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STRONG, REGULAR AND DENSE GENERATORS

by Reinhard BÖRGER and Walter THOLEN *

RÉSUMÉ. L'article développe une étude approfondie des générateurs en termes de familles "épics" et de leur hom-foncteurs (généralisés) induits, en insistant sur les différences subtiles entre les notions de générateur régulier et dense. Les applications données concernent des sous-catégories qui contiennent un générateur de leur supercatégorie, et une caractérisation des catégories où tout objet est coproduit d'objets dont les représentables préservent les coproduits.

Abstract

A comprehensive study of generators in terms of epic families as well as of their induced (generalized) hom-functors is given, with special emphasis on the subtle differences between the notions of regular and dense generator. Applications concern subcategories which contain a generator of their supercategory, and a characterization of categories with the property that every object is a coproduct of objects whose representables preserve coproducts.

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Introduction

The aim of this paper is to give a short survey of the notions occuring in the title, the subtle differences of which are a likely source of misunderstandings or even errors. Although we first present these notions colimit-free, most of the times we assume the existence of (sufficiently many) coproducts in order to be able to apply adjoint functor techniques. This way one arrives at various refined or simplified results involving generators.

A class $\mathcal{G}$ of objects in a category $\mathcal{A}$ is (extremally) generating iff the family $\mathcal{A}(\mathcal{G},A)$ of all morphisms $G \to A$, $G \in \mathcal{G}$, is (extremally) generating for all $A \in |\mathcal{A}|$. Equivalently one may say that the induced generalized hom-functor $U_{\mathcal{G}} : \mathcal{A} \to \mathrm{Set}^{\mathcal{G}}$ [=discrete power] is faithful (*and* conservative) or, if $\mathcal{G}$ is small and $\mathcal{A}$ has coproducts so that $U_{\mathcal{G}}$ has a left adjoint, that the co-units are (extremely) epic. Which are the correspoinding facts describing regular and dense generators? Clearly, the SGA4 notion of *strictly epimorphic family* is the natural family-extension of the notion of *regular epimorphism* as used by Kelly [18] and Gabriel and Ulmer [12]. However, to say that $\mathcal{A}(\mathcal{G},A)$ is strictly generating for every A is a strictly stronger statement than to say that all co-units are regular epimorphisms; whereas the latter statement was used in [12] to introduce the notion of regular generator, it seems little if at all known that the former statement means in fact density of the full subcategory $\bar{\mathcal{G}}$ with object-class $\mathcal{G}$ (cf. 1.4 below); we call $\mathcal{G}$ a dense generator of $\mathcal{A}$ in this case.

Another way of distinguishing the various notions of generator is to look at an *arbitrary* right adjoint functor $U : \mathcal{A} \to \mathcal{X}$ with left adjoint F . To say that the co-units are (extremally) epic means that $\mathcal{F} = \{FX | X \in |\mathcal{X}|\}$ is (extremally) generating in $\mathcal{A}$; but to say that the co-units are regularly epic means that $\mathcal{F}$ is dense in $\mathcal{A}$, whereas density of the functor $F : \mathcal{X} \to \mathcal{A}$ means that the co-units are isomorphisms, i.e. U is full and faithful (cf. 2.1). When applying this to $U_{\mathcal{G}}$ one arrives immediately at statements relating properties of the small set $\mathcal{G}$ to properties of the class $\mathcal{F} = \sqcup\mathcal{G}$ of all (small) coproducts of objects in $\mathcal{G}$, in particular: $\mathcal{G}$ is a regular generator of $\mathcal{A}$ iff $\sqcup\mathcal{G}$ is dense in $\mathcal{A}$ (cf. 3.5).

Our careful analysis of notions makes it easy to derive criteria for regular generators to be dense. We give simplified and partly strengthened versions of earlier results due to Gabriel and Ulmer [12] and Street [25], based on universality of coproducts or their preservation by $U_{\mathcal{G}}$ (cf. 4.2 and 5.2). Special attention is paid to the question of what it means that the class $\mathcal{Q}$ of *coprime objects* , i.e. objects G such that $\mathcal{A}(G,-) : \mathcal{A} \to \mathrm{Set}$ preserves coproducts for a category $\mathcal{A}$ with coproducts, gives a dense full subcategory (cf. 5.5). Here density of $\mathcal{Q}$ and of $\sqcup\mathcal{Q}$ *are* equivalent, a phenomenon we also observe in an entirely different situation: for an object A in a category $\mathcal{A}$ with binary products, we study "generating properties" of the class $\downarrow A$ of all objects D with $\mathcal{A}(D,A) \neq \emptyset$, which can be equivalently described by "epic properties" of the product projections $A \times B \to B$ (cf. 7.2). This also leads to criteria for the functor $A \times - : \mathcal{A} \to \mathcal{A}$ to be conservative.

Finally we look at a subcategory $\mathcal{B}$ which contains a strong or regular generator of its supercategory $\mathcal{A}$. Under suitable conditions on $\mathcal{A}$ we prove that completeness of $\mathcal{B}$ yields limit-preservation of $\mathcal{B} \hookrightarrow \mathcal{A}$, whereas cocompleteness of $\mathcal{B}$ implies its reflecivity (cf. 6.3 and 6.5). These results extend known properties of the category of compact Hausdorff spaces.

For various other aspects of generators we refer the reader to other recent papers. In [8] we characterize generators in terms of colimit-closures, emphasizing various set-theoretic subtleties involved; in [7] and [3] (as well as in [8]) criteria for totality involving generators and characterizations for total categories with generators are given. We also refer the reader to [23] where, under the set-theoretic *Vopěnka Principle* , the existence of a dense generator is derived from the existence of a (strong) generator with the property that every object is a colimit of generating objects.

We thank Bob Paré for drawing our attention to the notion of strictly epic family in SGA4.

1 The family approach to strong and dense generators

1.1 A family $(e_i : A_i \to B)_{i \in I}$ of morphisms in a category $\mathcal{A}$ with common codomain B is called *epic* if, for $u, v : B \to C$, $e_i u = e_i v$ for all $i \in I$, implies $u = v$; the family is *extremally epic* if, in addition, the e_i can factor jointly through a monomorphism $m : C \to B$ only if m is an isomorphism. Note that there is no restriction on the size of I : one may have $I = \emptyset$ in which case the family consists just of the object B , or I may be a proper class. If the coproduct $\sqcup A_i$ exists in $\mathcal{A}$, one has

$$(e_i)_{i \in I} \text{ (extremally) epic } \Leftrightarrow e : \sqcup A_i \to B \text{ (extremally) epic ;} \tag{1}$$

implication " $\Rightarrow$ " is actually a special case of a more general cancellation property for (extremally) epic families. Furthermore, "(extremally) epic" is a property under family expansion, that is: for $K \subseteq I$ one has

$$(e_i)_{i \in K} \text{ (extremally) epic } \Rightarrow (e_i)_{i \in I} \text{ (extremally) epic .} \tag{2}$$

In case $\mathcal{A}$ has pullbacks (of monomorphisms), "extremally epic" coincides with "strongly epic" (as used by Street [25]).

1.2 A family $(e_i : A_i \to B)_{i \in I}$ is called *strictly epic* (cf. [4]) if the e_i's form a colimit-cocone of the diagram given by all morphisms $x : D \to A_i$, $y : D \to A_j$ with $e_i x = e_j y$ for some $i, j \in I$; that means: given any family $(f_i : A_i \to C)_{i \in I}$ such that $e_i x = e_j y$ always implies $f_i x = f_j x$, then there is exactly one morphism $g : B \to C$ with $g e_i = f_i$ for all $i \in I$. Strictly epic morphisms (= singleton families) are also called *regular epimorphisms* (cf. Kelly [18], Gabriel-Ulmer [12]); a regular epimorphism is the coequalizer of its kernel pair if the latter exists. Considering (1) now, only

$$(e_i)_{i \in I} \text{ strictly epic } \Rightarrow e : \sqcup A_i \to B \text{ strictly epic} \tag{3}$$

is true in general, but " $\Leftarrow$ " not so: for X an infinite compact Hausdorff space, the family $(x : 1 \to X)_{x\in X}$ is not strictly epic in CompHaus, but the induced $\amalg_{x\in X} 1 \cong \beta(X_{\text{discrete}}) \to X$ is. Furthermore, (2) no longer holds for strictly epic families: in the category Cat of small categories, with $\mathbf{2} = \{0 \to 2\}$ and $\mathbf{3} = \{0 \to 1 \to 2\}$, let $E_0 : \mathbf{2} \to \mathbf{3}$ be the embedding, and let $E_1 : \mathbf{2} \amalg \mathbf{2} \to \mathbf{3}$ be the obvious regular epimorphism; then the induced $E : \mathbf{2} \amalg \mathbf{2} \amalg \mathbf{2} \to \mathbf{3}$ is no longer a regular epimorphism, hence (E_0, E_1) is not strictly epic. This example also shows that in Cat regular epimorphisms are not (right) cancellable (for another category with this property, see [18]). Finally we notice that one has the implications

$$\text{strictly epic} \Rightarrow \text{extremally epic} \Rightarrow \text{epic} \tag{4}$$

for families, none of which is an equivalence in general, even if one fixes the indexing system I.

1.3 A class $\mathcal{G}$ of objects in $\mathcal{A}$ is called (*extremally; densely* resp.) *generating* if, for every $A \in |\mathcal{A}|$, the family

$$\mathcal{A}(\mathcal{G}, A) = \bigcup_{G\in\mathcal{G}} \mathcal{A}(G, A)$$

is (extremally; strictly resp.) epic; in case $\mathcal{G}$ is small, we call $\mathcal{G}$ an (*extremal; dense* resp.) *generator* of $\mathcal{A}$. Generators are often called *separators*, and extremal generators are usually called *strong generators*, in defiance of the slight difference of notions of extremally and strongly epic family (see 1.1). Because of (2), a class $\mathcal{G}$ is (extremally) generating iff, for every $A \in |\mathcal{A}|$, there is *some* (extremally) epic family $e_i : G_i \to A)_{i\in I}$ with all $G_i \in \mathcal{G}$.

The notions of densely generating class and dense generator are justified by the following:

1.4 Proposition *A class $\mathcal{G}$ of objects in $\mathcal{A}$ is densely generating iff the full embedding $J : \bar{\mathcal{G}} \hookrightarrow \mathcal{A}$ with $|\bar{\mathcal{G}}| = \mathcal{G}$ is a dense functor.*

PROOF: For $\mathcal{G}$ densely generating, we first show that $\bar{\mathcal{G}}$ is dense in $\mathcal{A}$, i.e. that the cocone given by the comma category J/A (whose class of objects is $\mathcal{A}(\mathcal{G}, A)$) is a colimit-cocone for every $A \in |\mathcal{A}|$. This amounts to showing that a family $(f_h)_{h\in\mathcal{A}(\mathcal{G},A)}$ with common codomain that satisfies

$$h\,,\ hu \in \mathcal{A}(\mathcal{G}, A) \Rightarrow f_{hu} = f_h u \tag{5}$$

also satisfies

$$hx = h'y \text{ with } h\,,\ h' \in \mathcal{A}(\mathcal{G}, A) \Rightarrow f_h x = f_{h'} y\,. \tag{6}$$

Indeed, given the hypothesis of (6), in order show to $f_h x = f_{h'} y$ it suffices to show $f_h xz = f_{h'} yz$ for all morphisms z with domain in $\mathcal{G}$, since the densely generating class $\mathcal{G}$ is in particular generating. But (5) gives instantly

$$f_h xz = f_{hxz} = f_{h'yz} = f_{h'} yz\,.$$

Hence (5) $\Rightarrow$ (6). Trivially (6) $\Rightarrow$ (5), which shows the other part of the Proposition. □

2 Remarks on adjoint functors

2.1 Proposition *In each* (1), (2), *and* (3), *below the given statements* (i), (ii), *etc. are equivalent for a functor* $U : \mathcal{A} \to \mathcal{X}$ *with left adjoint* F *and co-unit* $\varepsilon : FU \to 1$ *:*

(1) (i) U *is faithful (and conservative, i.e. reflects isomorphisms),*

(ii) U *reflects (extremal) epimorphisms,*

(iii) *every* ε_A *is an (extremal) epimorphism,*

(iv) $\{FX|X \in \mathcal{H}\}$ *is (extremally) generating in* $\mathcal{A}$ *, for every (extremally) generating class* $\mathcal{H}$ *in* $\mathcal{X}$ *.*

(v) $\{FX|X \in |\mathcal{X}|\}$ *is (extremally) generating in* $\mathcal{A}$ *.*

(2) (i) U *has a full and faithful comparison functor into its Eilenberg-Moore category,*

(ii) U *reflects split epimorphisms into regular epimorphisms,*

(iii) *every* ε_A *is a regular epimorphism,*

(iv) $\{FX|X \in |\mathcal{X}|\}$ *is densely generating in* $\mathcal{A}$ *.*

(3) (i) U *is full and faithfull,*

(ii) *every* ε_A *is an isomorphism,*

(iii) F *is a dense functor.*

PROOF: (1) is easy and (essentially) known (see in particular [21] and [2], 1.5), so we can omit the proof here. For (2) (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii), see [24], Prop. 21.4.6; note in addition that (iii) is equivalent to the statement that, for very $A \in |\mathcal{A}|$,

$$FUFUA \underset{\varepsilon_{FUA}}{\overset{FU\varepsilon_A}{\rightrightarrows}} FUA \xrightarrow{\varepsilon_A} A \qquad (7)$$

is a coequalizer. We show (2) (iii) $\Rightarrow$ (iv):

Let $\mathcal{G} = \{FX|X \in \mathcal{H}\}$, and consider a family (f_h) with property (6) of 1.4. Then, with $f = f_{\varepsilon_A}$, one has $fx = fy$ whenever $\varepsilon_A x = \varepsilon_A y$, hence $f = g\varepsilon_A$ for a unique g . With $h^\# : X \to UA$ corresponding to $h : FX \to A$ by adjunction, one also has $gh = g\varepsilon_A \cdot Fh^\# = f_{\varepsilon_A} \cdot Fh^\# = f_{\varepsilon_A \cdot Fh^\#} = f_h$, by (5) of 1.4.

(2) (iv) $\Rightarrow$ (i): We must show that every $x : UA \to UB$ with $U\varepsilon_B \cdot UFx = x \cdot U\varepsilon_A$ is of the form $x = Ug$ for a unique $g : A \to B$ in $\mathcal{A}$. But with

$f_h := \varepsilon_B \cdot Fx \cdot FUh \cdot F\eta_X$ for every $h : FX \to A$ (with $X = X_h$ chosen for every $h \in \mathcal{A}(\mathcal{G}, A)$) , one has

$$Uf_h = x \cdot U\varepsilon_A \cdot UFUh \cdot UF\eta_X = x \cdot Uh \cdot U\varepsilon_{FX} \cdot UF\eta_X = x \cdot Uh$$

(independently of the choice of X). This implies (5) of 1.4 since, by part (1), U is faithful. Hence $f_h = gh$ with a unique g , in particular $g\varepsilon_A = f_{\varepsilon_A} = \varepsilon_B \cdot Fx$. Therefore $Ug = x$.

(3) is (partly) known, and the proof is left to the reader, as it will not be used in any theorems below. □

2.2 Corollary *For a dense functor $F : \mathcal{X} \to \mathcal{A}$ with a right adjoint, all co-units are regular epimorphisms, and $\{FX | X \in |\mathcal{X}|\}$ is densely generating in $\mathcal{A}$.*

PROOF: (3) (i) implies (2)(i) of Proposition 2.1.

2.3 Remarks

(1) For a left adjoint functor $F : \mathcal{X} \to \mathcal{A}$ with regularly epic co-units, and for $\mathcal{H}$ densely generating in $\mathcal{X}$, $\{FX \mid X \in \mathcal{H}\}$ need not be densely generating in $\mathcal{A}$, as the free group functor shows. The same example shows that, if $\{FX|X \in |\mathcal{X}|\}$ is densely generating in $\mathcal{A}$, the left adjoint F need not be dense (apply 2.1 (3)). Quite trivially one has, however, that any full functor F is dense as soon as $\{FX|X] \in |\mathcal{X}|\}$ is densely generating in $\mathcal{A}$.

(2) If a composition $F = GH$ (of functors) with G full and faithful is dense, both factors G and H must be dense too (cf. [20], Thm. 5.13). Conversely, if G and H are dense, F need not be dense, even when both G and H are full and faithfull; here is an easier example than the one due to Isbell (as mentioned in [12], 3.7): the empty category $\emptyset$ is dense in the terminal category $\{1\}$ which is densily embedded into **Set** , but $\emptyset$ is, of course, not dense in **Set** .

(3) It is known that, for $H \dashv F \dashv U$, one has U full and faithful if and only if H is full and faithful (cf. [11]). Hence 2.1 (3) and its dual give that *a functor with both a left and a right adjoint is dense iff it is condense* .

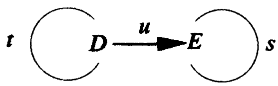
(4) Property (ii) of 2.1 (1) can be equivalently formulated as " U reflects split-epimorphisms into (extremal) epimorphisms", but in 2.1 (2) one may *not* say " U reflects regular epimorphisms" as the (monadic) functor U : **Cat** $\to$ **Gra** (directed graphs) shows.

(5) Schubert [24, Proposition 21.4.6] claims that, if A has coequalizers, one may add in 2.1 (2) the condition that U be conservative as another equivalent condition; however, U : **Cat** $\to$ **Set** shows that this is not true. Therefore, the equivalent statements of 2.1 (1) in the extremal case do not imply those of 2.1 (2).

(6) Im and Kelly [16], who state the equivalence (1) (i) $\Leftrightarrow$ (iii) of 2.1 in the extremal case only under additional assumptions on $\mathcal{A}$, mention the problem

whether a conservative functor $U : \mathcal{A} \to \mathcal{X}$ with left adjoint is necessarily faithful (cf. [16, Remark 4.4]). Certainly, one easily shows that this is true if $\mathcal{A}$ has equalizers. However, without any condition on $\mathcal{A}$ the assertion is false even when we have $\mathcal{X} = \mathsf{Set}$.

2.4 Example (*A conservative but non-faithful functor* $U : \mathcal{A} \to \mathsf{Set}$ *with left adjoint*). We first consider the finite category $\mathcal{D}$ with diagram sheme



and $su = ut = u$, $t^2 = 1_D$, and $s^2 = 1_E$. Let $\mathcal{A}$ be the formal coproduct completion of $\mathcal{D}$; so objects of $\mathcal{A}$ are small families $(A_i)_I$ of $\mathcal{D}$-objects, and an $\mathcal{A}$-morphism $f : (A_i)_I \to (B_j)_J$ is given by a family $(j_i)_I$ of elements in J and a family $(f_i : A_i \to B_{j_i})_I$ of $\mathcal{D}$-morphisms; composition is as in $\mathcal{D}$. Obviously, $\mathcal{D}$ is fully embedded into $\mathcal{A}$ in terms of "singleton families". The category $\mathcal{A}$ does have coproducts now. Therefore the representable functor

$$U = \mathcal{A}(D, -) : \mathcal{A} \to \mathsf{Set}$$

has a left adjoint (given by copowers of D in $\mathcal{A}$). U is not faithful since

$$Us : UE \to UE = \mathcal{A}(D, E) = \mathcal{D}(D, E) = \{u\}$$

is the identity map, so $Us = U1_E$, but $s \neq 1_E$.

In order to show that U is conservative, let $f : A \to B$ be a morphism in $\mathcal{A}$, given by families $(j_i)_I$ and $(f_i : A_i \to B_{j_i})_I$ as above, such that

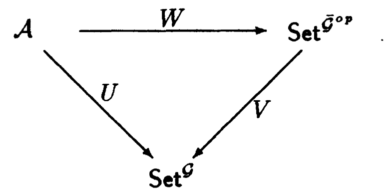
$$Uf : \mathcal{A}(D, (A_i)_I) \to \mathcal{A}(D, (B_j)_J)$$

is an isomorphism in Set . We claim that every f_i must be an isomorphism since, otherwise, we had $f_i = u$ for some $i \in I$, hence $fmt = fm$ for the coproduct injection $m : A_i \to A$, but $mt \neq m$; this would contradict the injectivity of Uf . So we must only show that the map $I \to J$, $i \mapsto j_i$, is bijective.

For $j \in J$, let $n : B_j \to B$ be the coproduct injection, and consider first the case $B_j = E$. Since Uf is bijective, one has a uniquely determined $\mathcal{A}$-morphism $x : D \to A$ with $fx = nu$, that is a pair (i, a) with $a : D \to A_i$ in $\mathcal{D}$ and $j_i = j$; since $f_i \neq u$, we have $A_i = B_j = E$, and a must necessarily be u . Hence i is uniquely determined by j . Similarly, in case $B_j = D$, one has a unique $\mathcal{A}$-morphism $y : D \to A$ with $fy = n$, hence a pair (i, b) with $b : D \to A_i$ in $\mathcal{D}$, $j_i = j$ and $f_i b = 1_D$ (in particular $A_i = B_j = D$ by the design of $\mathcal{D}$). So necessarily b must be f_i^{-1} , and it follows again that i is uniquely determined by j . Therefore f is an isomorphism in $\mathcal{A}$. □

3 The functorial description of strong, regular, and dense generators

3.1 Let $\mathcal{G}$ be a class of objects in the category $\mathcal{A}$, and let $\bar{\mathcal{G}}$ be the corresponding full subcategory of $\mathcal{A}$. We form the (possibly illegitimate) functor category $\mathbf{Set}^{\bar{\mathcal{G}}^{op}}$, whereas $\mathbf{Set}^{\mathcal{G}}$ is the product of $\mathcal{G}$-many copies of $\mathbf{Set}$. There is a restriction functor $V : \mathbf{Set}^{\bar{\mathcal{G}}^{op}} \to \mathbf{Set}^{\mathcal{G}}$ which is faithful and conservative and preserves and reflects (regular) epimorphisms; if the needed left Kan-extensions exist, then V has also a left adjoint and is then monadic. With the full embedding $J : \bar{\mathcal{G}} \hookrightarrow \mathcal{A}$, let $W = W_{\mathcal{G}} : \mathcal{A} \to \mathbf{Set}^{\bar{\mathcal{G}}^{op}}$ be the functor with $A \mapsto \mathcal{A}(J-, A)$, and put $U_{\mathcal{G}} := U := VW$,



i.e. $UA = (\mathcal{A}(G,A))_{G\in\mathcal{G}}$. In consideration of 1.4, the following is well-known:

3.2 Proposition (1) *$\mathcal{G}$ is (extremally) generating in $\mathcal{A}$ iff $U_{\mathcal{G}}$ or, equivalently, $W_{\mathcal{G}}$ is faithful (and conservative, i.e. reflects isomorphisms).*
(2) *$\mathcal{G}$ is densely generating in $\mathcal{A}$ iff $W_{\mathcal{G}}$ is full and faithfull.* □

3.3 The previous Proposition makes it look natural to consider the case that $U_{\mathcal{G}} = U$ is faithful and reflects regular epimorphisms; we call $\mathcal{G}$ a *regulary generating class* then, and a *regular generator* in case $\mathcal{G}$ is small. Hence regularity of a generating class $\mathcal{G}$ means that $f : A \to B$ in $\mathcal{A}$ is a regular epimorphism whenever $G \in \mathcal{G}$ is projective with respect to f .

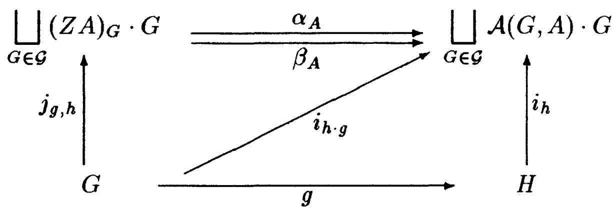
3.4 The functor $U_{\mathcal{G}} = U$ has a left adjoint F iff all coproducts

$$FX = \coprod_{G\in\mathcal{G}} X_G \cdot G \tag{8}$$

with $X = (X_G)_{G\in\mathcal{G}} \in \mathbf{Set}^{\mathcal{G}}$ exist; here $X_G \cdot G$ denotes the coproduct of X_G -many copies of G in $\mathcal{A}$. All properties of $\mathcal{G}$ considered before can be expressed in terms of the co-units $\varepsilon_A : FUA \to A$, $A \in |\mathcal{A}|$, which satisfy the identities $\varepsilon_A \cdot i_h = h$ for all $h \in \mathcal{A}(\mathcal{G}, A)$. In order to do that in the case of density, it is convenient to consider first the functor $Z : \mathcal{A} \to \mathbf{Set}^{\mathcal{G}}$, $A \mapsto ((ZA)_G)_{G\in\mathcal{G}}$ with $(ZA)_G = \coprod_{H\in\mathcal{G}} \mathcal{A}(G,H)\times\mathcal{A}(H,A)$, and the natural transformations $\alpha, \beta : FZ \to FU$ which are defined by

$$\alpha_A \cdot j_{g,h} = i_h \cdot g \quad \text{and} \quad \beta_A \cdot j_{g,h} = i_{hg} \tag{9}$$

for all $g \in \mathcal{A}(G,H), h \in \mathcal{A}(H,A),\ \ G,H \in \mathcal{G}$ (with $j_{g,h}$ and i_h canonical injections).



(10)

One easily checks that $\varepsilon\alpha = \varepsilon\beta$ holds (see also 5.1 below).

3.5 Theorem *For a small set $\mathcal{G}$ of objects in a category $\mathcal{A}$ with coproducts (of type* (8)*), in each* (1), (2), *and* (3) *below the given statements* (i), (ii), *etc. are equivalent:*

(1) (i) *$\mathcal{G}$ is a (strong) generator of $\mathcal{A}$,*

(ii) *$U_{\mathcal{G}}$ reflects (extremal) epimorphisms,*

(iii) *every ε_A is an (extremal) epimorphism,*

(iv) *the class $\sqcup\mathcal{G}$ of all small-indexed coproducts of objects in $\mathcal{G}$ is (extremally) generating in $\mathcal{A}$.*

(2) (i) *$\mathcal{G}$ is a regular generator of $\mathcal{A}$,*

(ii) *$U_{\mathcal{G}}$ reflects regular epimorphisms,*

(iii) *every ε_A is a regular epimorphism,*

(iv) *the class $\sqcup\mathcal{G}$ is densely generating in $\mathcal{A}$.*

(3) (i) *$\mathcal{G}$ is a dense generator of $\mathcal{A}$,*

(ii) *every ε_A is a coequalizer of α_A and β_A.*

PROOF: (1) and (2) follow from 2.1; for (2) (ii) note that regular epimorphisms in $\mathbf{Set}^{\mathcal{G}}$ split. (3) By the canonical construction of a colimit in terms of coproducts and coequalizers, (3) (ii) is equivalent to saying that the canonical cocone, given by the comma category J/A, with $J : \bar{\mathcal{G}} \hookrightarrow \mathcal{A}$, is a colimit, which is (3) (i). [Warning: 3.5 (3) does not follow from 2.1 (3).]

3.6 Remarks

(1) In 3.5 (1) one may equivalently say that every $A \in |\mathcal{A}|$ is an (extremal) quotient of an object in $\sqcup\mathcal{G}$. In 3.5 (2), however, to say that every A is a regular quotient of an object in $\sqcup\mathcal{G}$ is strictly weaker than the other statements, provided the negation of the set-theoretic Vopěnka Principle holds: cf. [1].

(2) In the presence of the coproducts (8) one may leave off the requirement that $U_{\mathcal{G}}$ be faithful in Definition 3.3 of a regular generator. However, in the characterization 3.2 of an extremal generator, the requirement that $U_{\mathcal{G}}$ be faithful is essential also in the presence of coproducts: in Example 2.4 we have seen that $U = U_{\{D\}} : \mathcal{A} \to \mathbf{Set}$ is conservative, but not faithful. These precautions are, of course, not needed when $\mathcal{A}$ has equalizers.

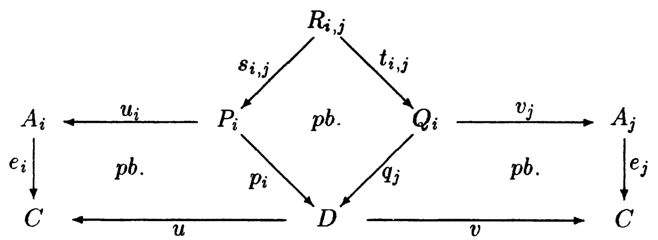
(3) Comparing (2) (iv) and (3) (i) as well as (2) (iii) and (3) (ii) in 3.5 shows the subtlety in difference between regularity and density, in particular in view of the fact that for a regular generator one does have the coequalizer representation (7) of 2.1 for ε_A . The equivalence (2) (iii) $\Leftrightarrow$ (iv) was first established by Kelly ([19], 5.24).

(4) None of the implications dense generator $\Rightarrow$ regular $\Rightarrow$ strong generator $\Rightarrow$ generator is a logical equivalence in general: the singleton space is a generator in **Top** but not strong; it is so in **CompHaus**, even regular, but not dense; **2** is a strong generator in **Cat**, but not regular (cf. 1.2). **Top** has in fact no strong generator (cf. [12], Staz 4.17), and **CompHaus** has no dense generator (cf. [19], 5.3); in [7], 5.6(3), we exhibit an example of a complete and cocomplete category with a strong generator, but without a regular one.

(5) In view of the fact that, for a left adjoint $F : \mathcal{X} \to \mathcal{A}$ with (extremally) epic counits, and for $\mathcal{H}$ (extremally) generating in $\mathcal{X}$, also $\{FX \mid X \in \mathcal{H}\}$ is (extremally) generating in $\mathcal{A}$ (cf. 2.1 (1)), one may ask whether the same holds true if "extremally" is replaced by "regularly". The left adjoint of U : **Cat** $\to$ **Gra** (cf. 2.3 (4)) shows that this is not the case.

4 Density of regular generators - by universality of coproducts

4.1 For a family $(e_i : A_i \to C)_{i \in I}$ of morphism in $\mathcal{A}$, and for a pair of morphisms $u, v : D \to C$, we consider the following pullback construction:



(11)

The family $(p_i)_{i \in I}$ is the *pullback of* $(e_i)_{i \in I}$ *along* v , and $(r_{i,j})_{i,j \in I}$ with $r_{i,j} = p_i s_{i,j} = q_j t_{i,j}$ is the *pullback of* $(e_i)_{i \in I}$ along (u, v) . It is easy to check that pullbacks of $(e_i)_{i \in I}$ along morphisms are epic if and only if pullbacks of $(e_i)_{i \in I}$

along pairs of morphisms are epic. *Pullbacks of a coproduct* are to be understood as pullbacks of the family of coproduct-injections.

4.2 Theorem *Let $\mathcal{G}$ be a regular generator of a category $\mathcal{A}$ with (small-indexed) coproducts of objects in $\mathcal{G}$. If pullbacks of these coproducts exist and are epic, then $\mathcal{G}$ is a dense generator of $\mathcal{A}$.*

PROOF: We must show that the family $\mathcal{A}(\mathcal{G}, A)$ is strictly epic for every $A \in |\mathcal{A}|$. So we consider a family $(f_h : G_h \to B)_{h \in \mathcal{A}(\mathcal{G},A)}$ that satisfies condition (6) of 1.4, and must only show that, in the notation of 3.4, the induced $f : FUA \to B$, satisfies $(\varepsilon_A u = \varepsilon_A v \Rightarrow fu = fv)$. So let $u, v : D \to FUA$ be given with $\varepsilon_A u = \varepsilon_A v$, and form the publlbacks (11) with (e_i) the family of coproduct injections $(i_h : G_h \to FUA)_{h \in \mathcal{A}(\mathcal{G},A)}$. Putting, with the notations of (11), $u_{h,h'} := u_h \cdot s_{h,h'}$, $v_{h,h'} := v_{h'} \cdot t_{h,h'}$ for $h, h' \in \mathcal{A}(\mathcal{G}, A)$, one has

$$h \cdot u_{h,h'} = \varepsilon_A \cdot u \cdot r_{h,h'} = \varepsilon_A \cdot v \cdot r_{h,h'} = h' \cdot v_{h,h'} ,$$

hence, by assumptioin on (f_h) , $f_h \cdot u_{h,h'} = f_{h'} \cdot v_{h,h'}$. Therefore

$$f \cdot u \cdot r_{h,h'} = f \cdot i_h \cdot u_{h,h'} = f \cdot i_{h'} \cdot v_{h,h'} = f \cdot v \cdot r_{h,h'} ,$$

hence $f \cdot u = f \cdot v$ since the family $(r_{h,h'})$ is epic by hypothesis. □

4.3 Corollary (Cf. [12], 3.7) *In a category with pullbacks and universal coproducts, every regular generator is dense.* □

4.4 Corollary *If in a cocomplete and finitely complete category $\mathcal{A}$ pullbacks of small strictly epic families are epic, then every strong generator of $\mathcal{A}$ is dense.*

PROOF: The only strictly epic families to be considered are families of coproduct injections (only with coproducts of objects in the given strong generator), and (singleton families of) regular epimorphisms. That pullbacks of the latter are epic means, in the presence of kernel pairs and coequalizers of such, that every extremal epimorphism is regular (cf. [27]). Hence the strong generator is actually regular, so 4.2 gives its density. □

4.5 Remark Street [25], Cor. 5, proves density of strong generators for a finitely complete category with intersections of *arbitrary* families of monomorphisms, and assuming that pullbacks of *arbitrary* extremally epic families are epic, but he does not need colimits. The absence of coproducts not only results in the necessity to consider large families, but to use much more complicated techniques than those used in the quite elementary proof of 4.2

4.6 Corollary *Let $\mathcal{B}$ be an extremally epireflective subcategory of a cocomplete and finitely complete category $\mathcal{A}$ such that pullbacks of small extremally epic families in $\mathcal{A}$ are epic in $\mathcal{A}$. Then every strong generator of $\mathcal{B}$ is dense in $\mathcal{B}$.*

PROOF: Since $\mathcal{B}$ is extremally epireflective, the injections of coproducts in $\mathcal{B}$ give extremally epic families in $\mathcal{A}$, hence their pullbacks are epic in $\mathcal{A}$ and, a fortiori, in $\mathcal{B}$. Since $\mathcal{A}$ has equalizers, extremal epimorphisms in $\mathcal{B}$ are extremally epic in $\mathcal{A}$, hence their pullbacks are epic in $\mathcal{A}$ and $\mathcal{B}$. Therefore, pullbacks of small extremally epic families in $\mathcal{B}$ are epic in $\mathcal{B}$, and one can apply 4.4. □

5 Density of regular generators - by hom-preservation of coprodutcs

5.1 For a set $\mathcal{G}$ of objects in a category $\mathcal{A}$ with the needed coproducts (8), we keep the notation of 3.1 and 3.4 and consider the natural transformations

$$\begin{aligned} \delta &: U \to Z, \quad (\delta_A)_G : h \mapsto (1_G, h)\,, \\ \rho &: Z \to UFU, \quad (\rho_A)_G : (g,h) \mapsto i_h \cdot g\,, \end{aligned}$$

(for all $A \in |\mathcal{A}|$, $G \in \mathcal{G}$) . Obviously one has

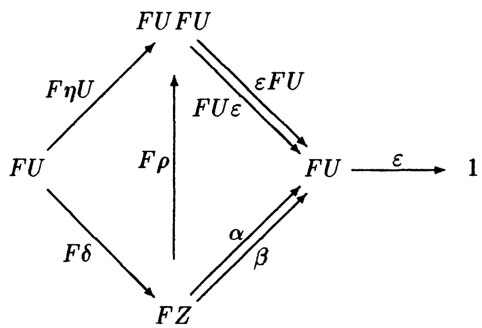
$$\rho \cdot \delta = \eta U\,, \tag{12}$$

with η the unit of the adjunction $F \dashv U$. Since $\varepsilon U \cdot \eta U = 1$, δ is a split-monomorphism. We call $\mathcal{G}$ *rigid* if $\mathcal{A}(G,G) = \{1\}$ and $\mathcal{A}(G,H) = \emptyset$ for all $G,H \in \mathcal{G}$, $G \neq H$; that is: if δ is an isomorphism.

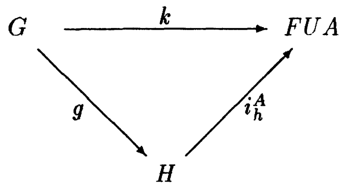
Routine diagram-chasing shows that

$$\varepsilon FU \cdot F\rho = \alpha \quad \text{and} \quad FU\varepsilon \cdot F\rho = \beta \tag{13}$$

Since $\varepsilon FU \cdot F\eta U = 1 = FU\varepsilon \cdot F\eta U$, (13) implies $\alpha \cdot F\delta = 1 = \beta \cdot F\delta$, and also $\varepsilon \cdot \alpha = \varepsilon \cdot \beta$ since $\varepsilon \cdot \varepsilon FU = \varepsilon \cdot FU\varepsilon$.



We say that $\mathcal{G}$ staisfies the *epi-condition* if ρ is an epimorphism; this means that every map $k : G \to FUA$ $(G \in \mathcal{G}, A \in |\mathcal{A}|)$ factors through some canonical injection of the coproduct FUA:



Equivanlently U transforms the injections of the coproduct FUA into an epic family in $\mathsf{Set}^{\mathcal{G}}$.

5.2 Proposition

(1) *If U preserves coproducts of objects in $\mathcal{G}$, or if $F \dashv U$ induces an idempotent monad, then $\mathcal{G}$ satisfies the epi-condition; idempotency of the induced monad is in fact equivalent to the epi-condition for $\mathcal{G}$ rigid.*

(2) *A regular generator that satisfies the epi-condition is dense.*

PROOF: (1) follows easily from (12) of 5.1, and (2) is a consequence of 3.5 (2) (iii) in conjunction with (7) of 2.1, and of 3.5 (3) (ii). □

5.3 In 5.2 (2) we actually do not need ρ_A to be epi but just $F\rho_A$ epi in $\mathcal{A}$. This is a necessary consequence of the epi-condition since the left adjoint F preserves epimorphisms. It means that, for every $A \in |\mathcal{A}|$, the family

$$(i_k^{FUA} : G_k \to FUFUA)_{k \in J(A)} \qquad (14)$$

with $J(A) = \{k \in \mathcal{A}(\mathcal{G}, FUA) \mid \exists (g,h) \in (ZA)_{G_k} : k = i_h^A \cdot g\}$ and $G_k =$ domain of k , is epic. So the condition $F\rho_A$ epi may be weaker than the epi-condition only if $J(A) \neq \mathcal{A}(\mathcal{G}, FUA)$.

In general, a subfamily $(i_\mu : A_\mu \to C)_{\mu \in M}$ of injections of a coproduct $C = \coprod_{\nu \in N} A_\nu$ with $M \subseteq N$ is epic if and only if every i_ν with $\nu \in N \setminus M$ is a *co-constant morphism*, that is: $u \cdot i_\nu = v \cdot i_\nu$ for all $u, v : C \to B$, $B \in |\mathcal{A}|$. One easily shows that any i_ν is co-constant iff there is exactly one morphism $C \to B$ for any B such that $\mathcal{A}(A_\mu, B) \neq \emptyset$ for all $\mu \in N$.

When we apply these observations to (13) and employ the same argumentation as in 5.2 (2), we see that we obtain only a marginal improvment of 5.2 (2):

Corollary *A regular generator $\mathcal{G}$ is dense in $\mathcal{A}$ if for all $G \in \mathcal{G}$ one of following conditions holds:*
(a) *every $k \in \mathcal{A}(G, FUA)$ with $A \in |\mathcal{A}|$ factors as $k = i_h^A \cdot g$ for some $g : G \to H$, $h : H \to A$ and $H \in \mathcal{G}$;*
(b) $|\mathcal{A}(G,B)| = 1$ *for any $B \in |\mathcal{B}|$ with $\mathcal{A}(H,B) \neq \emptyset$ for all $H \in \mathcal{G}$.* □

5.4 $A \in |\mathcal{A}|$ is called *coprime* if $\mathcal{A}(A, -)$ preserves (small-indexed) coproducts (cf. [14], [5]). Let $\mathcal{Q}$ be the class of coprime objects in $\mathcal{A}$, and let $\sqcup\mathcal{Q}$ be the class of coproducts of coprime objects in $\mathcal{A}$ (cf. 3.5). $\mathcal{A}$ is called *based* (cf. [14]) if $|\mathcal{A}| = \sqcup\mathcal{Q}$. We first note:

Proposition *For $\mathcal{A}$ with coproducts consider the statements*

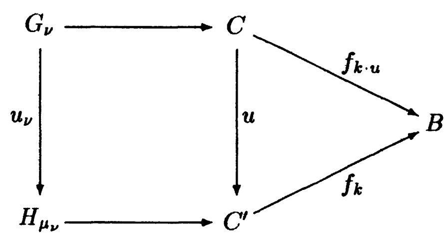
(i) *$\mathcal{Q}$ is densely generating in $\mathcal{A}$,*

(ii) *$\sqcup\mathcal{Q}$ is densely generating in ; $\mathcal{A}$,*

(iii) *$\mathcal{Q}$ is regularly generating in $\mathcal{A}$.*

One has (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) *whereas all are equivalent if $\mathcal{A}$ has only a small set of non-isomorphic coprime objects.*

PROOF: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) holds for any class $\mathcal{Q}$ in $\mathcal{A}$. (Only under the given smallness assumption one can apply 3.5 here, but these implications are easily shown directly, independently of the size of $\mathcal{Q}$.) (iii) $\Rightarrow$ (ii) follows from 3.5 (2). We show (ii) $\Rightarrow$ (i) since this is the only place where we refer to the definition of $\mathcal{Q}$. To this end, for $A \in |\mathcal{A}|$, let $(f_h)_{h \in \mathcal{A}(\mathcal{Q},A)}$ be a family that satisfies (5) of 1.4. Just using the coproduct-property, one can enlarge this family to $(f_k)_{k \in \mathcal{A}(\sqcup\mathcal{Q},A)}$, and one only needs that (5) of 1.4 remains true for the enlarged family. But for any $u : C \to C'$ with $C \cong \coprod_{\nu \in N} G_\nu$ and $C' \cong \coprod_{\mu \in M} H_\mu$ $(G_\nu, H_\mu \in \mathcal{Q})$ one has for every $\nu \in N$, since $\mathcal{A}(G_\nu, -)$ preserves the coproduct C' , uniquely determined $\mu_\nu \in M$ and $u_\nu : G_\nu \to H_{\mu_\nu}$ such that the square of



commutes. Routine diagram-chasing then gives that the triangle commutes as well for every $k \in \mathcal{A}(C', A)$, hence (5) holds. □

5.5 Theorem *For $\mathcal{A}$ with coproducts, let $J : \bar{\mathcal{Q}} \hookrightarrow \mathcal{A}$ be the full embedding of the coprime objects. Then $\mathcal{A}$ is based if and only if the following three conditions hold:*

(i) *for every $C \in |\mathcal{A}|$, there is only a small family of connected components of the comma-category J/C ;*

(ii) *for every $C \in |\mathcal{A}|$ and every connected component $\mathcal{K}$ of J/C , the colimit of $(\mathcal{K} \hookrightarrow J/C \to \mathcal{A})$ exists in $\mathcal{A}$;*

(iii) *J is dense (i.e., $\mathcal{Q}$ or, equivalently, $\sqcup\mathcal{Q}$ is densely generating).*

PROOF: First, let $\mathcal{A}$ be based. Hence $C \in |\mathcal{A}|$ is of the form $C \cong \coprod_{i \in I} G_i$ with $G_i \in \mathcal{Q}$ and I small. Any $f : A \to C$ with $A \in \mathcal{Q}$ factors uniquely through a coproduct-injection $u_i : G_i \to C$ for a uniquely determined index i, since $\mathcal{A}(A, -)$ preserves the coproduct. Hence f belongs to the same component as u_i, so the number of components of J/C is bounded by I; on the other hand, u_i is in fact a terminal object of its component $\mathcal{K}$, so the colimit of $(\mathcal{K} \hookrightarrow J/C \to A)$ is G_i. Therefore (a) and (b) hold, and (c) follows from 5.4. (One can also see directly that the canonical cocone of J/C is a colimit-cocone by decomposing J/C into its connected components, and by forming the coproduct of the colimits of these components, which is C.)

Vice versa, under hypothesis (b) one has for every $C \in |\mathcal{A}|$ and every connected component $\mathcal{K}$ of J/C a colimit $G_{\mathcal{K}}$ of $\mathcal{K} \hookrightarrow J/C \to \mathcal{A}$. As a connected colimit of coprime objects, $G_{\mathcal{K}}$ is coprime (cf. [5]). The restriction of the canonical cocone of J/C to $\mathcal{K}$ factors through the colimit $G_{\mathcal{K}}$ by a morphism $u_{\mathcal{K}} : G_{\mathcal{K}} \to C$; but since the canonical cocone of J/C is itself a colimit by (c), these $u_{\mathcal{K}}$ are the injections of a coproduct which, by (a), is small-indexed. Hence $C \in \sqcup\mathcal{Q}$. □

5.6 Remarks

(1) In the terminology of [5], the condition (b) of 5.5 equivalently means that $\mathcal{Q}$ is multicoreflective in $\mathcal{A}$.

(2) In the category of topological spaces, the coprime objects are the connected spaces; hence (a) and (b) hold, but (c) does not (since not every space is the coproduct of its connected components). Next, consider the full subcategory of Cat consisting of those small categories which are either finite or have infinitely many connected components; here the coprime objects are the finete connected categories, and one easily sees that (a) and (c) hold whereas (b) does not. We do not have an example of a legitimate category which satisfies (b) and (c), but not (a).

(3) The following are examples of based categories: G-sets (for a fixed monoid G), graphs, partially ordered sets, small categories, "metric" spaces (with non-expanding maps) in which $d(x, y) = \infty$ or $d(x, y) = 0$ for $x \neq y$ is permitted; in the latter category, the coprime objects are those non-empty spaces in which all distances are finite.

(4) Examples of *prime* objects (dual to coprime) in the category of unital rings are discussed in [6].

6 Subcategories which contain a generator of their supercategory

6.1 Proposition (H. Bernecker, private communication, 1976). *Let $\mathcal{B}$ be a full subcategory of a category $\mathcal{A}$ that contains an extremally generating class $\mathcal{G}$*

of $\mathcal{A}$. Then, for any diagram D in $\mathcal{B}$, the limits of D in $\mathcal{B}$ and in $\mathcal{A}$ coincide, provided both limits exist.

PROOF: We first show that, for (L,λ) , (K,κ) a limit of D in $\mathcal{A}$, $\mathcal{B}$ respectively, the canonical morphism $m : K \to L$ is a monomorphism: indeed, for $f,g : A \to K$ with $mf = mg$ and all $x \in \mathcal{A}(\mathcal{G}, A)$ one obtains $\kappa \cdot fx = \kappa \cdot gx$, hence $fx = gx$ by the limit-property in $\mathcal{B}$, and then $f = g$ since $\mathcal{G}$ is generating in $\mathcal{A}$.

Now it suffices to show that every $G \in \mathcal{G}$ is projective with respect to m , in order to conlude that m is an extremal epimorphism, hence an isomorphism. But by the limit-property of both κ and λ , it is immediate that every $y : G \to L$ factors (uniquely) through m . □

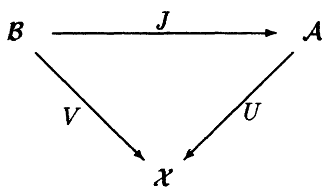
6.2 Remarks

(1) If, in 6.1, $\mathcal{G}$ is just generating, then m is a bimorphism, i.e. both monic and epic.

(2) Note that, in 6.1, $\mathcal{G}$ is assumed to be extremally generating in $\mathcal{A}$. But it is easy to show that it is also extremally generating in $\mathcal{B}$.

6.3 Corollary *Let the full subcategory $\mathcal{B}$ of the complete category $\mathcal{A}$ contain an extremally generating class $\mathcal{G}$ of $\mathcal{A}$. Then, if $\mathcal{B}$ is complete itself, the embedding $\mathcal{B} \hookrightarrow \mathcal{A}$ is continuous.* □

6.4 In order to derive even reflectivity for $\mathcal{B}$ in the setting of 6.1 we need colimits. For that we recall the "triangle theorems" on adjoint functors, the first of which is old (cf. [10], [15], [26]), the other fairly recent ([16],[17]). In both cases we are given a commutative diagram



in **Cat** with right-adjoint functors U and V . Then one has:

(1) If $\mathcal{B}$ has coequalizers and if the co-units belonging to U are regular epimorphism, then J has a left adjoint.

(2) If both $\mathcal{A}$ and $\mathcal{B}$ have coequalizers and small cointersections of strong epimorphisms, if $\mathcal{A}$ is weakly cowellpowered (= cwpd. w.r.t strong epis), and if the co-units belonging to U are strong epimorphisms, then J has a left adjoint.

These two facts are now applied in the situation that J is the embedding of a full subcategory, that $\mathcal{X} = \mathbf{Set}^{\mathcal{G}}$ for $\mathcal{G} \subseteq |\mathcal{B}| \subseteq |\mathcal{A}|$ small, and that U and V

are the generalized hom-functors induced by $\mathcal{G}$ (cf. 3.1). These functors have left-adjoints if $\mathcal{A}$ and $\mathcal{B}$ have sufficiently many coproducts. Hence, with 3.5 (1), (2) one obtains from (1) and (2) above:

6.5 Theorem

(1) *Let $\mathcal{B}$ contain a regular generator of the category $\mathcal{A}$. Then, if $\mathcal{A}$ has coproducts and $\mathcal{B}$ is cocomplete, $\mathcal{B}$ is reflective in $\mathcal{A}$.*

(2) *Let $\mathcal{B}$ contain a strong generator of the cocomplete and weakly cowellpowered category $\mathcal{A}$. Then $\mathcal{B}$ is reflective in $\mathcal{A}$ iff $\mathcal{B}$ is cocomplete.* □

The Theorem may be applied to all monadic categories over **Set**; for instance, for $\mathcal{A}$ = CompHaus the category of compact Hausdorff spaces one obtains:

6.6 Corollary (H. Müller; cf. [13], 13.1.3). *A full subcategory of* CompHaus *is reflective iff it is cocomplete (as a category in its own right) and contains at least one non-empty space.*

PROOF: Every non-empty space forms a regular generator of CompHaus. □

6.7 Remark The assumptions on $\mathcal{B}$ in each (1) and (2) of Theorem 6.5 guarantee that $\mathcal{B}$ is *total* (by Theorem 3.3 of [7]) which implies various strong (co)completeness properties, as well as the fact that $\mathcal{B}$ is *compact* (in the sense of Isbell; cf. [7]). Hence, in the situation of 6.5, one can derive not only the existence of a *left*-adjoint for $J : \mathcal{B} \hookrightarrow \mathcal{A}$, but also the existence of a *right*-adjoint for any functor $H : \mathcal{B} \to \mathcal{C}$ that preserves all colimits of $\mathcal{B}$. In particular: $\mathcal{B}$ is coreflective in any supercategory $\mathcal{C}$ such that the inclusion preserves colimits.

In this context we remind the reader of the following known fact which is very similar to 6.1 and which follows immediately from 2.1 (1), (2):

6.8 Corollary *Let $\mathcal{B}$ be a (full and replete) coreflective subcategory of $\mathcal{A}$.*

(1) *$\mathcal{B}$ is bicoreflective in $\mathcal{A}$ iff $\mathcal{B}$ contains a generating class of $\mathcal{A}$.*

(2) *$\mathcal{B}$ is equal to $\mathcal{A}$ iff $\mathcal{B}$ contains an extremally generating class of $\mathcal{A}$.*
□

6.9 Corollary *A coreflective subcategory of a monadic category over* Set *that contains the free object over 1, is the whole category.* □

In particular, CompHaus does not contain any coreflective subcategories other than the whole category or the one containing the empty space only.

7 An application to finite products

7.1 Let A be an object in $\mathcal{A}$ such that all products $A \times B$ exist in $\mathcal{A}$. This is the same as to say that the domain functor $F_A : \mathcal{A}/A \to \mathcal{A}$ has a right adjoint U_A ; this functor takes $B \in |\mathcal{A}|$ to the projection $A \times B \to A$. The co-units of the adjunction are given by the projections $A \times B \to B$. One therefore obtains from 2.1 a close relationship between "epic properties" of the projections and reflection properties of the functor

$$P_A = F_A U_A : \mathcal{A} \to \mathcal{A} \,,\; B \mapsto A \times B \,.$$

Since F_A is always faithful and conservative and reflects (split) epimorphisms, one has that P_A is faithful, conservative, or reflects (split-) epimorphisms into (extremal; regular) epimorphisms iff U_A has the respective property. Since

$$\downarrow A := \{D \in |\mathcal{A}| \mid \mathcal{A}(D,\mathcal{A}) \neq \emptyset\} = \{F_A f \mid f \in |\mathcal{A}/A|\} \,,$$

2.1 (1), (2) and 2.3 (4) give most of the following proposition.

7.2 Proposition *For an object A in $|\mathcal{A}|$, let all products $A \times B$, $B \in |\mathcal{A}|$, exist in $\mathcal{A}$. Then in both* (1) *and* (2) *assertions* (i) - (iv) *are equivalent:*

(1) (i) *every projection $A \times B \to B$ is extremally epic,*

(ii) *every projection $A \times B \to B$ is epic, and a morphism $f : B \to C$ is an isomorphism if $1_A \times f : A \times B \to A \times C$ is iso;*

(iii) *a morphism $f : B \to C$ is extremally epic if $1_A \times f : A \times B \to A \times C$ is extremally epic;*

(iv) *$\downarrow A$ is an extremally generating class in $\mathcal{A}$;*

(2) (i) *every projection $A \times B \to B$ is regularly epic;*

(ii) *a morphism $f : B \to C$ is regularly epic if $1_A \times f : A \times B \to A \times C$ is a split-epimorphism;*

(iii) *$\downarrow A$ is a regularly generating class in $\mathcal{A}$;*

(iv) *$\downarrow A$ is a densely generating class in $\mathcal{A}$;*

PROOF: Only (2) (iii) requires extra consideration; since trivially (iv) $\Rightarrow$ (iii), we just need to show (iii) $\Rightarrow$ (ii). By the definition of a regularly generating class, for that it suffices to show that $\mathcal{A}(D,f) : \mathcal{A}(D,B) \to \mathcal{A}(D,C)$ is a surjective map for every $D \in \downarrow A$, provided $1_A \times f$ is split-epic, i.e. if there is a morphism h with $(1_a \times f)h = 1_{A \times C}$. But then, by functoriality of $\mathcal{A}(D,-)$, and since $\mathcal{A}(D,-)$ preserves products,

$$1 \times \mathcal{A}(D,f) : \mathcal{A}(D,A) \times \mathcal{A}(D,B) \to \mathcal{A}(D,A) \times \mathcal{A}(D,C)$$

is a surjective map. But $\mathcal{A}(D,A)$ is not empty, so $\mathcal{A}(D,f)$ maps onto. □

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