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ON COMPLETE SAKS SPACES

by Armin FREI

Résumé. La catégorie $\mathcal{C}$ des espaces de Saks $(X, \|\|, \tau)$ dont la boule unité est τ-complète et des contractions linéaires τ-continues sur OX est symétrique fermée, bicomplète et auto-duale. Elle contient la catégorie Ban_1 comme sous-catégorie pleine codense et la categorie $\mathcal{W}$ des espaces de Saks ayant une boule unité τ-compacte comme sous-catégorie pleine dense.

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1. Introduction.

The category of dual Banach spaces and dual linear contractions is isomorphic to the category $\mathcal{W}$ of Saks spaces having compact unit balls, the isomorphism consisting in adding the w^*-topology to a dual Banach space. We embed the category $\mathcal{W}$ in a complete, cocomplete and self dual category $\mathcal{C}$, the full subcategory of the category $\mathcal{S}$ of Saks spaces consisting of complete Saks spaces, that is of the $\mathcal{S}$-objects having complete unit balls. The category $\mathcal{C}$ is endowed with an internal hom functor, inherited from an internal hom functor on $\mathcal{S}$ and with a tensor product which represents bilinear morphisms and which, together with the internal hom turns $\mathcal{C}$ into a closed category. This will provide a convenient base category for the theory of group representations [4]. In Section 2 we describe briefly the category $\mathcal{S}$ of Saks spaces and introduce its internal hom functor which gives rise to a duality functor on $\mathcal{C}$. In Section 3 we pay special attention to the full subcategories $\mathcal{B}$ and $\mathcal{W}$ of $\mathcal{S}$ consisting respectively of Banach spaces "with the norm taken twice" and of Saks spaces with compact unit ball. Section 4 is devoted to the complete Saks spaces and to those which are reflexive with respect to our duality and to the categories they form, in particular to the category $\mathcal{C}$. In Section 5 we construct the tensor product of $\mathcal{C}$. In [**5**] the Saks spaces are called spaces with mixed topology and in [**1**] an alternative description of the objects in $\mathcal{W}$, there called Waelbroek spaces is given.

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2. The category S of Saks spaces.

The objects of $\mathcal{S}$ are triples $\langle X, \| \|, \tau \rangle$ where $\langle X, \| \| \rangle$ is a Banach space and τ is a locally convex topology on X such that

i) τ is coarser than the $\| \|$-topology,
ii) the unit ball $OX = \{x \in X | \, \|x\| \leq 1\}$ is τ-closed.

In [**2**] and [**5**] the space $\langle X, \| \| \rangle$ is not necessarily complete. By i) the topology τ is defined by a family of seminorms p on X with $p < \| \|$. Axiom ii) is equivalent ([**2**], Lemma 13.1) to either of

ii') $\| \|$ is lower τ-semicontinuous,
ii") $\| \| = \sup\{p | p$ is a τ-continuous seminorm on X with $p \leq \| \|\}$

A morphism $f : \langle X_1, \| \|_1, \tau_1 \rangle \to \langle X_2, \| \|_2, \tau_2 \rangle$ in $\mathcal{S}$ is a contractive linear map whose restriction to OX_1 is τ_1-τ_2-continuous. An isomorphism is an isometry of Banach spaces which is a homeomorphism on the unit balls. We use the term "morphism" for morphisms in $\mathcal{S}$. We denote the objects of $\mathcal{S}$ simply by X, Y, $Z, \ldots$ and speaking of a map $f : X \to Y$ the term "bounded" refers to the norms, while "continuous" and "open" refer to the topology on the ball of the domain. Every topology considered is Hausdorff. By "family of seminorms on X" we always mean a family which defines the topology.

Given a set $\{X_t\}$ of $\mathcal{S}$-objects let ΠX_t denote the subspace of the product of the vectorspaces X_t consisting of those elements with $\|x\| = \sup_t \|x_t\| < \infty$. With this norm and with the topology induced by the product topology, ΠX_t is the product of $\{X_t\}$ in S, ([**5**], 13.7, 13.9). If f, $g : X \to Y$ are morphisms then the kernel $K = \{x \in X | f(x) = g(x)\}$ of $f - g$ is the equalizer in $\mathcal{S}$ of f and g. Thus

Proposition 2.1. *The category $\mathcal{S}$ is complete* ▮

Let $[X, Y]$ denote the vectorspace of all bounded linear maps from X to Y whose restrictions to OX are continuous, endowed with the usual norm $\|f\| = \sup_{x \in OX} \|f(x)\|$ and with the topology of uniform convergence on the compact subsets of OX. Thus the topology on $[X, Y]$ is generated by the family of seminorms $t_{Kq}(f) = \sup_{x \in K} qf(x)$, where q is a seminorm on Y and K a compact subset of OX. One verifies that $[X, Y]$ with the norm and the topology so defined is a Saks space and that

Proposition 2.2. *If $f : X \to Y$ and Z are in $\mathcal{S}$ then the composition maps $[Z,f] : [Z,X] \to [Z,Y]$ and $[f,Z] : [Y,Z] \to [X,Z]$ are morphisms. Thus $[-,-]$ is a bifunctor $\mathcal{S}^{op} \times \mathcal{S} \to \mathcal{S}$* ∎

Any Banach space, in particular the field $\mathbb{C}$, with the norm taken twice is an $\mathcal{S}$-object. Hence $(-)^* = [-,\mathbb{C}]$ is a functor $\mathcal{S}^{op} \to \mathcal{S}$ with $\|f^*\| = \|f\|$. We call X^* and f^* the dual of X and f respectively and draw attention to the fact that as a Banach space X^* is not the usual dual of the Banach space underlying X. Let η_X denote the natural map $X \to X^{**}$ taking $x \in X$ to the evaluation $\langle x, - \rangle$ at x. Each $\langle x, - \rangle$ is linear and bounded and is continuous as $f \to |f(x)|$ is a seminorm on X^*, thus lies in X^{**}. The map η_X is clearly linear and isometric but is not a morphism in general.

3. The full subcategories $\mathcal{B}$ and $\mathcal{W}$ of $\mathcal{S}$.

Let $\mathcal{B}$ be the full subcategory of $\mathcal{S}$ consisting of Banach spaces and $\mathcal{W}$ the one consisting of the objects whose unit ball is compact in the topology. We reserve the capital letters B and W for objects in $\mathcal{B}$ and in $\mathcal{W}$ respectively. One verifies that for an object B the topology on OB^* coincides with the w^*-topology. For an object W the topology on $[W,B]$ is just the norm topology as OW is cofinal in its compact subsets and $[W,B]$ is in $\mathcal{B}$. Thus

Proposition 3.1. *i) Every $[W,B]$, in particular W^*, lies in $\mathcal{B}$.*
ii) Every B^* lies in $\mathcal{W}$ ∎

The considerations above also show that both functors $\mathcal{B}^{op} \to \mathcal{W} \to \mathcal{B}^{op}$ giving the isomorphism $\mathcal{B}^{op} \cong \mathcal{W}$ of [**5**], 13.18 (our $\mathcal{W}$ is denoted A_C^1 there) are restrictions of our $(-)^*$. Thus

Theorem 3.2. *For all objects in $\mathcal{B} \cup \mathcal{W}$ the map $\eta_X : X \to X^{**}$ is an $\mathcal{S}$-isomorphism and $(-)^*$ gives an isomorphism of categories $\mathcal{B}^{op} \cong \mathcal{W}$* ∎

4. Complete objects, reflexive objects.

An object $\langle X, \| \, \|, \tau \rangle$ of $\mathcal{S}$ is said to be: *complete* if OX is complete in the uniformity of τ, *reflexive* if the map $\eta_X : X \to X^{**}$ is an isomorphism. The objects in $\mathcal{B} \cup \mathcal{W}$ are reflexive by Theorem 3.2 and are obviously complete. From the construction of limits as equalizers of morphisms between products we have immediately

Proposition 4.1. *The limit in $\mathcal{S}$ of a diagram consisting of complete objects is complete* ∎

Let X be an object in $\mathcal{S}$ and $\{p_i : X \to \mathbf{R}^+ | \sup p_i \leq \| \ \|\}$ a directed family of seminorms. We denote by B_i the Banach space obtained by the usual completion of the normed space $X/\ker p_i$ and by ψ_i the obvious morphism $X \to B_i$ with $p_i = \|\psi_i\|$. If $p_j \leq p_i$ there is a canonical morphism $\psi_{ij} : B_i \to B_j$ such that $\psi_{ij}\psi_i = \psi_j$. Then $\{\psi_{ij} : B_i \to B_j\}$ is a projective diagram in $\mathcal{S}$ and $\tilde{X} = \varprojlim B_i$ is complete. One verifies that the morphism $e_x : X \to \tilde{X}$ induced by the ψ_i is a dense embedding and that it has the universal property: Given a morphism $f : X \to Z$ with Z complete there is a unique morphism $g : \tilde{X} \to Z$ with $ge_x = f$. This justifies to call the morphism $e_x : X \to \tilde{X}$ the *completion* of X; if X is complete then e_x is clearly an isomorphism. This together with Proposition 4.1 gives (see also [**2**], I.3.8),

Proposition 4.2. *An object X in $\mathcal{S}$ is complete if and only if it is a limit of a diagram in $\mathcal{B}$* ∎

The family $\{\psi_i\}$ of morphisms constructed above has the following useful properties:

Proposition 4.3. *For any object Z in S*

i) *A map $f : Z \to X$ is a morphism if and only if every $\psi_i f$ is a morphism.*
ii) *A bounded map $f : Z \to X$ is in $[Z, X]$ if and only if $\psi_i f$ is in $[Z, B_i]$ for all i* ∎

We call a family $\{\psi_i : X \to B_i\}$ of morphisms so that $\{\|\psi_i\|\}$ is a family of seminorms a *generating family* for X. The family of all morphisms $X \to B$ is clearly generating.

For the sequel we need a few lemmas involving complete objects.

Lemma 4.4.

Let X be a complete object in $\mathcal{S}$ and K a compact subset of OX. There is an open morphism $\theta : W \to X$ with $K \subset \theta(OW)$.

Proof. The closed convex circled hull O of K is complete and precompact, hence compact. Let W be the closed linear span of O with the induced topology and the Minkowski functional $\| \|_O$ as norm. Hence $OW = O$ and O is closed in the topology of W. As $OW \subset OX$ the norm $\| \|_O$ is finer that the one induced from X, hence finer than the topology. By Lemma I.1.2 of **[2]** W is $\| \|_O$-complete. Thus W is an object in $\mathcal{W}$ and the injection $\theta : W \to X$ is contractive, continuos and open as the topology on OW is induced from that on X ∎

Lemma 4.5. *Let X be a complete and Y any object in $\mathcal{S}$. Let $\{\theta_i\}$ the family of all morphisms $W_i \to X$. The topology on $O[X,Y]$ is the initial topology σ defined by the family $\{\theta_i\}$.*

Proof. The "original" topology is clearly finer than σ. On the other hand let q be a seminorm on Y, K a compact set in OX and t_{Kq} the corresponding seminorm on $[W,Y]$. Let θ be as in Lemma 4.4. The map $s : g \mapsto \sup_{w \in OW} qg(w)$ is a seminorm on $[W,Y]$ and $r = s[\theta, Y]$ is a seminorm for σ an $[X,Y]$. Then for f in $O[X,Y]$ one has $\sup_{x \in K} qf(x) \leq \sup_{x \in \theta(OW)} qf(x) = \sup_{w \in OW} qf\theta(w) = r(f)$, hence σ is finer than the original topology on $[X,Y]$ ∎

Taking $Y = \mathbb{C}$ in Lemma 4.5 we obtain that $\{\|\theta_i^*\| : X \to \mathbf{R}^+\}$ is a family of seminorms on X^*. From this and Lemma 4.4 we have

Corollary 4.6. *Let X be a complete object in $\mathcal{S}$. There is a family $\{\theta_i : W_i \to X\}$ of morphisms having the properties:*

i) *$OX = \bigcup_i \theta_i(OW_i)$ and the $\theta_i|_{OW_i}$ are open.*

ii) *The family $\{\theta_i^*\}$ is generating for X^** ∎

We collect a few facts about the canonical map $\eta_X : X \to X^{**}$.

Lemma 4.7. *For every object X in $\mathcal{S}$*

i) *The canonical map η_X is linear isometric (hence injective) and open.*

ii) *The composition $X^* \overset{\eta_{X^*}}{\to} X^{***} \overset{\eta_X^*}{\to} X^*$ is the identity on X^*.*

In general η_X^* does not take values in X^* but it does on $\eta_{X^*}(X^*)$. However if η_X is a morphism or if η_{X^*} is surjective then η_X^* takes values in X^*.

Proof of Lemma 4.7.

i) The map η_X is clearly linear and isometric. To show that it is open let $\psi_i : X \to B_i$ be a generating family. A subbasic neighbourhood U in X is given by $\psi_i^{-1}(V)$ for an ϵ-neighbourhood V in B_i. Now $\eta_X(U) = (\psi_i^{**})^{-1}(\eta_{B_i}(V))$ which is open as η_{B_i} is an isomorphism.

ii) Let g be in X^* and f in X^{**}. Then $(\eta_{X^*}(g))(f) = f(g)$ hence $(\eta_X^*(\eta_{X^*}(g))(x) = (\eta_X(x))(g) = g(x)$ ∎

We are now ready to prove

Proposition 4.8. *If X is a complete object in $\mathcal{S}$ then η_X is surjective.*

Proof. Let the family $\{\theta_i\}$ be as in Corollary 4.6 and let f be an element of X^{**}. As f is continuous there are finitely many seminorms $\|\theta_i^*\|$, $i = 1, 2, \ldots, n$ and $c > 0$ such that for every h in OX^* one has

$$|f(h)| \leq c \max_i \|\theta_i^*(h)\| = c \max_i \|h\theta_i\|$$
$$= c \max_i \sup_{w \in OW_i} |h\theta_i(w)| = c \sup_{x \in K} |h(x)|$$

where $K = \bigcup_i \theta_i(OW_i)$ is compact in OX. Let $\theta : W \to X$ be the morphism corresponding to K as in Lemma 4.4. Then $c \sup_{x \in K} |h(x)| \leq c \sup_{w \in OW} |h\theta(w)| = c\|\theta^*(h)\|$, thus ker $\theta^* \subset$ ker f. Now θ^* maps X^* onto the (not necessarily complete) subspace $X^*/\text{ker}\ \theta^*$ of W^* and $f = X^* \to X^*/\text{ker}\ \theta^* \xrightarrow{t} \mathbb{C}$ where the linear map t is bounded. By the Hahn-Banach Theorem t has an extension to an element of W^{**} also denoted t. Then $f = X^* \xrightarrow{\theta^*} W^* \xrightarrow{t} \mathbb{C}$. Now for g in X^* one has $f(g) = t\theta^*(g) = t(g\theta) = g\theta(\eta_W^{-1}(t)) = g(\theta\eta_W^{-1}(t))$ where $\theta\eta_W^{-1}(t)$ is in X ∎

From the above and Lemma 4.7.i) we have

Corollary 4.9. *If X is a complete object in $\mathcal{S}$ then η_X^{-1} is a bijective isometric morphism* ∎

We now proceed to investigate the relationship between complete and reflexive $\mathcal{S}$-objects and the categories they form.

The full subcategory $\mathcal{R}$ of $\mathcal{S}$ consisting of reflexive objects is self dual. More precisely:

Proposition 4.10. *If X and Y are reflexive objects in $\mathcal{S}$ then the map $\Delta : [X,Y] \to [Y^*, X^*]$ taking f to f^* is a natural isomorphism.*

Proof. The map Δ is clearly linear and isometric; it is surjective as $\Delta\Delta \cong \mathrm{Id}$ by the naturality of η.

To show that it is bicontinuous we may, without loss of generality, show that the map $[X, Y^*] \to [Y, X^*]$ taking f to $f^*\eta_Y$ is bicontinuous. For this let $u_K(g) = \sup_{x \in K} |g(x)|$ and $v_C(h) = \sup_{y \in C} |h(y)|$ be seminorms in X^* and Y^* corresponding to the compact subsets K of OX and C of OY respectively. A seminorm t_{KC} on $[Y, X^*]$ is then given, on $f^*\eta_Y$, by $t_{KC}(f^*\eta_X) = \sup_{y \in C} u_K(f^*\eta_Y(y)) = \sup_{y \in C} \sup_{x \in K} |(\eta_Y(y)(f(x))| = \sup_{x \in K} \sup_{y \in C} |f(x)(y)| = \sup_{x \in K} v_C(f(x))$ where the last term is a seminorm on $[X, Y^*]$ evaluated at f. The naturality in both variables is obvious ∎

Let $\mathcal{C}$ denote the full subcategory of $\mathcal{S}$ consisting of all complete objects. We will show that $\mathcal{C}$ is a subcategory of $\mathcal{R}$ and is closed under the duality functor $(-)^*$, hence is self dual.

Theorem 4.11. *Every complete object in $\mathcal{S}$ is reflexive.*

Proof. If X is complete then η_X is a surjective open linear isometry by Lemma 4.7 i) and Proposition 4.8. In order to show that it is continuous let $\{\theta_i\}$ be the family of Corollary 4.6 i). As η_X and all η_{B_i} are bijective we have that $OX^{**} = \bigcup_i \theta^{**}(OW_i^{**})$. If U is an open subset of OX^{**}, then $U = \bigcup_i \theta_i^{**}((\theta_i^{**})^{-1}(U))$ and $\eta_X^{-1}(U) = \bigcup_i \eta_X^{-1}(\theta^{**}((\theta^{**})^{-1}(U))) = \bigcup_i \theta_i((\theta_i^{**}\eta_{W_i})^{-1}(U))$ which is open as the $\theta_{i|OW_i}$ are open ∎

Thus $\mathcal{C}$ is a subcategory of $\mathcal{R}$. In order to prove that $\mathcal{C}$ is closed under the duality functor $(-)^*$ we need some lemmas.

If X is an $\mathcal{S}$-object for which η_X is surjective, in particular if X is complete, then every morphism $\theta : W = B^* \to X^{**}$ is of the form $\theta = \psi^*$ for a morphism $\psi = X^* \to B$. Indeed one verifies that $\psi = \eta_B^{-1}\theta^*(\eta_X^{-1})^*$ does the trick.

As before we denote by $e_X : X \to \tilde{X}$ the completion of X.

Lemma 4.12. *If X is a complete object of $\mathcal{S}$ then the morphism $e^*_{X^*} : (\tilde{X}^*)^* \to X^{**}$ is an isomorphism.*

Proof. Let $\{\psi_i : X \to B_i\}$ be the family of all such morphisms. Taking the completion of X^* we obtain a diagram

$$\begin{array}{ccc} X^* & \xrightarrow{e_{X^*}} & \tilde{X}^* = \varprojlim B_i \\ & {\scriptstyle\psi_i}\searrow \quad \swarrow{\scriptstyle\lambda_i} & \\ & B_i & \end{array} \qquad \text{and dualizing} \qquad \begin{array}{ccc} (\tilde{X}^*)^* & \xrightarrow{e^*_{X^*}} & X^{**} \\ & {\scriptstyle\lambda_i^*}\nwarrow \quad \nearrow{\scriptstyle\psi_i^*} & \\ & B_i^* & \end{array}$$

As $\mathbb{C}$ is complete, $e^*_{X^*}$ is an isometric bijective morphism; it remains to show that it is open. By Theorem 4.11 the object X^{**} is complete and by the comment preceding this lemma the family $\{\psi_i^*\}$ consists of all morphisms $W \to X^{**}$. Let $\{\theta_j\}$ be the subfamily of $\{\psi_i^*\}$ as in Corollary 4.6 and let U be an open set in $(\tilde{X}^*)^*$. Then $U = \bigcup_j \lambda_j^*((\lambda_j^*)^{-1}(U))$ and $e^*_{X^*}(U) = \bigcup_j e^*_{X^*}\lambda_j^*((\lambda_j)^{-1}(U) = \bigcup_j \theta_j((\lambda_j^*)^{-1}(U))$ which is open as the $\theta_{j|OB^*j}$ are open ∎

The category $\mathcal{C}$ is closed under the functor $(-)^*$, more precisely

Theorem 4.13. *If X is a complete object in $\mathcal{S}$ then also X^* is complete.*

Proof. We consider the commutative diagram

$$\begin{array}{ccc} X^{***} & \xrightarrow{e^{**}_{X^*}} & (\tilde{X}^*)^{**} \\ {\scriptstyle\eta_{X^*}}\uparrow & & \uparrow{\scriptstyle\eta_{\tilde{X}^*}} \\ X^* & \xrightarrow[e_{X^*}]{} & \tilde{X}^* \end{array}$$

where $e^{**}_{X^*}$ is an isomorphism by Lemma 4.12, η^*_X and $\eta_{\tilde{X}^*}$ are isomorphisms by Theorem 4.11 and, by Lemma 4.7 ii) η_{X^*} is an isomorphism. Hence e_{X^*} is an isomorphism ∎

One verifies easily that for every object X in $\mathcal{S}$ the functor $[X, -] : \mathcal{S} \to \mathcal{S}$ commutes with limits. We use this to show that the category $\mathcal{C}$ is self enriched, that is

Theorem 4.14. *If X and Y are complete objects in $\mathcal{S}$ then also $[X, Y]$ is complete.*

Proof. Taking first $X = W$ we have $[W, Y] \cong [W, \varprojlim B_i] \cong \varprojlim [W, B_i]$ where the $[W, B_i]$ are in $\mathcal{B}$ by Proposition 3.1 and $[W, Y]$ is complete by Proposition 4.2. By Theorem 4.11 and Proposition 4.10 this entails that also $[X, B]$ is complete for every B. Finally $[X, Y] \cong [X, \varprojlim B_i] \cong \varprojlim [X, B_i]$ which is complete ∎

From Proposition 4.2 and the discussion preceding it we have that any object X in $\mathcal{C}$ can be represented as $X \cong \varprojlim B_i$ in $\mathcal{C}$ with limiting cone all morphisms $X \to B_i$, thus the subcategory $\mathcal{B}$ in codense in $\mathcal{C}$. The self-duality of $\mathcal{C}$ together with Theorem 4.13 then entails that the subcategory $\mathcal{W}$ is dense in $\mathcal{C}$.

Summarizing this section we have

Theorem 4.15. *The category $\mathcal{C}$ is*

i) *Complete and cocomplete,*
ii) *Self dual,*
iii) *Self enriched, and*
iv) *Contains the full subcategories $\mathcal{W}$ and $\mathcal{B}$ as full dense, respectively codense, subcategories.*

5. The tensor product for $\mathcal{C}$.

In this section we show that the functor $[-, (-)^*]^* : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a tensor product for $\mathcal{C}$. The following lemma will be useful in proving that certain maps are actually morphisms.

Lemma 5.1. *Let X and Y be complete Saks spaces and T a topological space. Then:*

i) *For a map $f : X \to Y$ the restriction $f|_{OX}$ is continuous if and only if $f\sigma_i|_{OW_i}$ is continuous for all morphisms $\sigma_i : W_i \to X$.*

ii) *For a map $g : X \times Y \to T$ the restriction $g|_{OX \times OY}$ is continuous if and only if $g(\sigma_i \times \psi_i)|_{OW_i \times OW_j}$ is continuous for all morphisms $\sigma_i : W_i \to X$ and $\psi_j : W_j \to Y$.*

Proof. i) Let U be an open subset of T and let $V = (f|_{OX})^{-1}(U)$. Let $\{\sigma_i\}$ be the family of morphisms of Corollary 4.6 Then $V = \bigcup_i \sigma_i(\sigma_i^{-1}(V)) = \bigcup_i \sigma_i((f\sigma_i|_{OW})^{-1}(U))$ which is open as the $\sigma_i|_{OW}$ are open. For ii) it suffices to observe that every morphism $\theta : W \to X \times Y$ can be written as $\theta = (p_1\theta \times p_2\theta)\delta$ where $\delta : W \to W \times W$ is the diagonal map and p_1, p_2 are the projections of $X \times Y$ onto its factors ▮

For objects X, Y and Z in $\mathcal{C}$ and a morphism f in $[X, [Y, Z]]$ we denote by $\hat{f}$ the map $Y \to \mathrm{Sets}(X, Z)$ given by $\hat{f}(y)(x) = f(x)(y)$.

Proposition 5.2. *If X, Y and Z are objects in $\mathcal{C}$ then the correspondence $f \to \hat{f}$ is a natural isomorphism $\Phi : [X, [Y, Z]] \to [Y, [X, Z]]$ in $\mathcal{C}$.*

Proof. The map $\hat{f}(y)$ is clearly linear and bounded for every y in Y and its continuity follows from that of f, thus $\hat{f}$ takes values in $[X, Z]$. The map $\hat{f}$ is clearly linear and $\|\hat{f}\| = \|f\|$. In order to show that $\hat{f}$ is continuous let $\{\theta_i : W_i \to X\}$ and $\{\omega_j : W_j \to Y\}$ be the families of all such morphisms. For each pair (i, j) we have two commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & [Y,Z] \\ {\scriptstyle \theta_i}\uparrow & & \downarrow{\scriptstyle [\omega_j,Z]} \\ W_i & \xrightarrow[\ g_{ij}\ ]{} & [W_j,Z] \end{array} \qquad\qquad \begin{array}{ccc} Y & \xrightarrow{\ \hat{f}\ } & [X,Z] \\ \uparrow{\scriptstyle \omega_j} & & \downarrow{\scriptstyle [\theta_i,Z]} \\ W_j & \xrightarrow[\ \hat{g}_{ij}\ ]{} & [W_i,Z] \end{array}$$

where g_{ij} is defined by commutativity. As all OW are compact the continuity of g_{ij} entails that of $\hat{g}_{ij}$, thus the $\hat{g}_{ij}$ are morphisms. Now the $\hat{f}\omega_j$ are continuous by Lemma 4.5 and $\hat{f}$ is continuous by Lemma

5.1.i). Thus the map Φ is well defined; it is clearly linear and bijective. In order to see that Φ is bicontinuous observe that a $[Y,[X,Z]]$-seminorm at $\hat{f}$ is an $[X,[Y,Z]]$-seminorm at f and vice-versa ∎

It is easy to see that $[X,Y^*]^*$ extends to a functor $\mathcal{C}\times\mathcal{C}\to\mathcal{C}$. This functor is symmetric, associative and has the object $\mathbb{C}$ as a unit. More precisely

Proposition 5.3. *There are natural isomorphisms*

$$\Gamma:[X,Y^*]^*\longrightarrow[Y,X^*]^*$$
$$\Omega:[[X,Y^*]^*,Z^*]^*\longrightarrow[X,[Y,Z^*]^{**}]^*$$
$$\Gamma:[X,\mathbb{C}^*]^*\longrightarrow X \;\blacksquare$$

Replacing the symbol $[X,Y^*]^*$ by $X\otimes Y$ the above isomorphisms take the more familiar forms $\Gamma: X\otimes Y\to Y\otimes X$, $\Omega:(X\otimes Y)\otimes Z\to X\otimes(Y\otimes Z)$ and $\Gamma: X\otimes\mathbb{C}\to X$. A tedious verification shows that Γ, Ω and Λ satisfy the coherence axioms [**3**] thus $\mathcal{C}$ is a monaidal category.

Proposition 5.4. *For every object Y in $\mathcal{C}$ the functor $[-,Y^*]^*$ is left adjoint to the functor $[Y,-]$. That is: there is an isomorphism $\Psi:[[X,Y^*]^*,Z]\to[X,[Y,Z]]$ in $\mathcal{C}$ which is natural in every variable.*

Proof. The isomorphism Ψ is given by the composition

$$[[X,Y^*]^*,Z]\xrightarrow{\Delta}[Z^*,[X,Y^*]^{**}]\xrightarrow{[Z^*,\eta^{-1}]}[Z^*,[X,Y^*]]\xrightarrow{\Phi}$$
$$[X,[Z^*,Y^*]]\xrightarrow{[X,\Delta^{-1}]}[X,[Y,Z]]$$

where Φ and Δ are the natural isomorphisms of Propositions 5.2 and 4.10 respectively ∎

With Propositions 5.3 and 5.4 we may add to the list of properties of the category $\mathcal{C}$ given in Theorem 4.15:

Theorem 5.5. *The category $\mathcal{C}$ in symmetric closed.*

Let X, Y and Z be Saks spaces. A *bilinear morphism* is a map $b: X\times Y\to Z$ which is bilinear, contractive and whose restriction to $OX \times OY$ is continuous. We conclude this section by showing that in the category $\mathcal{C}$ the tensor product $[X, Y^*]^*$ represents bilinear morphisms.

Theorem 5.6. *Let X, Y and Z be objects in $\mathcal{C}$. The map $\pi : X \times Y \to [X, Y^*]^*$ given by $\pi(x,y)(u) = u(x)(y)$, $u \in [X, Y^*]$ is a bilinear morphism of norm 1. Composition with π is a norm-preserving one to one correspondence between $[[X, Y^*]^*, Z]$ and the set $[X, Y; Z]$ of bilinear bounded maps $X \times Y \to Z$ whose restriction to $OX \times OY$ is continuous; in particular the morphisms correspond to the bilinear morphisms.*

Proof. The map π in obviously bilinear and of norm ≤ 1. In order to show that it is continuous we first take $X = W_1$ and $Y = W_2$. The object $[W_1, W_2^*]$ is in $\mathcal{B}$ by Proposition 3.1, hence the topology on $O[W_1, W_2^*]^*$ is w^*, which is defined by the seminorms $t_v(f) = |f(v)|$, v in $[W_1, W_2^*]$, f in $O[W_1, W_2^*]^*$. As OW_2 is compact, v, considered as a map $OW_1 \times OW_2 \to \mathbb{C}$ is continuous, hence there are seminorms p on W_1 and q on W_2 with $|\pi(w_1, w_2)(v)| = |v(w_1, w_2)| \leq p(w_1) \cdot q(w_2)$ for all (w_1, w_2) in $OW_1 \times OW_2$. Thus π is continuous. Let now $\{\sigma_i\}$ and $\{\psi_j\}$ be the families of all morphisms $W_i \to X$ and $W_j \to Y$ respectively. One verifies that the diagram

$$\begin{array}{ccc} [W_i, W_j^*]^* & \xrightarrow{[\sigma_i, \psi_j^*]^*} & [X, Y^*]^* \\ {\scriptstyle \pi_{ij}} \uparrow & & \uparrow {\scriptstyle \pi} \\ W_i \times W_j & \xrightarrow{\sigma_i \times \psi_j} & X \times Y \end{array}$$

commutes for all (i,j). The maps $[\sigma_i, \psi_j^*]^* \pi_{ij}$ are continuous and by Lemma 5.1 ii) also π is continuous.

We now consider the diagram

$$\begin{array}{ccc} [[X, Y^*]^*, Z] & \xrightarrow{\quad \Psi \quad} & [X[Y, Z]] \\ & {\scriptstyle \alpha} \searrow \quad \swarrow {\scriptstyle \beta} & \\ & [X, Y; Z] & \end{array}$$

where Ψ is the hom-tensor adjunction of Proposition 5.4, α is the composition with π and β takes an element f in $[X[Y,Z]]$ to $\hat{f}$ given by $\hat{f}(x,y) = f(x)(y)$. One verifies that the above diagram commutes. Let $\bar{b}$ be an element of $[[X,Y^*]^*,Z]$. Then $\alpha(\bar{b})$ is in $[X,Y;Z]$ as $\pi|_{OX\times OY}$ is continuous. On the other hand if b is in $[X,Y;Z]$, then $\beta^{-1}(b)$ is continuous, thus in $[X,[Y,Z]]$. Indeed if $\{p\}$, $\{q\}$ and $\{r\}$ are the families of seminorms on X, Y and Z respectively, the topology on $[Y,Z]$ is defined by the seminorms $t_{Cr}(v) = \sup_{y\in C} rv(y)$, C compact in OY, v in $[Y,Z]$. As $b|_{OX\times OY}$ is continuous there are seminorms p and q on X and Y with $rb(x,y) \leq p(x)\cdot q(y)$ for all (x,y) in $OX\times OY$, hence $t_{Cr}(\beta^{-1}(b))(x) = \sup_{y\in C} rb(x,y) \leq p(x)\cdot \max_{y\in C} q(y)$ for all x in OX. Thus β is a bijection from $[X[Y,Z]]$ onto $[X,Y;Z]$ and it is clearly norm preserving. As Ψ is an isomorphism it follows that the map α is a norm preserving bijection from $[[X,Y^*]^*,Z]$ onto $[X,Y;Z]$.

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