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**WHAT IS A DOUBLE CENTRAL EXTENSION?
(the question was asked by Ronald Brown)**

by George JANELIDZE

RÉSUMÉ. Cet article développe une notion d'extension centrale double comme cas particulier d'un objet normal dans une catégorie. Ceci répond à une question de R. Brown relative à une "théorie de Galois dans les catégories".

The theory of central extensions of groups can be considered as a particular case of the "Galois theory in categories" given in [2].

In a similar manner we obtain here the new notion of "double central extension" (as a particular case of a normal object in a category in the sense of [2]). This answers a question R. Brown asked the author in Tbilisi in 1987.

The idea comes from the generalization of the Hopf formula given by R. Brown and G. Ellis [1], but the method and results are independent of [1].

The results of this paper were reported at the International Category Theory Meeting, 1989, in Bangor.

The paper contains two sections. The first section recalls with some improvements a part of [2] and continues that work. The second section considers the *double central extensions of groups* of our title.

1. ON GALOIS THEORY IN CATEGORIES

Let \mathcal{E} be a category with pullbacks and \mathcal{F} a class of morphisms in \mathcal{E} containing all isomorphisms, closed under composition and such that for a pullback diagram

$$\begin{array}{ccc}
 C_1 \times_{C_0} C_2 & \xrightarrow{\pi_2} & C_2 \\
 \pi_1 \downarrow & & \downarrow \sigma_2 \\
 C_1 & \xrightarrow{\sigma_1} & C_2
 \end{array}$$

$\sigma_1, \sigma_2 \in \mathcal{E}$ implies $\pi_1 \in \mathcal{E}$ (and so $\pi_2 \in \mathcal{E}$).

For a fixed $(\mathcal{E}, \mathcal{E})$ we say that an object (C, σ) of $(\mathcal{E} \downarrow C_0)$ is an *extension* of C_0 if $\sigma \in \mathcal{E}$; the full subcategory of $(\mathcal{E} \downarrow C_0)$ with objects all extensions of C_0 is denoted by $\mathcal{E}(C_0)$.

For an extension (C, σ) of C_0 there is the composition functor $U^\sigma : \mathcal{E}(C) \rightarrow \mathcal{E}(C_0)$ defined by $(A, \alpha) \mapsto (A, \sigma\alpha)$ and its right adjoint: the pullback functor $P^\sigma : \mathcal{E}(C_0) \rightarrow \mathcal{E}(C)$, defined by

$$(A, \alpha) \mapsto (C \times_{C_0} A, \text{proj}_1).$$

A "Galois structure" (Definition 3.1 of [2]) consists of an adjunction

$$(I, H, \eta, \varepsilon) : \mathcal{E} \rightarrow \mathcal{X}$$

and classes \mathcal{E} and \mathcal{Z} of morphisms of \mathcal{E} and \mathcal{X} respectively such that: $(\mathcal{E}, \mathcal{E})$ satisfies the conditions above; $(\mathcal{X}, \mathcal{Z})$ satisfies the same conditions; $I(\mathcal{E}) \subset \mathcal{Z}$; $H(\mathcal{Z}) \subset \mathcal{E}$; ε is an isomorphism¹; $\eta_C \in \mathcal{E}$ for each $C \in \text{Ob } \mathcal{E}$.

We consider a fixed Galois structure Γ and so assume that $\mathcal{E}, \mathcal{X}, I, H, \eta, \varepsilon, \mathcal{E}$ and \mathcal{Z} are fixed.

Let $I^C : (\mathcal{E} \downarrow C) \rightarrow (\mathcal{X} \downarrow I(C))$ be the functor induced by I . It has the right adjoint H^C constructed as follows: for $(X, \varphi) \in \text{Ob}(\mathcal{X} \downarrow I(C))$ consider the pullback diagram:

¹⁾

this is not necessary, but it holds in the case considered in section 2.

$$\begin{array}{ccc}
 C \times_{HI(C)} H(X) & \xrightarrow{\text{proj}_2} & H(X) \\
 \text{proj}_1 \downarrow & & \downarrow H(\varphi) \\
 C & \xrightarrow{\eta_C} & HI(C)
 \end{array}$$

and

$$H^C : (X, \varphi) \mapsto (C \times_{HI(C)} H(X), \text{proj}_1) .$$

Moreover I^C itself induces a functor $\mathfrak{E}(C) \rightarrow \mathcal{Z}(I(C))$ and H^C induces a functor $\mathcal{Z}(I(C)) \rightarrow \mathfrak{E}(C)$; denote them by $I^{C, \Gamma}$ and $H^{C, \Gamma}$ respectively.

Let (C, σ) and (A, α) be extensions of C_0 ; (A, α) is split over (C, σ) with respect to Γ if the canonical morphism

$$P^\sigma((A, \alpha)) \rightarrow H^{C, \Gamma} I^{C, \Gamma} P^\sigma((A, \alpha))$$

is an isomorphism; the full subcategory of $\mathfrak{E}(C_0)$ with objects all extensions split over (C, σ) with respect to Γ is denoted by $\text{Spl}_\Gamma((C, \sigma))$ (3.2 of [2]).

Clearly we have

PROPOSITION 1.1. *Let (C, σ) and (A, α) be extensions of C_0 . If the counit $I^{C, \Gamma} H^{C, \Gamma} \rightarrow I_{\mathcal{Z}(I(C))}$ is an isomorphism, then the following conditions are equivalent:*

- (a) (A, α) is split over (C, σ) with respect to Γ ;
- (b) the morphism

$$\langle \text{proj}_1, \eta_{C \times_{C_0} A} \rangle : C \times_{C_0} A \rightarrow C \times_{HI(C)} HI(C \times_{C_0} A)$$

is an isomorphism;

(c) there exists $(X, \varphi) \in \text{Ob} \mathcal{Z}(I(C))$ such that the objects $(C \times_{C_0} A, \text{proj}_1)$ and $(C \times_{HI(C)} H(X), \text{proj}_1)$ of $\mathfrak{E}(C)$ are isomorphic. □

After that we can prove

PROPOSITION 1.2. *If (C, σ) is split over (C, σ) with respect to Γ and the counit above is an isomorphism, then*

$U^\sigma H^{C,\Gamma}((X,\varphi))$ is split over (C,σ) with respect to Γ for each $(X,\varphi) \in \text{Obz}(I(C))$.

PROOF. The diagram

$$\begin{array}{ccc}
 C \times_{C_0} C & \xrightarrow{\langle \text{proj}_1, \eta_C \times_{C_0} C \rangle} & C \times_{HI(C)} HI(C \times_{C_0} C) \\
 \text{proj}_2 \downarrow & & \downarrow \text{proj}_2 \\
 C & & HI(C \times_{C_0} C) \\
 \eta_C \downarrow & & \downarrow HI(\text{proj}_2) \\
 HI(C) & \xlongequal{\quad\quad\quad} & HI(C)
 \end{array}$$

commutes and so we can write

$$(C \times_{C_0} C) \times_{HI(C)} H(X) \cong C \times_{HI(C)} HI(C \times_{C_0} C) \times_{HI(C)} H(X)$$

and this isomorphism commutes with the first projection. The left side is isomorphic to $C \times_{C_0} (C \times_{HI(C)} H(X))$ and the right side to

$$C \times_{HI(C)} H(I(C \times_{C_0} C) \times_{I(C)} X),$$

and these isomorphisms commute with the first projection too. Now the implication (c) \Rightarrow (a) of 1.1 yields that

$$U^\sigma H^{C,\Gamma}((X,\varphi)) = (C \times_{HI(C)} H(X), \sigma \cdot \text{proj}_1)$$

is split over (C,σ) with respect to Γ . □

From this proposition it follows that Definition 3.3 of [2] is equivalent to the following one:

DEFINITION 1.3. An extension (C,σ) of C_0 is Γ -normal if the following conditions hold:

- (a) the counit $I^{C,\Gamma} H^{C,\Gamma} \rightarrow 1_{Z(I(C))}$ is an isomorphism;
- (b) the functor P^σ is monadic;
- (c) (C,σ) is split over (C,σ) with respect to Γ . □

We shall write $(A, \alpha) \leq_{\Gamma} (C, \sigma)$ if (A, α) is split over (C, σ) with respect to Γ and (C, σ) satisfies the conditions (a) and (b) of 1.3.

DEFINITION 1.4.

- (a) An extension (A, α) of C_0 is a Γ -covering if there exists an extension (C, σ) of C_0 such that $(A, \alpha) \leq_{\Gamma} (C, \sigma)$;
- (b) An extension (C, σ) of C_0 is a *weak universal Γ -covering* if it is a Γ -covering and for each extension (A, α) of C_0 which is a Γ -covering one has $(A, \alpha) \leq_{\Gamma} (C, \sigma)$. □

The full subcategory of $\mathfrak{E}(C_0)$ with objects all Γ -coverings is denoted by $\text{Spl}(\Gamma, C_0)$. If there exists a weak universal Γ -covering (C, σ) then we can write

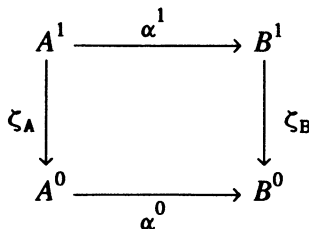
$$\text{Spl}(\Gamma, C_0) = \text{Spl}_{\Gamma}((C, \sigma)) .$$

Moreover, clearly (C, σ) is a Γ -normal extension of C_0 and so using the Theorem 3.6 of [2] we obtain the following description of this category:

THEOREM 1.5. *Let (C, σ) be an extension of C_0 which is a weak universal covering and let $G = \text{Gal}_{\Gamma}((C, \sigma))$ be the Galois groupoid of (C, σ) in the sense of [2]. Then the category $\text{Spl}(\Gamma, C_0)$ is equivalent to the full subcategory of \mathcal{X}^G with objects all internal functors $F = (F_0, \pi, \xi) : G \rightarrow \mathcal{X}$ such that (F_0, π) is an extension of $I(C)$.* □

2. DOUBLE CENTRAL EXTENSIONS

Let Γ be the following Galois structure: \mathfrak{E} is the full subcategory of the category $\text{Ar}(\text{Groups})$ with objects all surjective homomorphisms of groups; if A is an object of \mathfrak{E} then we write $A = (A^0, \zeta_A, A^1)$, where $\zeta_A : A^1 \rightarrow A^0$ is a group epimorphism, and a morphism $\alpha : A \rightarrow B$ in \mathfrak{E} is a commutative square



in the category of groups:

\mathfrak{E} consists of all morphisms $\alpha : A \rightarrow B$ such that for each $b^1 \in B^1$ and $a^0 \in A^0$ with $\alpha^0(a^0) = \zeta_B(b^1)$ there exists $a^1 \in A^1$ with $\zeta_A(a^1) = a^0$ and $\alpha^1(a^1) = b^1$, i.e. the homomorphism

$$\langle \zeta_A, \alpha^1 \rangle : A^1 \rightarrow A^0 \times_{B^0} B^1$$

is surjective;

\mathfrak{X} is the full subcategory of \mathfrak{E} with objects all central extensions;

$\mathfrak{Z} = \mathfrak{X} \cap \mathfrak{E}$;

I is the "centralization", i.e.

$$I(A) = \left[\begin{array}{c} A^1 / [\text{Ker } \zeta_A, A^1] \\ \downarrow \bar{\zeta}_A \\ A^0 \end{array} \right],$$

where $\bar{\zeta}_A$ is the homomorphism induced by ζ_A , i.e.

$\bar{\zeta}_A(\text{cls}(a^1)) = \zeta_A(a^1)$ for each $a^1 \in A^1$;

H is the inclusion functor;

$\eta_A : A \rightarrow HI(A)$ is

$$\begin{array}{ccc} A^1 & \xrightarrow{\quad} & A^1 / [\text{Ker } \zeta_A, A^1] \\ \zeta_A \downarrow & & \downarrow \bar{\zeta}_A \\ A^0 & \xlongequal{\quad} & A^0 \end{array},$$

where the upper arrow is the canonical epimorphism;

$\varepsilon_X : IH(X) \rightarrow X$ is

$$\begin{array}{ccc} X^1 / \{1\} & \xrightarrow{\cong} & X^1 \\ \bar{\zeta}_X \downarrow & & \downarrow \zeta_X \\ X^0 & \xlongequal{\quad} & X^0 \end{array}.$$

LEMMA 2.1. *Let (C, σ) and (A, α) be extensions of C_0 . Then the following conditions are equivalent:*

- (a) (A, α) is split over (C, σ) with respect to Γ ;
 (b) the diagram

$$\begin{array}{ccc}
 C^1 \times_{C_0^1} A^1 & \longrightarrow & C^1 \times_{C_0^1} A^1 / [(\text{Ker } \zeta_C) \times_{C_0^1} (\text{Ker } \zeta_A), C^1 \times_{C_0^1} A^1] \\
 \downarrow & & \downarrow \\
 C^1 & \longrightarrow & C^1 / [\text{Ker } \zeta_C, C^1]
 \end{array}$$

is a pullback;

- (c) the canonical homomorphism

$$[(\text{Ker } \zeta_C) \times_{C_0^1} (\text{Ker } \zeta_A), C^1 \times_{C_0^1} A^1] \rightarrow [\text{Ker } \zeta_C, C^1]$$

is an isomorphism.

PROOF. (a) \Leftrightarrow (b) follows easily from (a) \Leftrightarrow (b) of 1,1 and (b) \Leftrightarrow (c) follows from a well known property of group extensions. \square

DEFINITION 2.2. An extension (A, α) of C_0 is a *double central extension* if the commutants

$$[\text{Ker } \zeta_A \cap \text{Ker } \alpha^1, A^1], [\text{Ker } \zeta_A, \text{Ker } \alpha^1]$$

are trivial groups, i.e.

$$ka_1 = a_1k, k_{\zeta_A} k_{\alpha^1} = k_{\alpha^1} k_{\zeta_A}$$

for each $k \in \text{Ker } \zeta_A \cap \text{Ker } \alpha^1$, $a_1 \in A^1$, $k_{\zeta_A} \in \text{Ker } \zeta_A$, $k_{\alpha^1} \in \text{Ker } \alpha^1$. \square

LEMMA 2.3.

- (a) If (A, α) is split over (C, σ) with respect to Γ , then (A, α) is a double central extension;
 (b) If (A, α) is a central double extension and for an extension (C, σ) there exists a morphism $\gamma : C \rightarrow A$ with $\alpha\gamma = \sigma$, then (A, α) is split over (C, σ) with respect to Γ .

PROOF.

(a) Let $k, a_1, k_{\zeta_A}, k_{\alpha^1}$ be as above. First consider k and a_1 . Choose $c_1 \in C^1$ with $\sigma^1(c_1) = \alpha^1(a_1)$ and consider the element

$$t = [(1, k), (c_1, a_1)] \in [(\text{Ker } \zeta_C) \times_{C_0^1} (\text{Ker } \zeta_A), C^1 \times_{C_0^1} A^1];$$

its image in $[\text{Ker } \zeta_C, C^1]$ is $[1, C^1] = 1$ and so $t = 1$ by the condition (c) of 2.1. Hence $[k, a_1] = 1$, i.e. $ka_1 = a_1k$.

Now consider k_{ζ_A} and k_{α^1} .

Choose $c_1 \in C^1$ with $\sigma^1(c_1) = \alpha^1(k_{\zeta_A})$ and $\zeta_C(c_1) = 1$ (this is possible because

$$\zeta_{C_0}(\alpha^1(k_{\zeta_A})) = \alpha^0(\zeta_A(k_{\zeta_A})) = 1 = \sigma^1(1)).$$

Consider the element

$$t = [(c_1, k_{\zeta_A}), (1, k_{\alpha^1})] \in [(\text{Ker } \zeta_C) \times_{C_0^1} (\text{Ker } \zeta_A), C^1 \times_{C_0^1} A^1];$$

its image in $[\text{Ker } \zeta_C, C^1]$ is $[c_1, 1] = 1$ and so $t = 1$ by the condition (c) of 2.1. Hence $[k_{\zeta_A}, k_{\alpha^1}] = 1$, i.e.

$$k_{\zeta_A} k_{\alpha^1} = k_{\alpha^1} k_{\zeta_A}.$$

(b) We will prove that the condition (c) of 2.1 is satisfied. It is sufficient to prove that the composition

$$\begin{aligned} [(\text{Ker } \zeta_C) \times_{C_0^1} (\text{Ker } \zeta_A), C^1 \times_{C_0^1} A^1] &\rightarrow [\text{Ker } \zeta_C, C^1] \rightarrow \\ &\rightarrow [(\text{Ker } \zeta_C) \times_{C_0^1} (\text{Ker } \zeta_A), C^1 \times_{C_0^1} A^1], \end{aligned}$$

where the second arrow is defined by $c_1 \mapsto (c_1, \sigma^1(c_1))$ (clearly correct!), is the identity map. Thus it is sufficient to prove that for each $k_C \in \text{Ker } \zeta_C, k_A \in \text{Ker } \zeta_A, c_1 \in C^1$ and $a_1 \in A^1$ with $\sigma^1(k_C) = \alpha^1(k_A)$ and $\sigma^1(c_1) = \alpha^1(a_1)$ we have

$$[k_A, a_1] = [\gamma^1(k_C), \gamma^1(c_1)] .$$

To prove this, first observe that

$$\alpha^1(k_A) = \alpha^1(\gamma^1(k_C)), \alpha^1(a_1) = \alpha^1(\gamma^1(c_1))$$

and so there exist $k, k' \in \text{Ker } \alpha^1$ with

$$\gamma^1(k_C) = k_A k, \gamma^1(c_1) = a_1 k'$$

Moreover $k \in \text{Ker } \zeta_A \cap \text{Ker } \alpha^1$ because $\gamma^1(k_C)$ and k_A are in $\text{Ker } \zeta_A$. After that we have

$$[\gamma^1(k_C), \gamma^1(c_1)] = [k_A k, a_1 k'] = k_A k a_1 k' k^{-1} k_A^{-1} k'^{-1} a_1^{-1} .$$

By the condition $[\text{Ker } \zeta_A \cap \text{Ker } \alpha^1, A^1] = \{1\}$ we can write $k a_1 k' k^{-1} = a_1 k'$ and by the condition $[\text{Ker } \zeta_A, \text{Ker } \alpha^1] = \{1\}$ we can write $k' k_A^{-1} k'^{-1} = k_A^{-1}$. Thus we have

$$\begin{aligned} k_A k a_1 k' k^{-1} k_A^{-1} k'^{-1} a_1^{-1} &= k_A a_1 k' k_A^{-1} k'^{-1} a_1^{-1} \\ &= k_A a_1 k_A^{-1} a_1^{-1} = [k_A, a_1] , \end{aligned}$$

which completes the proof. □

LEMMA 2.4. *The conditions (a) and (b) of 1.3 hold for any extension (C, σ) .*

PROOF. 1.3(a): For an extension (X, φ) of $I(C)$ (in \mathfrak{A}) consider the pullback

$$\begin{array}{ccc} D & \xrightarrow{\pi_2} & X^1 \\ \pi_1 \downarrow & & \downarrow \varphi^1 \\ C^1 & \xrightarrow[\eta_C]{} & C^1 / [\text{Ker } \zeta_C, C^1] ; \end{array}$$

we need to prove that π_2 induces an isomorphism

$$D / [\text{Ker } \zeta_X \pi_2, D] \cong X^1 ,$$

i.e. that $\pi_2(d) = 1$ implies $d \in [\text{Ker } \zeta_X \pi_2, D]$ for each $d \in D$.

Let d be an element of D with $\pi_2(d) = 1$. Then $\pi_1(d)$ is in $[\text{Ker } \zeta_C, C^1]$ and so we can write

$$d = ([k_1, c_1] \dots [k_n, c_n], 1),$$

where k_1, \dots, k_n are in $\text{Ker } \zeta_C$. Choose $x_1, \dots, x_n \in X^1$ with $\phi^1(x_i) = c_i[\text{Ker } \zeta_C, C^1]$ and $y_1, \dots, y_n \in X^1$ with

$$\phi^1(y_i) = k_i[\text{Ker } \zeta_C, C^1], \zeta_X(y_i) = 1$$

for $i = 1, \dots, n$; we have

$$d = [(k_1, y_1), (c_1, x_1)] \dots [(k_n, y_n), (c_n, x_n)]$$

and so d is in $[\text{Ker } \zeta_X \pi_2, D]$.

1.3(b) easily follows from the fact that for each group epimorphism $\mathfrak{B} \rightarrow \mathfrak{A}$ the pullback functor

$$(\text{Groups } \downarrow A) \rightarrow (\text{Groups } \downarrow B)$$

is monadic. □

Now let (C, σ) be the extension of C_0 constructed by the following three steps:

1⁰, $\sigma^0 : C^0 \rightarrow C_0^0$ is an arbitrary epimorphism with a free C_0^0 ;
 2⁰, after that consider an arbitrary epimorphism from a free group F to the group $C_0^1 \times_{C_0^0} C^0$ and denote the compositions

$$F \longrightarrow C_0^1 \times_{C_0^0} C^0 \xrightarrow{\text{proj}_1} C_0^1,$$

$$F \longrightarrow C_0^1 \times_{C_0^0} C^0 \xrightarrow{\text{proj}_2} C^0$$

by π_1, π_2 respectively.

3⁰, (C, σ) is the "double centralization": C^1 is the factor group of F by the commutants $[\text{Ker } \pi_1 \cap \text{Ker } \pi_2, F]$ and $[\text{Ker } \pi_1, \text{Ker } \pi_2]$, and σ^1, ζ_C are induced by π_1, π_2 respectively.

Clearly (C, σ) is a double central extension of C^0 , and

if (A, α) also is a double central extension of C_0 , then there exists a morphism $\gamma : C \rightarrow A$ with $\alpha\gamma = \sigma$. Using the lemmas above we obtain:

THEOREM 2.5. *The extension (C, σ) constructed above is a weak universal covering and for an extension (A, α) of C_0 the following conditions are equivalent:*

- (a) (A, α) is a double central extension;
- (b) (A, α) is split over (C, σ) with respect to Γ ;
- (c) (A, α) is a Γ -covering;
- (d) (A, α) is a Γ -normal extension. □

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