$$H^p_G(K', M^q) \Rightarrow H^{p+q}_G(BW, M)$$

such that, for every cell s' of K',

$$M^q(s') = H^q_G(G \times_{G_{k'}} BW(k'), M)$$

where k' is an arbitrary point of s'. If, for every point k' of K', the space $BW(k')$ is $G_{k'}$-contractible, then $K'' \cong K'$ and we obtain a spectral sequence

$$\lim_{[s] \in S/G}{}^p \quad H^q_G(G \times_{G_s} U(w_n), M) \Rightarrow H^{p+q}_G(K', M).$$

If both of the above assumptions are fulfilled, then there is an isomorphism

$$H^*_G(K', M) \cong H^*_G(BW, M).$$

Thus, in this case, for every abelian group A, there is an isomorphism:

$$H^*(K'/G, A) \cong H^*(BW/G, A).$$

If G is a trivial group then we obtain 12.1 from Chapter IX of [3].

In 2.10 we consider the special case of 2.9 where W is a G-poset of subgroups of G and $U(H) = K'^H$. This example is a generalization of 11.2 from [3], Ch.IX (in the case where Γ is a finite group). We obtain the result of [3], if we assume that W is the set of all nontrivial subgroups of G and that, for every $H \in W$, K'^H is acyclic. We consider also the set of all nontrivial p-subgroups of G. From our considerations, it follows that there is an isomorphism

$$H^*(BG, \mathbf{F}_p) \cong H^*(EG \times_G K', \mathbf{F}_p)$$

whenever, for every nontrivial p-subgroup H of G, K'^H is $\mathbf{F}_p$-acyclic and for every point k of K', the Sylow p-subgroups of G_k are nontrivial. In the case where K' is a realization of a finite dimensional simplicial complex, this fact is a consequence of the results of [13].

From the results of Sections 1 and 2 it follows that some well-known spectral sequences have common origin which can be expressed in terms of Bredon cohomology groups. In Section 3 we will apply the results of Section 2 to investigate properties of functors defined on certain categories which were described in [11] and [12]. Let $Sub - G$ be the G-poset of all subgroups of G. A G-poset map

$$d : W \longrightarrow Sub - G$$

will be called admissible if, for every $w \in W$, $dw \subseteq G_w$. For any admissible map d, we define a certain category $W(d)$. In the case where W is a G-poset of subgroups of G and d is the identity map, $W(d)$ is a category of orbits of G. Using the results of Section 2 we construct in 3.7 a free locally contractible $W(d)^{op}$-CW-complex (in the sense of Dror, [6]). For any functor

$$M : W(d)^{op} \longrightarrow Ab,$$

we obtain a spectral sequence (3.10)

$$\lim_{[s] \in S/G}{}^{p} \quad H^q(G_s/dw_0, M(w_0)) \Rightarrow \lim_{W(d)}{}^{p+q} M.$$

This is a generalization of the spectral sequence 1.2 from [10]. Corollary 3.10 is independently proved in [12] using different methods. This fact is also related to the result 17.18 of [7]. As corollary, one can show using 3.11 that if F is is a G-poset of subgroups of G and M is a generic G-coefficient system such that, for every $H \in F$, $M(G/H)$ is a projective NH/H-module, then there exists a natural number m such that, for all $n \geq m$,

$$H^n_G(EF, M) = 0,$$

where EF is a classifying space of F.

1 General results

We need the following definitions and notation. Some of them was used in [1] and [11]. Assume that K is a G-CW-complex. This means that K is a CW-complex and that G acts cellulary on K in such a way that, for every subgroup H of G, the fixed point set K^H is a subcomplex of K. The G-set of all open n-cells of K will be denoted by S_nK. For a cell s of K, let $K(s)$ be the smallest subcomplex of K which contains s. We will use the notation $s \subseteq s'$ in the case where $K(s) \subseteq K(s')$.

The poset of all open cells of K and the category associated to this poset will be denoted by SK. The group G acts on SK in such a way that the condition $s \subseteq s'$ implies that $gs \subseteq gs'$. It is clear that $S(K/G) = (SK)/G$.

By $\mathcal{P}(K)$ we will denote the poset of all subcomplexes of K. The group G acts on $\mathcal{P}(K)$ and SK can be considered as a G-subposet of $\mathcal{P}(K)$. By $\mathcal{P}(K)_G$ we will denote the category whose objects are subcomplexes of K and whose morphisms are the maps $K' \longrightarrow K''$, which are the compositions of the inclusions $K' \subseteq g^{-1}K''$ and of the maps $[g] : g^{-1}K'' \longrightarrow K''$ given by the operation by the elements of G. The full subcategory of $\mathcal{P}(K)_G$ whose object set is equal to SK will be denoted by SK_G. We will consider SK as a subcategory of SK_G and $\mathcal{P}(K)$ as a subcategory of $\mathcal{P}(K)_G$. The category $\mathcal{P}(K)_G$ is contained in the category CW of of all CW-complexes.

It is clear that SK_G is a full subcategory of the category $\mathcal{K}$ of all finite CW-subcomplexes of K, defined by Bredon in [1], Ch.I.2. and that $\mathcal{K}$ is a full subcategory of $\mathcal{P}(K)_G$. For any two cells s and s' of K,

$$\mathrm{Mor}_{SK_G}(s, s') = \{g \in G : gs \subseteq s'\}/G_s = \coprod_{[g]\in G/G_s} \mathrm{Mor}_{SK}(gs, s')$$

where $G_s = \{g \in G : gs = s\}$ is the isotropy group of the action of G on SK at the point s. It follows from the defition of a G-CW-complex that, if k belongs to an open cell s, then $G_k = G_s$.

By a local coefficient system on K we will mean a functor L from SK_G to the category Ab of abelian groups. Let

$$\overline{c}_*(K) : SK_G \longrightarrow Ab_c$$

be the covariant functor from SK_G to the category Ab_c of chain complexes of abelian groups such that, for every cell s of K , $\overline{c}_*(K)(s)$ is equal to the ordinary cellular homology chain complex $C_*(K(s), \mathbf{Z})$ with coefficients in the ring $\mathbf{Z}$ of integers. By $\mathbf{Z}(X)$ we denote the free abelian group with the basis equal to X. It follows from [11] that, for every natural number n,

$$\overline{c}_n(K) = \coprod_{[s]\in(S_nK)/G} \mathbf{Z}(\mathrm{Mor}_{SK_G}(s, -)),$$

so $\overline{c}_*(K)$ is a chain complex of projective objects in the category (SK_G, Ab) of functors from SK_G to Ab. The n-th Bredon cohomology group $H^n_G(K, L)$ is equal to the n-th cohomology group of the cochain complex of the natural transformation groups

$$C^*_G(K, L) = \mathrm{Hom}_{SK_G}(\overline{c}_*(K), L).$$

It follows from the definition that

$$C^n_G(K, L) = \prod_{[s]\in S_nK/G} L(s).$$

Assume that H is a subgroup of G and that K' is an H-subcomplex of K. Let $j : SK'_H \longrightarrow SK_G$ denote the natural inclusion of categories. We will use the notation

$$H^n_H(K', L) = H^n_H(K', Lj).$$

A G-CW-complex K will be called *special* if, for every cell s of K, the subcomplex $K(s)$ is contractible. If K is special then $\overline{c}_*(K)$ is a projective resolution of the constant functor $\mathbf{Z}_K$ such that $\mathbf{Z}_K(s) = \mathbf{Z}$, $\mathbf{Z}_K(s \subseteq s') = id_{\mathbf{Z}}$. In this case

$$H^*_G(K, L) = \lim_{SK_G}{}^* \ L.$$

Let $IK : SK_G \longrightarrow CW$ be the natural inclusion of categories such that, for every cell s of K, $IK(s) = K(s)$. It is easy to check that IK is a free SK_G-CW-complex (in the sense of Dror [6]) whose cells correspond to the elements of the set SK/G. As a functor to the category Set of sets

$$IK = \coprod_{s \in SK} \coprod_{k \in s} \mathrm{Mor}_{SK}(s, -) = \coprod_{[s] \in SK/G} \coprod_{k \in s} \mathrm{Mor}_{SK_G}(s, -).$$

This also implies that IK after the restriction to SK is a free SK-CW-complex. It is obvious that

$$\overline{c}_*(K) = C_*(IK(-), \mathbf{Z}).$$

Let Top denote the Steenrod category of compactly generated Hausdorff spaces (k-spaces). We will consider the category CW of CW-complexes as a subcategory of Top. If K and K' are CW-complexes then the topology of the product $K \times K'$ is equal to the topology of the product of k-spaces and $S(K \times K') = SK \times SK'$.

We will also need the definition of the product of functors over a category which is described, for example, in [6], 2.16. Let $\mathcal{C}$ be a small category and let T' (T) be a covariant (contravariant) functor from $\mathcal{C}$ to Top. Then the product $T' \times_{\mathcal{C}} T$ of T' and T over $\mathcal{C}$ is defined to be the co-equalizer of the two obvious maps

$$\Big(\coprod_{(c,c') \in \mathrm{Ob}(\mathcal{C} \times \mathcal{C})} \coprod_{f \in \mathrm{Mor}_{\mathcal{C}}(c,c')} T'(c) \times T(c') \Big) \rightrightarrows \coprod_{d \in \mathrm{Ob}\mathcal{C}} T'(d) \times T(d).$$

This means that $T' \times_{\mathcal{C}} T$ is the coend of

$$T'(-) \times T(-) : \mathcal{C} \times \mathcal{C}^{op} \longrightarrow Top.$$

For every object c of $\mathcal{C}$

$$\mathrm{Mor}_{\mathcal{C}}(c, -) \times_{\mathcal{C}} T = T(c),$$

$$T' \times_{\mathcal{C}} \mathrm{Mor}_{\mathcal{C}}(-, c) = T'(c).$$

Let $B : Cat \longrightarrow Top$ be the classifying space functor. It is easy to see that

$$\operatorname*{hocolim}_{\mathcal{C}} T = B(\mathcal{C}/-) \times_{\mathcal{C}} T.$$

Here $\mathcal{C}/c$ is the category of all morphisms of $\mathcal{C}$ of the form $c' \longrightarrow c$.

Assume that G' is a finite group. Let $G' - Top$ denote the category of compactly generated Hausdorff G'-spaces. There is the natural underlying functor from the category $G' - Top$ to Top. If

$$T : SK_G^{op} \longrightarrow G' - Top,$$

then we will consider the space $IK \times_{SK_G} T$ as a G'-space. For any $g' \in G'$, $s \in SK$, $k \in K(s)$ and $t \in T(s)$ we have $g'[k,t] = [k, g't]$. We will use the notation

$$K\{T\} = IK \times_{SK_G} T.$$

Assume that H is a subgroup of G'. Let $T^H : SK_G^{op} \longrightarrow Top$ be the functor such that, for every $s \in SK$, $T^H(s) = T(s)^H$. Then

$$K\{T\}^H = K\{T^H\}.$$

A natural transformation $\phi : T \longrightarrow T''$ induces the G'-space map $K(\phi)$. If ϕ is a local weak G'-homotopy equivalence, i.e. if $\phi(s)$ is a weak G'-homotopy equivalence whenever $s \in SK$, then one can prove, using methods and results of [6], that, for any subgroup H of G', $K\{\phi\}^H$ is a weak equivalence. This implies that $K\{\phi\}$ is a G'-homotopy equivalence.

In the case where $T = pt$ is the constant functor such that, for every cell s of K, $pt(s)$ is a one-point G'-space pt, we have that $K\{pt\} = K/G$. If T is locally G'-contractible, then there is a weak G'-homotopy equivalence

$$K\{T\} \cong K/G.$$

Let $T/G' : SK_G \longrightarrow Top$ be the functor such that, for $s \in SK$,

$$(T/G')(s) = T(s)/G'.$$

Then $K\{T\}/G' = K\{T/G'\}$. If $T(s)/G'$ is contractible whenever $s \in SK$, then we have a homotopy equivalence

$$K\{T\}/G' \cong K/G.$$

If K is a special G-CW-complex, then IK is a free locally contractible SK_G-CW-complex and there is a G'-homotopy equivalence

$$IK \times_{SK_G} T \cong \operatorname*{hocolim}_{SK_G} T.$$

Assume that T is a contravariant functor from SK_G to CW. Then the product $IK \times_{SK_G} T$ has the structure of a CW-complex such that

$$S_n(IK \times_{SK_G} T) = \coprod_{i=0}^{n} \coprod_{[s]\in S_i K/G} S_{n-i}T(s).$$

The category of all G'-CW-complexes and G'-cellular maps will be denoted by $G' - CW$. Let T be a contravariant functor from SK_G to the category $G' - CW$. Then $IK \times_{SK_G} T$ has the structure of a G'-CW-complex.

Assume that L is a local G'- coefficient system on $IK \times_{SK_G} T$. For every open cell s of K, let

$$\lambda(s) : K(s) \times T(s) \longrightarrow K\{T\}$$

denote the structural map. By L_s we will denote the local G'-coefficient system on $T(s)$ such that, for every open cell s' of $T(s)$

$$L_s(s') = L(\lambda(s)(s \times s')).$$

Let $H^q_{G'}(T, L)$ be the local G-coefficient system on K such that, for every cell s of K,

$$H^q_{G'}(T, L)(s) = H^q_{G'}(T(s), L_s).$$

1.1 Proposition *There exists a spectral sequence*

$$H^p_G(K, H^q_{G'}(T, L)) \Rightarrow H^{p+q}_{G'}(IK \times_{SK_G} T, L).$$

Proof. It is easy to check that

$$\mathrm{Hom}_{SK\{T\}_{G'}}(\overline{c}_n(K\{T\}), L) = \prod_{i=0}^{n} \prod_{[s]\in S_i K/G} \prod_{[s']\in S_{n-i}T(s)/G'} L(s').$$

Thus the cochain complex

$$\mathrm{Hom}_{SK\{T\}_{G'}}(\overline{c}_*(K\{T\}), L)$$

is equal to the cochain complex associated to the bicomplex

$$\mathrm{Hom}_{SK_G}(\overline{c}_*(K), L_0^*),$$

where L_0^* is the cochain complex of local G-coefficient systems on K such that

$$L_0^*(s) = \mathrm{Hom}_{ST(s)_{G'}}(\overline{c}_*(T(s)), L_s)$$

whenever $s \in SK$. Now, we can take the appropriate spectral sequence associated to this bicomplex. □

Proposition 1.1 implies that, in particular, if K is a special G-CW-complex, then there exists a spectral sequence

$$\lim_{SK_G}{}^p \; H^q_{G'}(T,L) \Rightarrow H^{p+q}_{G'}(IK \times_{SK_G} T, L).$$

Let us consider now the case where L is determined by a generic coefficient system. Let O_G be the category of canonical G-orbits. Its objects are the G-sets G/H, where H is a subgroup of G and its morphisms are the G-maps. For a subgroup H of G, by i_H we will denote the functor $O_H \longrightarrow O_G$ such that $i_H(H/H') = G/H'$. There exists a functor

$$\theta_K(G) : SK_G^{op} \longrightarrow O_G$$

such that, for every cell s of K, $\theta_K(G)(s) = G/G_s$. Contravariant functors from O_G to Ab will be called generic coefficients systems for G. A generic G-coefficient system $M : O_G^{op} \longrightarrow Ab$ defines the local G-coefficient system $M\theta_K(G)$ on K and

$$H^*_G(K,M) = H^*_G(K, M\theta_K(G)).$$

Bredon proved in [1] that $H^*_G(K,M)$ is equal to the n-th cohomology group of the cochain complex $\mathrm{Hom}_{O_G}(c_*(K), M)$, where $c_*(K)$ is the chain complex of contravariant functors from O_G to Ab such that, for every subgroup H of G,

$$c_*(K)(G/H) = C_*(K^H, \mathbf{Z}).$$

1.2 Corollary *Let M be a G'-generic coefficient system. Then, for any functor $T : SK_G^{op} \longrightarrow G' - CW$, there exists a spectral sequence*

$$H^p_G(K, H^q_{G'}(T,M)) \Rightarrow H^{p+q}_{G'}(IK \times_{SK_G} T, M).$$

where

$$H^q_{G'}(T,M)(s) = H^q_{G'}(T(s), M)$$

whenever s is a cell of K.

Proof. This result is a consequence of 1.1 because if $L = M\theta_{K\{T\}}(G')$, then $L_s = M\theta_{T(s)}(G')$. □

We will now describe the case where T is determined by a functor

$$T_o : O_G \longrightarrow G' - CW.$$

Let $I_K : O_G^{op} \longrightarrow CW$ be equal to the functor $\mathrm{Map}_G(-,K)$, where $\mathrm{Map}_G(-,-)$ denotes the set of G-maps. Hence $I_K(G/H) = K^H$ whenever H is a subgroup of G and

$$IK \times_{SK_G} T_o\theta_K(G) = I_K \times_{O_G} T_o.$$

1.3 Corollary *Let M be a generic G'-coefficient system. Then there exists a spectral seqence*

$$H^p_G(K, h^q_{G'}(T_o, M)) \Rightarrow H^{p+q}_{G'}(I_K \times_{O_G} T_o, M),$$

where $h^q_{G'}(T_o, M)$ is the generic G-coefficient system such that

$$h^q_{G'}(T_o, M)(G/H) = H^q_{G'}(T_o(G/H), M).$$

Proof. It is easy to check that

$$h^q_{G'}(T_o, M)\theta_K(G) = H^q_{G'}(T_o\theta_K(G), M).\square$$

In the following examples $G = G'$.

1.4 Examples (i) Assume that T_o is the functor such that, for every subgroup H of G, $T_o(G/H) = G/H \times K_0$ where K_0 is a G-CW-complex. Then

$$I_K \times_{O_G} T_o = K \times K_0.$$

Let M be the constant coefficient system such that, for every subgroup H of G, $M(G/H) = \mathbf{Z}$. Then

$$h^q_G(T_o, M)(G/H) = H^q((G/H \times K_0)/G, \mathbf{Z}) = H^q(K_0/H, \mathbf{Z}).$$

If $K_0 = EG$ is a universal free G-CW-complex then

$$h^q_G(T_o, M)(G/H) = H^q(H, \mathbf{Z})$$

and 1.3 gives us the spectral sequence described in [3], Ch.VII.7.

(ii) Let K be a G-CW-complex and let $T_o(G/H)$ be equal to the fiber product of two projections $\pi : K \longrightarrow K/G$ and $\pi_0 : (G/H \times K)/G \longrightarrow K/G$. As a topological space, $T_o(G/H)$ is a subspace of the product $K \times (K/H)$ in the category of k-spaces. The group G acts on $T_o(G/H)$ by the action on the first coordinate. In this case, we obtain the spectral sequence described in [11], 1.18. $\square$

In order to give other examples of spectral sequences of the type described in 1.1, in the case where G' is equal to G, we recall the following notation and simple facts from [11]. Let $SK[G]$ be the category whose objects are the same as SK_G and SK and whose morphism sets are defined in such a way that

$$\mathrm{Mor}_{SK[G]}(s, s') = \{g \in G : gs \subseteq s'\}.$$

The composition of morphisms of $SK[G]$ is determined by the multiplication in G. We have two inclusions of categories $\imath_G : SK \longrightarrow SK_G$, and $\imath[G] : SK \longrightarrow SK[G]$ and the projection $\rho_G : SK[G] \longrightarrow SK_G$ such that $\imath_G = \rho_G \imath[G]$. It is clear that

$$\mathrm{Mor}_{SK[G]}(s,-)\imath[G] = \coprod_{g\in G} \mathrm{Mor}_{SK}(gs,-),$$

$$\mathrm{Mor}_{SK_G}(s,-)\imath_G = \coprod_{[g]\in G/G_s} \mathrm{Mor}_{SK}(gs,-).$$

Assume that Y (X) is a covariant (contravariant) functor from $SK[G]$ to Top. Then there exists a natural action of G on the topological space

$$Y \times_{SK} X = Y\imath[G] \times_{SK} X\imath[G].$$

For any $s \in SK$, $y \in Y(s)$ and $x \in X(s)$ we have

$$g[y,x] = [Y([g]), X([g^{-1}])],$$

where $[g^{-1}] : gK(s) \longrightarrow K(s)$ and $[g] : K(s) \longrightarrow gK(s)$ are the maps determined by the operation by g^{-1} and g on K.

It is clear that

$$Y \times_{SK[G]} X = (Y \times_{SK} X)/G.$$

Let $X_G : SK_G \longrightarrow Top$ be the functor such that $X_G(s) = X(s)/G_s$ whenever $s \in SK$. Assume that Y'' is a covariant functor from SK_G to Top. Then

$$Y''\rho_G \times_{SK[G]} X = Y'' \times_{SK_G} X_G.$$

If $X = X''\rho_G$, where X'' is a contravariant functor from SK_G to Top, then $X_G = X''$ and

$$Y''\rho_G \times_{SK[G]} X''\rho_G = Y'' \times_{SK_G} X'' = (Y''\imath_G \times_{SK} X''\imath_G)/G = (Y'' \times_{SK} X'')/G.$$

The functor $B(SK/-)$ can be extended in a natural way to a functor from $SK[G]^{op}$ to Top. In this case we obtain a natural G-action on

$$\operatorname*{hocolim}_{SK} X = B(SK/-) \times_{SK} X.$$

Assume now that K is a special G-CW-complex. Then $IK\imath_G$ is a free locally contractible SK-CW-complex and there are homotopy equivalences

$$\operatorname*{hocolim}_{SK_G} X'' \cong IK \times_{SK_G} X'' \cong (IK\imath_G \times_{SK} X''\imath_G)/G \cong (\operatorname*{hocolim}_{SK} X''\imath_G)/G.$$

Hence, in this case, the classifying space BSK_G of the category SK_G is homotopy equivalent to K/G.

Let us consider now a functor $U : SK[G]^{op} \longrightarrow Top$. We will use the notation

$$U[K] = IK\imath_G \times_{SK} U\imath[G].$$

Then

$$U[K]/G = IK\rho_G \times_{SK[G]} U = IK \times_{SK_G} U_G.$$

The functor U defines the functor $T_U : SK_G^{op} \longrightarrow G - Top$ such that as G-spaces,

$$T_U(s) = \coprod_{[g]\in G/G_s} U(gs)$$

and

$$U[K] = IK \times_{SK_G} T_U = K\{T_U\}.$$

By pt we will denote the constant contravariant functor from $SK[G]$ to Top such that $pt(s)$ is equal to a one-point space pt. The natural transformation $\pi_U : U \longrightarrow pt$ induces the G-map $p_U : U[K] \longrightarrow K$. For every subgroup H of G,

$$(U[K])^H = IK^H \times_{SK^H} U^H$$

where $U^H : (SK^H)^{op} \longrightarrow Top$ is the functor such that $U^H(s) = U(s)^H$ whenever $s \in SK^H$.

If for every cell s of K, $U(s)$ is G_s-contractible, then p_U is a G-homotopy equivalence. Indeed, if for every subgroup H of G and $s \in SK^H$, $U(s)^H$ is contractible, then $(p_U)^H$ is a homotopy equivalence whenever H is a subgroup of G and this implies that p_U is a G-homotopy equivalence.

If for every cell s of K, $U(s)/G_s$ is contractible, then we have a homotopy equivalence

$$U[K]/G \cong K/G.$$

This is a consequence of the fact that $T_U(s)/G = U(s)/G_s$.

If K is a special G-CW-complex, then

$$U[K] \cong \operatorname*{hocolim}_{SK} U\imath[G]$$

as G-spaces and

$$U[K]/G \cong \operatorname*{hocolim}_{SK_G} U_G \cong (\operatorname*{hocolim}_{SK} U\imath[G])/G.$$

Assume that U is a contravariant functor from $SK[G]$ to CW such that, for every cell s of K, $U(s)$ is a G_s-CW-complex. Then $U[K]$ is a G-CW-complex such that

$$S_n U[K] = \coprod_{i=0}^{n} \coprod_{s\in S_i K} S_{n-i} U(s).$$

Let L be a local G-coefficient system on $U[K]$. For every cell s of K, let $L(s)$ be the local G_s-coefficient system on $U(s)$ such that $L(s)(s') = L(s')$ and let

$$H^q(U, L) : SK_G \longrightarrow Ab$$

be the local G-coefficient system on K such that

$$H^q(U, L)(s) = H^q_{G_s}(U(s), L(s)) = H^q_G(T_U(s), L_s).$$

1.5 Proposition *Let K be a G-CW-complex. Assume that*

$$U : SK[G]^{op} \longrightarrow CW$$

is a functor such that $U(s)$ is a G_s-CW-complex whenever s is a cell of K.

(i) *Let L be a local G-coefficient system on $U[K]$. Then there exists a spectral sequence*

$$H^p_G(K, H^q(U, L)) \Rightarrow H^{p+q}_G(U[K], L).$$

In particular, if K is a special G-CW-complex, then there exists a spectral sequence

$$\lim_{SK_G}{}^p\ H^q(U, L) \Rightarrow H^{p+q}_G(\operatorname{hocolim}_{SK} U, L).$$

(ii) *Let M be a generic G-coefficient system. Then there exists a spectral sequence*

$$H^p_G(K, H^q(U, M)) \Rightarrow H^{p+q}_G(U[K], M)$$

where

$$H^q(U, M)(s) = H^q_{G_s}(U(s), Mi_{G_s}) = H^q_G(G \times_{G_s} U(s), M).$$

Proof. The statement (i) is an immediate consequence of 1.1. The statement (ii) follows from (i) because if $L = M\theta_{U[K]}(G)$, then for every cell s of K and s' of $U(s)$, $L(s)(s') = M(G/(G_s)_{s'})$, so $L(s) = Mi_{G_s}\theta_{U(s)}(G_s)$. □

If $L = L''p_U$, where L'' is a local G-coefficient system on K, then $L(s)(s')$ is equal to $L''(s)$ so $L(s)$ is a constant G_s-coefficient system on $U(s)$ and

$$H^q(U, L)(s) = H^q(U(s)/G_s, L''(s)).$$

If, additionally, $U(s)/G_s$ is $L''(s)$-acyclic whenever $s \in SK$, then

$$H^*_G(K, L'') \cong H^*_G(U[K], L).$$

If, for every $s \in SK$, the projection $G \times_{G_s} U(s) \longrightarrow G/G_s$ induces an isomorphism $M(G/G_s) \cong H^*(G \times_{G_s} U(s), M)$, then

$$H^*_G(K, M) \cong H^*_G(U[K], M).$$

Assume now that K and K' are G-CW-complexes and that Y is a G-CW-subcomplex of $K \times K'$. By U_Y we will denote the contravariant functor from $SK[G]$ to the category $\mathcal{P}(K')_G$ such that, for every cell s of K,

$$U_Y(s) = \{k' \in K' : (K(s) \times \{k'\}) \subseteq Y\}.$$

It is clear that $U_Y(s)$ is a G_s-CW-subcomplex of K'. If $s \subseteq s_0$, then $U_Y(s \subseteq s_0)$ is equal to the inclusion $U_Y(s_0) \subseteq U_Y(s)$. For an element g of G, $U_Y(gs) = gU_Y(s)$ and $U_Y(g)$ is an operation by g^{-1}. One can easily check that $U_Y[K] = Y$. Similarly, let

$$U'_Y : SK'[G]^{op} \longrightarrow \mathcal{P}_G$$

be the functor such that, for every cell s' of K',

$$U'_Y(s') = \{k \in K : (\{k\} \times K'(s')) \subseteq Y\}.$$

Then $U'_Y[K'] = Y$.

If for every cell s of K, $U_Y(s)$ is G_s-contractible, then there is a G-homotopy equivalence $Y \cong K$. If, additionally, for every cell s' of K', $U'_Y(s')$ is $G_{s'}$-contractible, then K is G-homotopy equivalent to K'.

Proposition 1.5 yields the following fact.

1.6 Corollary *Let L be a local G-coefficient system on Y. Then there exist two spectral sequences*

$$H^p_G(K, H^q(U_Y, L)) \Rightarrow H^{p+q}_G(Y, L)$$

and

$$H^p_G(K', H^q(U'_Y, L)) \Rightarrow H^{p+q}_G(Y, L).\square$$

It is easy to check that in this case $L(s)(s') = L(s \times s')$.

1.7 Example Assume that M is a generic G-coefficient system. Then there exists a spectral sequence

$$H^p_G(K, H^q(U_Y, M)) \Rightarrow H^{p+q}_G(Y, M)$$

where $H^q(U_Y, M)$ is the local G-coefficient system such that, for every cell s of K,

$$H^q(U_Y, M)(s) = H^q_G(G \times_{G_s} U_Y(s), M).$$

In particular, if for every cell s of K, $U_Y(s)$ is G_s-contractible, then

$$H^*_G(K, M) = H^*_G(Y, M).$$

In this case there exists a spectral sequence

$$H^p_G(K', H^q(U'_Y, M)) \Rightarrow H^{p+q}_G(K, M).\square$$

1.8 Example Assume that L' is a local G'-coefficient system on K'. Let $L = L'p$ where p denote the restriction of the structural map $K \times K' \longrightarrow K'$ to Y. Then

$$H^q(U_Y, L)(s) = H^q_{G_s}(U_Y(s), L')$$

and

$$H^q(U'_Y, L)(s') = H^q(U'_Y(s')/G_{s'}, L'(s')).$$

In particular, let A be an abelian group. Then there exists a spectral sequence

$$H^p_G(K, H^q(U_Y, A)) \Rightarrow H^{p+q}(Y/G, A)$$

where

$$H^q(U_Y, A)(s) = H^q(U_Y(s)/G_s, A).$$

Hence

$$H^*(K/G, A) = H^*(K'/G, A)$$

whenever, for every cell s of K and s' of K', the CW-complexes $U_Y(s)/G_s$ and $U'_Y(s')/G_{s'}$ are A-acyclic. $\square$

1.9 Example Let K and K' be G-CW-complexes. Assume that

$$U : SK[G]^{op} \longrightarrow \mathcal{P}(K')_G$$

is a functor such that, for every $s \in SK$ and $g \in G$, $U(gs) = gU(s)$, $U(g)$ is the operation by g^{-1} and $U(s \subseteq s')$ is the inclusion $U(s') \subseteq U(s)$. Then there is an G-injection $\mathcal{U} : U[K] \longrightarrow K \times K'$ such that $\mathcal{U}[k, k'] = (k, k')$ whenever $k \in K(s)$ and $k' \in U(s)$. Let

$$U' : SK'[G]^{op} \longrightarrow CW$$

be the functor such that, for every cell s' of K'

$$U'(s') = \{k \in K : K'(s') \subseteq U(s_k)\}$$

where s_k is the open cell of K which contains k. If $Y = \mathcal{U}(U[K])$, then $U'_Y = U'$ and $U_Y = U$.

Let L be a local G-coefficient system on K'. Assume that, for every cell s' of K', the map $U'(s')/G_{s'} \longrightarrow pt$, where pt is a one-point space, induces the isomorphism

$$L(s') \longrightarrow H^*(U'(s')/G_{s'}, L(s')).$$

Then, by 1.8, there is a spectral sequence

$$H_G^p(K, H^q(U, L)) \Rightarrow H_G^{p+q}(K', L)$$

where

$$H^q(U, L)(s) = H_{G_s}^q(U(s), L).\square$$

2 Diagrams over G-posets

Assume that W is a poset and that G acts on W preserving order. By BW we will denote the classifying space of the category associated to W, which is constructed in the following way. Let

$$W(n) = \{(w_0, \ldots, w_n) \in W^{n+1} : w_0 \subset \ldots \subset w_n\}$$

where $w \subset w'$ means that $w \subseteq w'$ and $w \neq w'$, and let $W' = \coprod_{n=0} W(n)$. Then W' can be considered as a simplicial complex whose elements are the subsets $\{w_0, \ldots, w_n\}$ of W such that $(w_0, \ldots, w_n) \in W(n)$ and BW is the polyhedron associated to W'. The action of G on W induces the G-action on W' and BW. It is clear that BW is a special G-CW-complex and that G-poset of cells of BW is equal to W'.

In this Section we will apply the results of Section 1 to the case where K is equal to BW. We begin with a definition of a regular G-CW-complex and then describe some properties of such CW-complexes. We will use them later because BW is a regular G-CW-complex. Next we specialize the main results of Section 1 to the case $K = BW$ and give certain examples. In particular, we will consider the case where W is a G-poset of subgroups of G. As application, we obtain a certain sufficient condition for a G-CW-complex K such that the natural projection $EG \times_G K \longrightarrow BG$ induces an isomorphism of cohomology groups with coefficients in the field $\mathbf{F}_p$ of integers *mod p* .

We will call a G-CW-complex K *regular* if, for any two cells s and s' of K and for any element g of G, the conditions $s \subseteq s'$ and $s \subseteq gs'$ imply that $s = gs$. One can easily check that BW is a regular G-CW-complex. We will need the following facts.

2.1 Lemma *Assume that K is a regular G-CW-complex. Then the projection $\nu : SK_G \longrightarrow S(K/G)$ such that $\nu(s) = [s]$, is a natural equivalence of categories.*
Proof. In this case, for any cell s of K, there is a natural equivalence of functors

$$\mathrm{Mor}_{SK_G}(s,-) \longrightarrow \mathrm{Mor}_{SK/G}(\nu(s),\nu(-))$$

and this yields the result. □

2.2 Corollary *Let K be a regular G-CW-complex. Then, for any functor $T : SK_G^{op} \longrightarrow CW$,*

$$IK \times_{SK_G} T = I(K/G) \times_{SK/G} T/G$$

where, for every cell s of K,

$$(T/G)([s]) = (\coprod_{s'' \in Gs} T(s''))/G = T(s)/G_s.$$

In particular, if K is a special regular G-CW-complex, then

$$\operatorname*{hocolim}_{SK_G} T \cong \operatorname*{hocolim}_{SK/G} T/G$$

and

$$K/G \cong B(SK/G).$$

Proof. The result follows from the fact that there exist isomorphisms

$$\mathrm{Mor}_{SK_G}(s,-) \times_{SK_G} T = (\coprod_{[g] \in G/G_s} \mathrm{Mor}_{SK}(gs,-) \times_{SK} T)/G$$

$$= (T/G)([s]) = \mathrm{Mor}_{S(K/G)}([s],-) \times_{S(K/G)} (T/G). \square$$

The next result can be proved using similar arguments.

2.3 Corollary *Let K be a regular G-CW-complex. Then, for any functor $L : SK_G \longrightarrow Ab$,*

$$H_G^*(K,L) = H_{(e)}^*(K/G, L/G)$$

where $L/G : S(K/G) \longrightarrow Ab$ is the local coefficient system on K/G such that, for every cell s of K,

$$(L/G)([s]) = (\coprod_{s'' \in Gs} L(s''))/G.$$

In particular, if K is additionally a special G-CW-complex, then

$$\lim_{SK_G}{}^* L = \lim_{S(K/G)}{}^* L/G. \square$$

Assume now that $U : SK[G]^{op} \longrightarrow CW$ is a functor such that, for every $s \in SK$, $U(s)$ is a G_s-CW-complex. If K is a regular G-CW-complex, then there is a G-CW-isomorphism

$$(IK)\imath_G \times_{SK} U\imath[G] = I(K/G) \times_{SK/G} N_U$$

where $N_U : S(K/G)^{op} \longrightarrow G - CW$ is the functor such that, for every $s \in K$,

$$N_U([s]) = (\coprod_{s'' \in Gs} U(s''))/G = G \times_{G_s} U(s) = T_U(s).$$

(The functor T_U was defined before Proposition 1.5.)

We will now apply the above results to the case where K is equal to BW and W is a G-poset. It is clear that BW is a special, regular G-CW-complex. We will use the notation $SW = SBW = W'$, $BW/G = (BW)/G$, and $SW/G = (SW)/G$. Corollary 2.3 implies that, for every local G-coefficient system L on BW,

$$H^*_G(BW, L) = H^*(BW/G, L/G) = \lim_{SW/G}{}^* L/G.$$

From Corollary 2.2 we obtain that

$$BW/G \cong B(SW/G).$$

The following fact is now an immediate consequence of 1.5.

2.4 Corollary *Let $U : SW[G]^{op} \longrightarrow CW$ be the functor such that, for every $s \in SW$, $U(s)$ is a G_s-CW-complex. Then, for any local G-coefficient system L on $U[BW]$, there is a spectral sequence*

$$H^p(BW/G, H^q_G(N_U, L)) \Rightarrow H^{p+q}_{G_s}(IBW \times_{SW} U, L)$$

where

$$H^q_G(N_U, L)([s]) = H^q_G(N_U([s]), L_s) = H^q_{G_s}(U(s), L(s)).$$

Morever, there are G-homotopy equivalences

$$\operatorname{hocolim}_{SW} U \cong IBW \times_{SW} U \cong \operatorname{hocolim}_{SW/G} N_U. \square$$

By $W[G]$ we will denote the category whose objects are the elements of W and whose morphism sets are defined in such a way that

$$\mathrm{Mor}_{W[G]}(w, w') = \{g \in G : gw \subseteq w'\}.$$

Let $q_W : SW \longrightarrow W$ be the G-poset map such that

$$q_W(w_0, \ldots, w_n) = w_n.$$

Then $q_W[G]$ will denote the functor $SW[G] \longrightarrow W[G]$ induced by q_W.

We will now consider the following example. Let D be a G-set and let $P(D)$ be the G-poset of all non-empty subsets of D. Then the poset $SP(D)/G$ is conically contractible ([9], 1.5). Thus the space $(BP(D))/G$ is contractible because it is homotopy equivalent to $B(SP(D)/G)$. We will consider the category $D[G]$ whose objects are the elements of D and whose morphism sets are defined as follows

$$\mathrm{Mor}_{D[G]}(d, d') = \{g \in G : gd = d'\}.$$

2.5 Example Assume that

$$X : D[G]^{op} \longrightarrow CW$$

is a functor such that, for every $d \in D$, $X(d)$ is a G_d-CW-complex. We define the functor

$$P(X) : P(D)[G]^{op} \longrightarrow CW,$$

in such a way that, for every subset D_0 of D,

$$P(X)(D_0) = \prod_{d \in D_0} X(d).$$

If $g \in G$ and $(x_d)_{d \in D_0} \in P(X)(D_0)$, then $P(X)(g)(x) \in P(X)(g^{-1}D_0)$ and

$$P(X)(g)(x) = (X(g)x_{gd})_{d \in g^{-1}D_0}.$$

If $D_0 \subseteq D_1$, then $P(X)(D_0 \subseteq D_1)$ is the natural projection. It is clear that $PX(D_0)$ is a G_{D_0}-CW-complex.

Let $X_0 = P(X)q_{P(D)}$. Then, for every element $\delta = (D_0, \ldots, D_n)$ of $SP(D)$, $G_\delta = \bigcap_{i=0}^n G_{D_i}$ and

$$X_0(\delta) = \prod_{d \in D_n} X(d)$$

is a G_δ-CW-complex. Let

$$\mathop{*}_{d \in D} X(d) = X_0[BP(D)] \cong \operatorname*{hocolim}_{\delta \in SP(D)} \prod_{d \in D_n} X(d).$$

We will say that this G-CW-complex is the equivariant join of X over the G-set D.

Let M be a generic G-coefficient system. Then there exists a spectral sequence

$$\lim_{[\delta] \in SP(D)/G}{}^p \quad H^q_G(G \times_{G_\delta} \prod_{d \in D_n} X(d), M) \Rightarrow H^{p+q}_G(\mathop{*}_{d \in D} X(d), M). \square$$

The following example describes a special case of 2.5.

2.6 Example Let H be a subgroup of G and let V be an orthogonal representation of H. By V_0 we will denote the orthogonal representation of G induced from V. Hence $V_0 = \mathbf{R}(G) \otimes_{\mathbf{R}(H)} V$ where $\mathbf{R}$ denote the field of real numbers. Let $SV = \{v \in V : ||v|| = 1\}$ be the unit sphere of V. It is well known that there exists a H-CW-complex structure on SV. Assume that $D = G/H$ and that $X(gH)$ is equal to the subcomplex gSV of $G \times_H SV$. Then it is easy to check that SV_0 is G-homeomorphic to the G-CW-complex

$$\mathop{*}_{[g] \in G/H} gSV.$$

Let $\mathcal{S}_H$ denote the right G-poset of all non-empty left H-subsets of G. It is clear that $\mathcal{S}_H = P(G/H)$. The elements of $S\mathcal{S}_H$ will be denoted by $\alpha = (A_0, \ldots, A_n)$. There exists a G-homotopy equivalence

$$\operatorname*{hocolim}_{[\alpha] \in (S\mathcal{S}_H)/G} G \times_{G_\alpha} \mathrm{Map}_H(A_n, SV) \cong SV_0.$$

For any generic G-coefficient system M, there is a spectral sequence

$$\lim_{[\alpha] \in (S\mathcal{S}_H)/G}{}^p \quad H^q_G(G \times_{G_\alpha} \mathrm{Map}_H(A_n, SV), M) \Rightarrow H^{p+q}_G(SV_0, M). \square$$

The following example gives a generalization of the spectral sequence described in [3], Ch.VII.4.

2.7 Example Let D be a G-set. Assume that K' is a G-CW-complex and that $X : D[G]^{op} \longrightarrow CW$ is a functor such that, for every $d \in D$, $X(d)$ is a G_d-CW-subcomplex of K' and, for every $g \in G$, $X(gd) = gX(d)$ and $X(g)$ is the operation by g^{-1}. We define the functor

$$X_I : P(D)[G]^{op} \longrightarrow CW$$

in such a way that

$$X_I(D_0) = \bigcap_{d \in D_0} X(d)$$

whenever D_0 is a subset of D. Let $X_i = X_I q_{P(D)}[G]$. Hence, if $\delta = (D_0, \ldots, D_n)$, then $X_i(\delta) = X_I(D_n)$. We will now apply 1.9 to $U = X_i$, $K = BP(D)$. In this case, for any $s' \in SK'$, $U'(s') = BP(D_{s'})$ where

$$D_{s'} = \{d \in D : K'(s') \subseteq X(d)\}.$$

If the set $D_{s'}$ is not empty, then $U'(s')$ is $G_{s'}$-contractible.

Assume now that, for every point k of K', there is $d \in D$ such that $k \in X(d)$. Then, for every cell s' of K', the set $D_{s'}$ is non-empty. Hence there is a G-homotopy equivalence

$$K' \cong \operatorname*{hocolim}_{[\delta] \in SP(D)/G} G \times_{G_s} X_I(D_n).$$

We have a spectral sequence

$$\lim_{[\delta] \in SP(D)/G}{}^{p} H^q_{G_\delta}(X_I(D_n), L) \Rightarrow H^{p+q}_G(K', L)$$

whenever L is a local G-coefficient system on K'. If M is a generic G-coefficient system, then there is a spectral sequence

$$\lim_{[\delta] \in SP(D)/G}{}^{p} H^q_G(G \times_{G_\delta} X_I(D_n), M) \Rightarrow H^{p+q}_G(K', M).$$

In particular, for any abelian group A, there is a spectral sequence

$$\lim_{[\delta] \in SP(D)/G}{}^{p} H^q(X_I(D_n)/G_\delta, A) \Rightarrow H^{p+q}(K'/G, A).$$

If $X_I(D')$ is $G_{D'}$-contractible whenever $D' \in P(D)$, then K' is G-contractible. If $X_I(D_n)/G_\delta$ is contractible whenever $\delta \in SP(D)$, then K'/G is contractible. □

2.8 Example Let us consider the case where D is a G-set of subgroups of G. For any nonempty subset D_0 of D, by $H(D_0)$ we will denote the subgroup of G generated by all elements of D_0. Let X be the functor such that $X(H) = K'^H$, whenever H belongs to D. Then $X_I(D_0) = K'^{H(D_0)}$. If, for any D_0, $K'^{H(D_0)}/NH(D_0)$ is contractible, then K'/G is acyclic. □

The next example generalizes the spectral sequence described in [3], Ch.IX.12.

2.9 Example Assume that W is a G-poset and that K' is a G-CW-complex. Let $U_0 : W^{op} \longrightarrow \mathcal{P}(K')$ be a G-poset map. Then we define the functor

$$U : W[G]^{op} \longrightarrow \mathcal{P}(K')_G$$

in such a way that, for every $w \in W$, $U(w) = U_0(w)$, for every $g \in G$, $U(g)$ is the operation by g^{-1} and $U(w \subseteq w')$ is the inclusion $U(w') \subseteq U(w)$. By the same letter U, we will denote the extension of this functor to the category $SW[G]$ in such a way that, for $\omega = (w_0, \dots, w_n)$, $U(\omega) = U(w_n)$. Let $K = BW$. We will now consider Example 1.9 in this case. We have G-homotopy equivalences

$$U[K] \cong \operatorname*{hocolim}_{W} U_0 \cong \operatorname*{hocolim}_{[\omega] \in SW/G} G \times_{G_\omega} U(w_n).$$

For any element k' of K', let

$$W(k') = \{w \in W : k' \in U(w)\}.$$

It is clear that $W(k')$ is a $G_{k'}$-subposet of W and that if $w \in W(k')$ and $w \subseteq w'$, then $w' \in W(k')$. for any cell s' of K', we will use the notation $W(s') = W(k')$, where k' belongs to the open cell s'. Then $U'(s') = BW(s')$. Thus there is a G-homotopy equivalence

$$U[K] \cong IK' \times_{SK'} BW(-).$$

If for every $w \in W$, $U_0(w)$ is G_ω-contractible, then there is a G-homotopy equivalence

$$U[K] \cong BW.$$

In this case, for any generic G-coefficient system M, there exists a spectral sequence

$$H^p_G(K', H^q_G(G \times_{G_-} BW(-), M)) \Rightarrow H^{p+q}_G(BW, M).$$

If for every element ω of SW, $U(w_n)/G_\omega$ is contractible, then there exist homotopy equivalences

$$U[K]/G \cong BW/G \cong B(SW/G).$$

Let L be a local G-coefficient system on BW such that, for every $\omega \in SW$, the map

$$L(\omega) \longrightarrow H^*_G(U(w_n)/G_\omega, L(\omega))$$

is an isomorphism. Then there is a spectral sequence

$$H^p_G(K', H^q_{G_-}(BW(-), L)) \Rightarrow H^{p+q}_G(BW, L).$$

Assume now that, for every point k' of K', $BW(k')$ is $G_{k'}$-contractible. Then we have a G-homotopy equivalence

$$U[K] \cong K'.$$

In this case, for any generic G-coefficient system M, there exists a spectral sequence

$$\lim_{[\omega] \in SW/G}^p \quad H^q_G(G \times_{G_\omega} U(w_n), M) \Rightarrow H^{p+q}_G(K', M).$$

If, for every point k' of K', $BW(k')/G_{k'}$ is contractible, then there is a homotopy equivalence

$$U[K]/G \cong K'/G.$$

Assume that L is a local G-coefficient system on K' such that, for every $s' \in SK'$, the map $BW(s') \longrightarrow pt$ induces the isomorphism

$$L(s') \longrightarrow H^*_G(BW(s')/G_{s'}, L(s')).$$

Then there is a spectral sequence

$$\lim_{[\omega] \in SW/G}^p \quad H^q_{G_\omega}(U(w_n), L) \Rightarrow H^{p+q}_G(K', L).\square$$

2.10 Example Let W be a G-poset of subgroups of G and let

$$U_0 : W^{op} \longrightarrow \mathcal{P}(K')$$

be the G-poset map such that, for every element H of W, $U_0(H) = K'^H$. If $k' \in K'$, then

$$W(k') = W(G_{k'}) = \{H \in W : H \subseteq G_{k'}\}.$$

(i) Let us consider the following conditions.

(α) For every $k' \in K'$, $G_{k'} \in W$.

(β) The poset W has the smallest element H_0 and $H_0 \subseteq G_{k'}$, whenever $k' \in K'$.

(γ) For every $k' \in K'$, there exists $H \in W$ such that $H \subseteq G_{k'}$, and if $H, H' \in W$, then $H \cap H' \in W$.

If one of this conditions holds then, for every $k' \in K'$, $W(k')$ is $G_{k'}$-contractible. We will consider this case in (ii).

(ii) Assume that, for every $k' \in K'$, $W(k')$ is $G_{k'}$-contractible. Then we have G-homotopy equivalences

$$\operatorname*{hocolim}_{H \in W} K'^H \cong \operatorname*{hocolim}_{\omega \in SW} K'^{H_n} \cong K'.$$

(An example of such situation is described in [4]. In that case G is a p-group, which is not an elementary p-group, K' is a G-CW-complex with nontrivial isotropy groups and W is the G-poset of all nontrivial and proper subgroups of G.)

For any generic G-coefficient system M, there is a spectral sequence

$$\lim_{[\omega] \in SW/G}^p \quad H^q_G(G \times_{G_\omega} K'^{H_n}, M) \Rightarrow H^{p+q}_G(K', M).$$

If, for every $H \in W$, K'^H is NH-contractible, then K' is G-homotopy equivalent to BW. If, for every $\omega \in SW$, K'^{H_n}/G_ω is contractible, then K'/G is homotopy equivalent to BW/G.

(iii) Let A be an abelian group. Assume that, for every $k' \in K'$, the map $A \longrightarrow H^*(BW(k')/G_{k'}, A)$ is an isomorphism. Then there is a spectral sequence

$$\lim_{[\omega] \in SW/G}{}^{p} \quad H^q((K'^{H_n})/G_\omega, A) \Rightarrow H^{p+q}(K'/G, A).$$

If, additionally, for every $\omega \in SW$, $(K'^{H_n})/G_\omega$ is A-acyclic, then

$$H^*(BSW/G, A) = H^*(BW/G, A) = H^*(K'/G, A).$$

(iv) Assume that, for every $k' \in K'$, $BW(k')/G_{k'}$ is contractible. Then there is a homotopy equivalence

$$\operatorname*{hocolim}_{[\omega] \in SW/G} \quad (K'^{H_n})/G_\omega \cong K'/G. \square$$

2.11 Example The set of all nontrivial p-subgroups of G will be denoted by $W_p(G)$. Assume that K' is a nonempty G-CW-complex and that

$$K' = \bigcup_{H \in W_p(G)} K'^H.$$

(i) Assume that, for every nontrivial p-subgroup H of G, K'^H is NH/H-contractible. Then, by methods similar to those used by R.Oliver in the proofs of Proposition 3 and Theorem 1 of [8], one can prove that K'/G is contractible. Indeed, let $M_p(G')$ denote the set of all Sylow p-subgroups of G' and let

$$K'_H = \{k \in K' : H \in M_p(G_k)\}$$

and $K'_s(H) = K'^H \setminus K'_H$. Then $K'_s(H)$ is a NH/H-subcomplex of K'^H and, as is proved in [8], the natural projection

$$f : (K'^H/NH)/(K'_s(H)/NH) \longrightarrow (GK'^H/G)/(GK'_s(H)/G)$$

is, in fact, a homeomorphism. This implies that if G_p is a Sylow p-subgroup of G, then $GK'^{G_p}/G = K'^{G_p}/NG_p$ is contractible. We will use the induction on $|G_p|$. If $|G_p| = p$, then $K' = GK'^{G_p}$ and the result is proved. It follows from the inductive assumptions that, for every $H \in W_p(G) \setminus M_p(G)$, $K'_s(H)/NH$ is contractible. Now we can use the induction on $|\bigcup_{k \in K'} M_p(G_k)|$.

In particular, $BW_p(G)/G$ is *contractible*. If H is a subgroup of G, then $W_p(G)(H)$ is equal to $W_p(H)$. Thus we may apply (iii) and (iv) of 2.10.

(ii) Assume that, for every $\omega \in SW_p(G)$, $(K'^{H_n})/G_\omega$ is contractible (A-acyclic). Then K'/G is contractible (A-acyclic).

(iii) Assume that K' is a finite dimensional G-CW-complex such that, for every nontrivial p-subgroup H of G, K'^H is $\mathbf{Z}$-acyclic. From the Smith theory ([2], Ch.III) it follows that K'^{H_n}/G_ω is $\mathbf{Z}$-acyclic, whenever $\omega \in SW_p(G)$. Thus K'/G is $\mathbf{Z}$-acyclic.

(iv) Assume that, for every nontrivial p-subgroup H of G, K'^H is $\mathbf{F}_p$-acyclic. Then, for every $\omega \in SW_p(G)$, $(K'^{H_n})/G_\omega$ is $\mathbf{F}_p$-acyclic, Thus, by (ii), K'/G is $\mathbf{F}_p$-acyclic. This is a generalization of one of the results of P.Webb, who in [13] considered the case where K' is a geometrical generalization of a finite dimensional G-simplicial complex.

(v) Let $A_p(G)$ denote the poset of all nontrivial elementary abelian p-subgroups of G. Then $BA_p(G)$ is G-homotopy equivalent to $BW_p(G)$ and $BA_p(G)/G$ is $\mathbf{Z}$-acyclic. If H is a subgroup of G, then $A_p(G)(H) = A_p(H)$. Thus we may apply 2.10.(iii). If, for all $\omega \in SA_p(G)$, $(K'^{H_n})/G_\omega$ is A-acyclic, then K'/G is A-acyclic. If K' is a finite dimensional complex, then the condition that K'^H is $\mathbf{Z}$-acyclic ($\mathbf{F}_p$-acyclic) whenever $H \in A_p(G)$ implies that K'/G is $\mathbf{Z}$-acyclic ($\mathbf{F}_p$-acyclic). □

2.12 Corollary *Assume that K' is a G-CW-complex such that, for every nontrivial p-subgroup H of G, K'^H is $\mathbf{F}_p$-acyclic. Suppose that the Sylow p-subgroups of G_k are nontrivial whenever $k \in K$. Then the natural map*

$$EG \times_G K' \longrightarrow BG$$

induces an isomorphism of the cohomology groups with coefficients in $\mathbf{F}_p$.

Proof. This result follows from 2.11 and from the next lemma. □

2.13 Lemma *Let K' be a G-CW-complex such that, for every nontrivial p-subgroup H of G, K'^H is $\mathbf{F}_p$-acyclic. Then the map $EG \longrightarrow pt$ induces the isomorphism*

$$H^*(CK'/G, K'/G; \mathbf{F}_p) \longrightarrow H^*(EG \times_G CK', EG \times_G K'; \mathbf{F}_p)$$

where CK' is the cone of K'.

Proof. Let G_p be a Sylow p-subgroup of G and let

$$K'' = \bigcup_{H \in W_p(G_p)} K'^H.$$

Then K'' is a G_p-subcomplex of K' and the map

$$H^*(K'/G_p, K''/G_p; \mathbf{F}_p) \longrightarrow H^*(EG \times_{G_p} K', EG \times_{G_p} K''; \mathbf{F}_p)$$

is an isomorphism. It follows from the assumptions that the maps

$$H^*(CK'/G_p, \mathbf{F}_p) \longrightarrow H^*(K''/G_p, \mathbf{F}_p)$$

and

$$H^*(EG \times_{G_p} CK', \mathbf{F}_p) \longrightarrow H^*(EG \times_{G_p} K'', \mathbf{F}_p)$$

are isomorphisms. Hence the map

$$H^*(CK'/G_p, K'/G_p; \mathbf{F}_p) \longrightarrow H^*(EG \times_{G_p} CK', EG \times_{G_p} K'; \mathbf{F}_p)$$

is also an isomorphism. Now the lemma follows from the existence of the transfer map. □

2.14 Example There are isomorphisms

$$H^*(EG \times_G BW_p(G), \mathbf{F}_p) \cong H^*(BG, \mathbf{F}_p) \cong H^*(EG \times_G BA_p(G), \mathbf{F}_p)$$

(see [13]) and

$$H^*(EG \times_G E_p(G), \mathbf{F}_p) \cong H^*(BG, \mathbf{F}_p)$$

where $E_p(G)$ is a G-CW-complex whose all isotropy groups belong to $W_p(G)$ and whose fixed point subcomplexes $E_p(G)^H$ are contractible whenever $H \in W_p(G)$.

3 CW-diagrams over categories associated to G-posets

Let $Sub - G$ denote the G-poset of all subgroups of G. Assume that W is a G-poset. A G-poset map $d : W \longrightarrow Sub - G$ will be called admissible if, for every $w \in W$, dw is a subgroup of G_w. Such maps are considered for example in [5]. It follows from the definition that, in fact, dw is a *normal* subgroup of G_w. By $W(d)$ we will denote the category whose object set is equal to W and whose morphism sets are defined in such a way that

$$\mathrm{Mor}_{W(d)}(w, w') = \mathrm{Mor}_{W[G]}(w, w')/dw' = \coprod_{[g] \in G/dw'} \mathrm{Mor}_W(w, gw')$$

whenever $w, w' \in W$. Such categories was considered in [11] and [12].

In this Section we apply the results of Sections 1 and 2 to study homotopy colimits and derived functors of inverse limits on categories of the form $W(d)$. To this end we describe free locally conctractible $W(d)^{op}$-CW-complexes in 3.7. We begin with some results (3.1 and 3.2) from [11] and [12] which are needed in our considerations,

It is clear that if d is the constant map such that, for $w \in W$, $dw = (e)$, then $W(d) = W[G]$. By $\imath(d)$ we will denote the natural inclusion $W \subseteq W(d)$ and by $\rho(d)$ the natural projection $W[G] \longrightarrow W(d)$. If

$$\imath[G] : W \longrightarrow W[G]$$

is the natural inclusion, then $\imath(d) = \rho(d)\imath[G]$.

3.1 Examples (i) Let $W = Sub - G$ and let d be the identity map. Then $W(d) = O_G$. If F is a G-subposet of $Sub - G$, then the subcategory $F(d)$ of O_G will be denoted by O_F.

(ii) Let $W = SK^{op}$ and let $ds = G_s$ whenever $s \in SK$. Then $W(d) = SK_G^{op}$. □

3.2 Lemma (i) *Assume that* $X : W(d) \longrightarrow Top$ *and* $Y : W(d)^{op} \longrightarrow Top$. *Then*

$$X \times_{W(d)} Y = X\rho(d) \times_{W[G]} Y\rho(d) = (X\imath(d) \times_W Y\imath(d))/G,$$

where G *acts on* $X\imath(d) \times_W Y\imath(d)$ *in such a way that*

$$g[x, y] = [X([g])x, Y([g])^{-1}y]$$

whenever $g \in G$, $w \in W$, $x \in X(w)$, $y \in Y(w)$, *and* $[g] : w \longrightarrow gw$.

(ii) *Assume that* $M, N : W(d)^{op} \longrightarrow Ab$. *Then*

$$\mathrm{Hom}_{W(d)}(M, N) = \mathrm{Hom}_{W[G]}(M\rho(d), N\rho(d)) = \mathrm{Hom}_W(M\imath(d), N\imath(d))^G,$$

where G *acts on* $\mathrm{Hom}_W(M\imath(d), N\imath(d))$ *in such a way that*

$$(gf)(m) = N([g])^{-1}(f(M([g])m))$$

whenever $g \in G$, $w \in W$, $m \in M(w)$ *and* $[g] : g^{-1}w \longrightarrow w$. □

Assume now that $\alpha : SK \longrightarrow W^{op}$ is a G-poset map. By

$$X(\alpha) : W^{op} \longrightarrow Top$$

we will denote the functor such that, for every $w \in W$,

$$X(\alpha)(w) = \bigcup_{s \in SK(w)} K(s),$$

where
$$SK(w) = \{s \in SK : w \subseteq \alpha(s)\}.$$
If $w \subseteq w'$, then $X(\alpha)(w \subseteq w')$ is equal to the inclusion
$$X(\alpha)(w') \subseteq X(\alpha)(w).$$

3.3 Lemma (i) *$X(\alpha)$ is a free W^{op}-CW-complex.*

(ii) *For any functor $Y : W \longrightarrow Top$,*
$$X(\alpha) \times_W Y = IK \times_{SK} Y\alpha.$$
In particular,
$$K = \operatorname*{colim}_W X(\alpha).$$

(iii) *For any functor $M : W^{op} \longrightarrow Ab$,*
$$\mathrm{Hom}_W(C_*(X(\alpha)(-), \mathbf{Z}), M) = \mathrm{Hom}_{SK}(C_*(K(-), \mathbf{Z}), M\alpha).$$

Proof. Let $u : Top \longrightarrow Set$ denote the underlying functor. It follows from the definition that $uX(\alpha)$ is the left Kan extension of uIK along α and that
$$uX(\alpha) = \coprod_{k \in K} \mathrm{Mor}_W(-, \alpha(s_k))$$
where s_k is the open cell which contains k. This implies the fact that $X(\alpha)$ is a free W^{op}-CW-complex and that the statement (ii) is true. The construction of $X(\alpha)$ yields that it is a CW-complex and that the set of open W^{op}-cells of $X(\alpha)$ is equal to SK. The statement (iii) follows from the fact that
$$C_*(X(\alpha)(-), \mathbf{Z}) = \coprod_{s \in SK} \mathbf{Z}(\mathrm{Mor}_W(-, \alpha(s))).\square$$

3.4 Corollary *Assume that $d : W \longrightarrow Sub - G$ is an admissible G-poset map. Let $\alpha : SK \longrightarrow W^{op}$ be a G-poset map such that, for every $s \in SK$, $G_s = d\alpha(s)$. Then the following is true.*

(i) *There is a functor $\alpha(d) : SK_G \longrightarrow W(d)^{op}$ such that $\alpha(d)(s) = \alpha(s)$ whenever $s \in SK$.*

(ii) *The functor $X(\alpha)$ can be extended to a free $W(d)^{op}$-CW-complex $X(\alpha(d))$.*

(iii) *For any functor $Y : W(d) \longrightarrow Top$,*
$$X(\alpha(d)) \times_{W(d)} Y = (IK \times_{SK} Y\alpha)/G = IK \times_{SK_G} Y_G$$
where $Y_G = Y\alpha(d)$.

(iv) *For any functor* $M : W(d)^{op} \longrightarrow Ab$,

$$H^n_G(K, M\alpha(d)) = H^n(\mathrm{Hom}_{W(d)}(C_*(X(\alpha(d))(-), \mathbf{Z}), M)).$$

Proof. (i) We may define $\alpha(d)$ in such a way that if $[g] : s \longrightarrow gs$, then $\alpha(d)([g])$ is equal to the map $[g^{-1}] : g\alpha(s) \longrightarrow \alpha(s)$.

(ii) This follows from 3.3 (i) and from the fact that

$$\mathrm{Mor}_{W(d)}(-, w)\imath(d) = \coprod_{[g] \in G/dw} \mathrm{Mor}_W(-, gw).$$

The statements (iii) and (iv) are immediate consequence of 3.2 and 3.3. □

Assume that $X : W(d)^{op} \longrightarrow Top$ is a functor such that the restriction $X\imath(d)$ is a free CW-diagram over W^{op}. Then $X(w)$ can be considered as a subcomplex of $X(w')$ whenever $w' \subseteq w$. Let

$$K(X) = \underset{W}{colim}\ X.$$

The CW-complex $K(X)$ has the natural structure of a G-CW-complex and for every $w \in W$, $X(w)$ is a G_w-subcomplex of K^{dw}. By $\alpha(X)$ we will denote the G-poset map from $SK(X)$ to W^{op} such that $\alpha(X)(s)$ is the greatest element of W, which satisfies the condition that $K(X)(s) \subseteq X(w)$. Let

$$SK(w) = \{s \in SK(X) : w \subseteq \alpha(X)(s)\}.$$

Then

$$X(w) = \bigcup_{s \in SK(w)} K(X)(s).$$

3.5 Lemma *Let X be a $W(d)^{op}$-CW-complex such that $X\imath(d)$ is a free W^{op}-diagram. Assume that, for every $s \in SK$, $G_s = d\alpha(X)(s)$. Then the following is true.*

(i) *The functor X is a free $W(d)^{op}$-CW-complex.*

(ii) *Suppose that $X(w)$ is contractible whenever $w \in W$. Then, for any functor $Y : W(d) \longrightarrow Top$,*

$$\underset{W(d)}{\mathrm{hocolim}}\ Y \cong IK(X) \times_{SK(X)_G} Y_G.$$

(iii) *Let R be a commutative ring. If $X(w)$ is R-acyclic whenever $w \in W$, then, for any functor $M : W(d)^{op} \longrightarrow R - Mod$,*

$$\underset{W(d)}{\lim{}^*}\ M = H^*_G(K(X), M\alpha(X)(d)).$$

Proof. The statement (i) follows from 3.4 (ii). The assertion (ii) is a consequence of 3.4 (iii) because in this case $X(w)$ is a free locally contractible $W(d)^{op}$-CW-complex. The statement (iii) follows from 3.4 (iv), because the assumptions imply that $C_*(X(-), \mathbf{Z})$ is a projective resolution of the constant functor $\mathbf{Z}$ in the category of contravariant functors from $W(d)$ to the category $R - Mod$ of R-modules. □

3.6 Corollary *Let $X : W[G]^{op} \longrightarrow CW$ be a functor such that the restriction $X\imath[G]$ is a free W^{op}-CW-complex.*

(i) *Assume that $d : W \longrightarrow Sub - G$ is an admissible G-poset map. If, for any cell s of $K(X)$, $G_s = d\alpha(X)(s)$, then there exists a free $W(d)^{op}$-complex X' such that $X'\rho(d) = X$.*

(ii) *Assume that, for every $w \in W$, $X(w)$ is contractible. Then there exists a homotopy equivalence*

$$EG \times_G K(X) \cong EG \times_G BW.$$

Proof. (i) It follows from 3.4 and 3.5 that we may take

$$X' = X(\alpha(X)(d)).$$

(ii) From (i), we obtain that $EG \times X$ is a free locally contractible $W[G]^{op}$-CW-complex. One can consider the particular case where $X(w) = B(w\backslash W)$ and $w\backslash W = \{w' \in W; w \subseteq w'\}$. In this case $K(X) = BW$, as G-CW-complexes. The results of Dror [6], imply that there is a homotopy equivalence

$$\underset{W[G]}{colim}\ (EG \times X) \cong \underset{W[G]}{colim}\ (EG \times B(-\backslash W)).$$

Now, it is sufficient to use the fact that

$$\underset{W[G]}{colim}\ (EG \times X) \cong EG \times_G K(X). \square$$

Let K be a G-CW-complex and let $\alpha : SK \longrightarrow W^{op}$ be a G-poset map. Assume that $d : W \longrightarrow Sub - G$ is an admissible G-poset map. By

$$Y(d\alpha) : SK^{op}(d\alpha) \longrightarrow CW$$

we will denote the functor such that, for every $s \in SK$,

$$Y(d\alpha)(s) = E(G_s/d\alpha(s)),$$

where E is a functor from the category Gr of groups to CW such that, for every group G', EG' is a universal free G'-CW-complex. If $[g^{-1}] : gs \longrightarrow s$ is a morphism of $SK(d\alpha)$, then

$$Y(d\alpha)([g]) : E((gG_sg^{-1}/g(d\alpha(s))g^{-1}) \longrightarrow E(G_s/d\alpha(s))$$

is equal to the map induced by the conjugation by g^{-1}. We will consider the functor $E(\alpha) : W^{op} \longrightarrow CW$ such that, for any $w \in W$,

$$E(\alpha)(w) = IK \times_{SK} (\mathrm{Mor}_W(w, \alpha(-)) \times Y(d\alpha)).$$

One can easily check that

$$E(\alpha)(-) = IX(\alpha) \times_{SX(\alpha)(-)} Y(d\alpha).$$

3.7 Proposition *Assume that, for every $w \in W$, $X(\alpha)(w)$ is contractible. Then the functor $E(\alpha)$ can be extended to a free locally contractible $W(d)^{op}$-CW-complex.*

Proof. It follows from the definition that, for every $w \in W$, the functor $Y(d\alpha)$ after the restriction to $SX(\alpha)(w)$ is locally contractible. Thus there is a homotopy equivalence $E(\alpha)(w) \cong X(\alpha)(w)$ whenever $w \in W$. Hence the assumptions imply that $E(\alpha)$ is locally contractible. The functor $E(\alpha)$ can be extended to a functor from $W[G]^{op}$ to CW in such a way that, for any morphism $[g] : g^{-1}w \longrightarrow w$ of $W[G]$,

$$E(\alpha)([g])([k, (w \subseteq \alpha(s), y)]) = [g^{-1}k, (g^{-1}w \subseteq \alpha(g^{-1}s), Y(d\alpha)[g]y)]$$

whenever $s \in SK$, $k \in K(s)$, $y \in Y(d\alpha)(s)$. It is clear that

$$K(E(\alpha)) = IK \times_{SK} Y(d\alpha),$$

as G-CW-complexes. Let $p : K(E(\alpha)) \longrightarrow K$ be the natural projection and let $Sp : SK(E(\alpha)) \longrightarrow SK$ be the G-poset map induced by p. Then

$$\alpha(E(\alpha)) : SK(E(\alpha)) \longrightarrow W^{op}$$

is equal to the composition αSp. From the definition of $Y(d\alpha)$, it follows that, for every cell s of $K(E(\alpha))$,

$$G_s = d\alpha(Sp(s)) = d\alpha(E(\alpha))(s).$$

Now, it is sufficient to apply 3.6. □

3.8 Corollary *Let K, α, $X(\alpha)$, and $Y(d\alpha)$ be the same as in 3.7. Then we have the following:*

(i) *For any functor $U : W(d) \longrightarrow Top$, there is a homotopy equivalence*

$$\operatorname*{hocolim}_{W(d)} U \cong IK \times_{SK_G} U_E$$

where $U_E : SK_G^{op} \longrightarrow Top$ *is the functor such that, for every* $s \in SK$,

$$U_E(s) = E(G_s/d\alpha(s)) \times_{G_s/d\alpha(s)} U(\alpha(s)).$$

(ii) *For any functor* $M : W(d)^{op} \longrightarrow Ab$, *there is a spectral sequence*

$$H^p_G(K, M^q) \Rightarrow \lim_{W(d)}{}^{p+q} M$$

where $M^q : SK_G \longrightarrow Ab$ *is the functor such that, for every* $s \in SK$,

$$M^q(s) = H^q(G_s/d\alpha(s), M(\alpha(s))).$$

Proof. The results of [6] imply that

$$\operatorname*{hocolim}_{W(d)} U \cong E(\alpha) \times_{W(d)} U.$$

Let $K' = K(E(\alpha))$ and $\alpha' = \alpha(E(\alpha))$. From 3.4 we obtain that

$$E(\alpha) \times_{W(d)} U = (IK' \times_{SK'} U\alpha')/G.$$

From the definition of $E(\alpha)$, it follows that, as G-CW-complexes,

$$IK' \times_{SK'} U\alpha' = IK \times_{SK} U'',$$

where, for any cell s of K,

$$U''(s) = E(G_s/d\alpha(s)) \times U(\alpha(s)).$$

Now, we may use the results of Section 2.

(ii) This is a consequence of 3.4 and 1.2. □

3.9 Corollary *Assume that* K *is a* G*-CW-complex such that, for every cell* s *of* K, $K(s)$ *is contractible. Let* $S = SK^{op}$ *and let* $d : S \longrightarrow Sub - G$ *be an admissible* G*-poset map. Then the following is true.*

(i) *For any functor* $U : S(d) \longrightarrow Top$, *there exist homotopy equivalences*

$$\operatorname*{hocolim}_{S(d)} U \cong \operatorname*{hocolim}_{S_G} U_E \cong IK \times_{SK_G} U_E,$$

where

$$U_E(s) = E(G_s/ds) \times_{G_s/ds} U(s).$$

(ii) *For any functor* $M : S(d)^{op} \longrightarrow Ab$, *there is a spectral sequence*

$$H^p_G(K, M^q) \Rightarrow \lim_{S(d)}{}^{p+q} M$$

where

$$M^q(s) = H^q(G_s/ds, M(s)).$$

Proof. This result is a consequence of 3.8 and the results of Section 1. □

3.10 Corollary *Let* W *be a* G*-poset and let* $d : W \longrightarrow Sub-G$ *be an admissible* G*-poset map. Then we have the following:*

(i) *For any functor* $U : W(d) \longrightarrow Top$, *there exist homotopy equivalences*

$$\operatorname*{hocolim}_{W(d)} U \cong \operatorname*{hocolim}_{SW_G} U_E \cong IBW \times_{SW_G} U_E \cong$$

$$\operatorname*{hocolim}_{[\omega] \in SW/G} (E(G_\omega)/dw_0) \times_{G_\omega/dw_0} U(w_0)) \cong$$

$$(\operatorname*{hocolim}_{\omega \in SW} (E(G_\omega)/dw_0) \times U(w_0)))/G.$$

(ii) *For any functor* $M : W(d)^{op} \longrightarrow Ab$, *there is a spectral sequence*

$$E_2^{p,q} \Rightarrow \lim_{S(d)}{}^{p+q} M$$

such that

$$E_2^{p,q} = H^p_G(BW, M^q) = \lim_{[\omega] \in SW/G}{}^{p} H^q(G_\omega/dw_0, M(w_0)).$$

where $M^q : SW \longrightarrow Ab$ *is the functor such that*

$$M^q(\omega) = H^q(G_\omega/dw_0, M(w_0)).$$

Proof. These results follow from 3.8 in the case where $K = BW$ and

$$\alpha : SBW \longrightarrow W^{op}$$

is given by $\alpha(w_0, \ldots, w_n) = w_0$. It is clear that

$$X(\alpha)(w_0, \ldots, w_n) = B(W/w_0),$$

where $W/w_0 = \{w \in W : w \subseteq w_0\}$. This implies that $X(\alpha)$ is locally contractible. Hence the assumptions of 3.8 are fulfilled. □

One can check that the $E_2^{p,q}$-groups of the above spectral sequence are isomorphic to the $E_2^{p,q}$-groups of the spectral sequence described in [7], 17.18 in the case where $\Gamma = W(d)$, M is the constant functor and $N = M$. The spectral sequence from [7] is constructed by homological methods which generalize the construction of the spectral sequence from [10], 1.2. The Corollary 3.10 is proved in [11] using methods of the theory of categories.

3.11 Example Let F be a G-poset of subgroups of G and let d be equal to the identity map. Let

$$EF = \operatorname*{hocolim}_{[\omega] \in SF/G} E((NH_0 \cap \ldots \cap NH_n)/H_0)$$

where $\omega = (H_0, \ldots, H_n)$ and $H_0 \subset H_1 \subset \ldots \subset H_n$ are elements of F. Then the isotropy groups of all points of EF belong to F and EF^H is contractible whenever $H \in F$. Thus, for any functor $M : O_F^{op} \longrightarrow Ab$,

$$H_G^*(EF, M) = \lim_{O_F}{}^* M.$$

Assume that K is a G-CW-complex satisfying the following two conditions:
(i) K^H is R-acyclic whenever $H \in F$
(ii) $G_k \in F$ whenever $k \in K$.

Then $\overline{c}_*(K) \otimes R$ is a projective resolution in the functor category $(O_F^{op}, R - Mod)$. Hence, for any functor $M : O_F^{op} \longrightarrow R - Mod$,

$$H_G^*(K, M) = \lim_{O_F}{}^* M.$$

This implies that there exists a spectral sequence

$$\lim_{[H_0 \subset \ldots \subset H_n] \in SF/G}{}^p H^q((NH_0 \cap \ldots \cap NH_n)/H_0, M(G/H_0)) \Rightarrow H_G^{p+q}(K, M).$$

This spectral sequence is a generalization of the spectral sequence described in [10], 1.2. $\square$

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