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NILPOTENT CROSSED MODULES AND Pro-*C* COMPLETIONS

by F.J. KORKES and T. PORTER

RÉSUMÉ. Cet article donne deux résultats sur les pro-*C* complétions de groupes simpliciaux, où *C* est une sous-catégorie pleine de la catégorie des groupes finis; on étudie leurs relations avec les pro-*C* complétions de modules croisés et les actions nilpotentes.

In a earlier paper [2], we have considered the algebraic problem of studying the pro-*C* completion of a crossed module and gave a "cofinality condition" that guaranteed that the pro-*C* completion of the crossed module could be performed levelwise, i.e., by completing the two groups making up the crossed module individually.

It is a result of Mac Lane and Whitehead that certain equivalence classes of crossed modules correspond via an equivalence of categories with homotopy 2-types. i.e., with homotopy types that have zero homotopy groups above level 2. Any connected homotopy type, X, can be represented by a simplicial group G(X), and any simplicial group G gives rise to a reduced simplicial set W(G), so that these two functors, G and W, give an equivalence on homotopy categories. The work of Bousfield and Kan [1] on p-profinite completions of homotopy types includes the possible use of this adjointness to define such a completion functor algebraically. They also study nilpotent actions and nilpotent fibrations and the interplay of these ideas with their completion processes.

In this paper we investigate the connection between pro-*C* completions of crossed modules, levelwise constructions of pro-*C* completion of simplicial groups and nilpotent actions.

1. PRELIMINARIES.

In what follows *C* will denote a non-trivial full subcategory of the category of finite groups. This category is assumed to be closed under the formation of subgroups. quotients and finite products. For any group G we let $\Omega(G)$ be the directed set of normal subgroups N of G such that $G/N \in \mathcal{C}$. With this notation the group theoretic pro-$\mathcal{C}$ completion of G is given by

$$\hat{G} = \varprojlim G/N$$

where the inverse limit is taken over the normal subgroups $N \in \Omega(G)$.

A *crossed module* consists of two groups C and G, an action of G on C and a homomorphism $\partial: C \to G$ satisfying the two conditions:

CM1) for all $c \in C$ and $g \in G$,

$$\partial({}^{g}c) = g\partial(c)g^{-1},$$

CM2) for all $c_1, c_2 \in C$

$${}^{\partial(c_1)}c_2 = c_1 c_2 c_1^{-1}.$$

Morphisms of crossed modules are pairs of homomorphisms preserving the action and giving a commutative square in the obvious way.

A crossed module is said to be a pro-$\mathcal{C}$ crossed module if both groups are pro-$\mathcal{C}$ groups, that is inverse limits of groups in $\mathcal{C}$, and the action and homomorphism are continuous in the inverse limit topology. Morphisms between two pro-$\mathcal{C}$ crossed modules are continuous morphisms of the algebraic crossed modules underlying them.

There is an obvious forgetful functor from the category of pro-$\mathcal{C}$ crossed modules to that of crossed modules, and in [2] we showed that that functor had a left adjoint which is a pro-$\mathcal{C}$ completion functor. This completion functor is not just the group theoretic pro-$\mathcal{C}$ completion of the two groups. The "bottom" group (G in the above) is sent to its pro-$\mathcal{C}$ completion, but the "top" group on which G acts does not, in general, go to its pro-$\mathcal{C}$ completion. This does happen, however, if the crossed module satisfies the following cofinality condition:

Let $\Omega_G(C)$ be the directed subset of $\Omega(C)$ given by those $N \in \Omega(C)$ which are G-equivariant. We say (C, G, ∂) satisfies the *cofinality condition* if $\Omega_G(C)$ is a cofinal subset of $\Omega(C)$.

We will also need some facts about simplicial groups. We recall that given a simplicial group $G_{.}$, the *Moore complex* (NG, ∂) of $G_{.}$ is the chain complex defined by

$$(NG)_n = \bigcap \{\operatorname{Ker} d_i^n \mid i \neq 0\}$$

with $\partial_n: NG_n \to NG_{n-1}$ given by $\partial_n = d_0^n$ restricted to $(NG)_n$. The image of ∂_{n+1} is normal in G_n and the homotopy groups of G. can be calculated using this complex (NG,∂); in fact,

$$\pi_n(G.) \approx \bigcap_{i=0}^{n} \text{Ker } d_i^n / (d_0^{n+1}(\bigcap_{i=1}^{n+1} \text{Ker } d_i^{n+1})).$$

From G. we can also form a crossed module

$$\partial: (NG_1/d_0 NG_2) \to G_0,$$

in which the "boundary" map ∂ is induced by that of the Moore complex. We will denote this crossed module by M(G.,1) as it represents the 2-type of G. The cokernel of this crossed module is $\pi_0(G.)$ and the kernel is $\pi_1(G.)$ as is easily seen from the quoted facts about the Moore complex.

Given a simplicial group G., its pro-***C*** completion will be taken to be the pro-***C*** simplicial group obtained by applying the group theoretical pro-***C*** completion in each dimension. (Although this ties in with the definition of Bousfield and Kan, see [1] page 109, the reader should be warned that it does not necessarily coincide with the Artin-Mazur type completion which aims to pro-***C*** complete the homotopy groups rather than an algebraic model of the homotopy type. The two definitions will coincide in the presence of finiteness conditions.)

Finally we recall (again cf. [1]) the notion of a nilpotent action of a group G on a group C. An action of G on C is said to be *nilpotent* if there is a finite sequence

$$C = C_1 \supset \cdots \supset C_j \supset \cdots \supset C_n = \{e\}$$

of subgroups of C such that for each j

(i) C_j is closed under the action of G,

(ii) C_{j+1} is normal in C_j and C_j/C_{j+1} is abelian,

and (iii) the induced G-action on C_j/C_{j+1} is trivial.

We will say that the *G-nilpotent length* of C in this case is less than or equal to n (denoted $\lambda_G \leq n$).

2. pro-*C* COMPLETIONS OF SIMPLICIAL GROUPS AND OF CROSSED MODULES.

Bousfield and Kan proved ([1] page 113): "the homotopy type of $R_\infty X$ in dimensions $\leq k$" depends only on" the homotopy type of X in dimensions $\leq k$". Here R_∞ is a completion functor and we will only be looking at this for the case of a pro-***C*** completion. For $k=1$, as the 1-type of X is determined by the fundamental group, this is clear and relatively uninteresting. The next simplest case of this is given by the 2-type and thus, by Mac Lane and Whitehead, by a crossed module. One way of assigning a crossed module to a 2-type is via simplicial groups and the M(−,1) construction recalled earlier.

Our situation is not identical with that studied by Bousfield and Kan but we can view their result in a different light representing the 2-type by a crossed module, so it is natural to ask what is the exact relationship between the representing crossed module $M(\hat{G}_.,1)$ of the 2-type of the pro-***C*** completion of a simplicial group $G_.$, and $M(G_.,1)^\sim$, the crossed module pro-***C*** completion of $M(G_.,1)$ as introduced by us in [2]. The answer is as nice as it could be.

PROPOSITION. *There is a natural isomorphism*

$$M(G_.,1)^\sim \approx M(\hat{G}_.,1).$$

PROOF. The nerve functor

$$E: \mathrm{CMod} \to \mathrm{Simp.Groups}$$

is defined as follows: If $M = (C,G,\partial)$ is a crossed module, then $E(M)$ is the simplicial group given as the nerve of the associated cat^1-group, which is an internal category in the category of groups (see Loday [3]). In dimension 0, $E(M)$ is just G, in dimension 1, it is $C \rtimes G$, and in higher dimensions it is a multiple semi-direct product with many copies of C.

This simplicial group has Moore complex isomorphic to

$$\cdots \to 0 \to 0 \to \cdots \to C \to G,$$

i.e., essentially giving us back M. Now let $T_{1]}$ be the full (reflexive) subcategory of the category Simp.Groups defined by the condition that $G_.$ is in it if and only if the Moore complex of $G_.$ has trivial terms in dimensions 2 and above, i.e., $N(G)_i = \{1\}$ for each $i \geq 2$. The reflector $t_{1]}: \mathrm{Simp.Groups} \to T_{1]}$ is defined by the condition that $N(t_{1]}G_.)$ is the same as the truncation of $N(G_.)$ given by:

$N(G)_0$	in dimension 0,
$N(G)_1/d_0N(G)_2$	in dimension 1,
1	in dimensions ≥ 2.

One easily checks that $t_{1]}G_.$ is isomorphic to $EM(G_.,1)$ and that $M(-,1)$ and E set up an equivalence of categories between CMod and $T_{1]}$. Of course a similar thing happens with pro-***C*** crossed modules and a reflexive subcategory of pro-***C*** simplicial groups.

(The notation we will use for the various categories will, it is hoped, be self explanatory.)

Suppose that G. is a (discrete/abstract) simplicial group and M is a pro-$\boldsymbol{C}$ crossed module, then

$$\mathrm{Simp.Groups}(G_., U(EM)) \approx \mathrm{Simp.pro\text{-}}\boldsymbol{C}(\hat{G}, EM)$$

since EM is a simplicial pro-$\boldsymbol{C}$ group. This set is itself naturally isomorphic to $T_{11}^{\mathcal{C}}(t_{11}^{\mathcal{C}}(\hat{G}), EM)$. Since

$$M(t_{11}^{\mathcal{C}}(\hat{G}), 1) \approx M(\hat{G}, 1),$$

this gives a natural isomorphism

$$\mathrm{Simp.Groups}(G_., U(EM)) \approx \mathrm{pro\text{-}}\boldsymbol{C}.\mathrm{CMod}(M(\hat{G}, 1), M).$$

The forgetful functor U: pro-$\boldsymbol{C}$ → Groups, or more exactly its simplicial and crossed modules extensions, satisfies UE ≈ EU, so one also has

$$\mathrm{Simp.Groups}(G_., U(EM)) \approx \mathrm{Simp.Groups}(G_., E(UM))$$
$$\approx \mathrm{CMod}(M(G_., 1), UM)) \approx \mathrm{pro\text{-}}\boldsymbol{C}.\mathrm{CMod}(M(G_., 1)^{\sim}, M)).$$

We thus have that there is a natural isomorphism

$$M(G_., 1)^{\sim} \approx M(\hat{G}_., 1)$$

as required.

This clarifies and extends Bousfield and Kan's result in the case $k=2$, since for a reduced homotopy type X, the pro-$\boldsymbol{C}$ completion $W\hat{G}X$ of X has a 2-type represented by $M(\hat{G}X_., 1)$ which is isomorphic to the pro-$\boldsymbol{C}$ completion of the crossed module $M(GX_., 1)$, that represents the 2-type of X.

3. NILPOTENT CROSSED MODULES, COFINALITY CONDITIONS AND pro-$\boldsymbol{C}$ COMPLETIONS.

Crossed modules occur in the work of Loday [3], linked closely to the study of fibrations: if p: E→B is a fibration with connected fibre F then the induced map from $\pi_1(F)$ to $\pi_1(E)$ makes $(\pi_1(F), \pi_1(E), p_*)$ into a crossed module. The preservation of certain crossed module structures by termwise pro-$\boldsymbol{C}$ completion is thus reminiscent of the preservation of nilpotent fibrations by completions as exemplified by the nilpotent fibration lemma of Bousfield and Kan [1], and suggests there should be a link between nilpotent actions and cofinality conditions. The link is the following:

PROPOSITION. *If M = (C,G,∂) is a crossed module in which the action of G on C is nilpotent, then M satisfies the cofinality* *condition and hence* $M^{\sim}$ *is isomorphic to* $(\hat{C},\hat{G},\hat{\partial})$.

PROOF. The proof is by induction on the G-nilpotent length of C. First we note that if $W \lhd C$ is such that C/W is in $\mathcal{C}$, it is sufficient to prove that $\cap {}^{g}W = V$, say, is such that C/V is in $\mathcal{C}$. If $\lambda_G = 1$, the group C is trivial. If $\lambda_G(C) = 2$, then the group C is abelian with trivial G-action. In neither case is there any difficulty.

Next suppose we have that the conclusion holds provided that $\lambda_G(C) < n$, more precisely we assume that if W is normal in C and $C/W \in \mathcal{C}$, then $V = \cap {}^{g}W$ is also such that C/V is in $\mathcal{C}$. Now if C is such that $\lambda_G(C) = n$, there is a sequence

$$C = C_1 \supset \cdots \supset C_j \supset \cdots \supset C_n = \{e\}$$

as in the definition of Section 1. Taking the normal subgroup C_2 we get a short exact sequence

$$1 \to C_2 \to C_1 \xrightarrow{p} C_1/C_2 \to 1$$

in which $\lambda_G(C_2) < n$ and C_1/C_2 is abelian with trivial G-action.

Now suppose $W \lhd C$ is such that $W/C \in \mathcal{C}$. For any $g \in G$, $p({}^{g}W) = p(W)$, since the G-action on C_1/C_2 is trivial. Moreover

$${}^{g}W \cap C_2 = {}^{g}(W \cap C_2),$$

since C_2 is closed under the G-action. Thus setting $V = \cap {}^{g}W$, we get

$$p(V) = p(W) \text{ and } V \cap C_2 = {}^{g}(W \cap C_2).$$

As $C_2/C_2 \cap W \in \mathcal{C}$, we apply the induction hypothesis to conclude that $C_2/C_2 \cap V \in \mathcal{C}$. Similarly the quotient of C_1/C_2 by $p(V)$ is in $\mathcal{C}$ as it is the same as that by $p(W)$. The group C/V is thus part of an exact sequence, the other groups of which are in $\mathcal{C}$, hence it also is in $\mathcal{C}$ as required.

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