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(E,M)-FUNCTORS AND M-UNIVERSAL INITIAL COMPLETIONS

by Hubertus W. BARGENDA

RÉSUMÉ. Une M-complétion initiale universelle d'une catégorie concrète (A,U) sur X, où M est une collection de A-sources, est un foncteur concret $\mathrm{E}:(A,\mathrm{U})\to(A_M,\mathrm{U}_M)$ dans une catégorie initialement complète qui transforme les M-sources en sources U_M-initiales et qui est universel pour cette propriété. On donne un critère pour que E soit plein et une condition pour que ce soit un adjoint à droite. Pour des M spéciaux, on en déduit divers résultats connus (par exemple pour des foncteurs topologiquement algébriques) ou nouveaux (e.g., pour des foncteurs essentiellement algébriques).

0. INTRODUCTION.

Various types of functors studied in Categorical Topology and Algebra are instances of (E,M)-functors, e.g., topological (or initially complete), topologically algebraic, regular, essentially algebraic functors (for some definitions, see 1). Since H. Herrlich's guiding paper on initial completions [6] (see also [7]), it has been an objective to determine MacNeille completions and universal initial completions of given concrete categories (for a survey, see [8] 1.3, [12], [3]). Moreover, in [11] Herrlich and Strecker discovered an interesting connection between topologically algebraic functors (introduced by Y.H. Hong in [13] as a generalization of topological as well as algebraic functors) and their universal initial completions: a concrete category (A,U) is topologically algebraic iff its universal initial completion is reflective. This result will now be extended to (E,M)-functors. For this purpose, the concept of a universal initial completion of a concrete category (A,U) over X is generalized relative to a given arbitrary conglomerate M of A-sources, called an *M-universal initial completion*

$$\mathrm{E}:(A,\mathrm{U})\to(A_M,\mathrm{U}_M).$$

This completion is new only in the sense that we don't demand that M consists only of U-initial sources. So, the completion

(A_M, U_M) is a slight generalization of the concept of a *universal* (Δ,Γ)-*completion* where $\Gamma = \{X\text{-sources}\}$ and Δ is any conglomerate of U-initial sources, as described by Andrée C. Ehresmann [4]. The construction and universal property of E: $(A,U) \to (A_M, U_M)$ are obtained by the "same" technique used in [4] and [6]. E is not a completion in the usual sense, i.e., E need not be a full embedding. But we shall prove that E is a full embedding iff M consists only of U-initial sources.

We establish a general correspondence between an (E, M)-factorization structure of U and the right adjointness of the M-universal initial completion of (A,U). Our main result is that $U: A \to X$ is an (E,M)-functor iff $E: (A,U) \to (A_M, U_M)$ is right adjoint and M is (as we shall say) U-*restrictive*. From this, the above mentioned characterization of topologically algebraic functors follows for $M = \{\text{U-initial sources}\}$, but for other choices of M we obtain new characterizations. In particular, the case $M = \{A\text{-monosources}\}$ is interesting: in [10], Herrlich introduced the concept of an *essentially algebraic category* (A,U) as a very general notion of an "algebraic" category. It will turn out that a concrete category (A,U) is essentially algebraic iff it has a full and reflective {monosources}-universal initial completion. Some examples of {monosources}-universal initial completions will be determined.

1. TERMINOLOGY.

In this paper, let (A,U) denote a *concrete category* over a fixed (base) category X, i.e., a pair (A,U) where $U: A \to X$ is a faithful and amnestic functor (*amnestic* means that an A-isomorphism f is an A-identity if Uf is an X-identity). A *concrete functor* $F: (A,U) \to (B,V)$ between concrete categories over X is a functor $F: A \to B$ with $U = VF$. An *extension* is a full concrete embedding.

A U-*morphism* is a pair (f, A), where $f: X \to UA$ is an X-morphism and A an A-object. We often write $f: X \to A$ for a U-morphism. A U-*epi(morphism)* is a U-morphism $e: X \to A$ such that for each pair $(a,b): A \rightrightarrows B$ of A-morphisms $(Ua)e = (Ub)e$ implies $a = b$.

A U-*source* on X is a pair (X,S) where X is an X-object and $S = (f_i: X \to A_i)_{i \in I}$ is a family of U-morphisms indexed by a class I. We usually write $(f_i: X \to A_i)_I$ or $(f_i)_I$ for (X,S). We say that $f: X \to A$ *belongs* to, or is a *member* of $(f_i)_I$ provided there is some $i \in I$ with $f = f_i$. If U is the identity functor on A, then a U-source is called an *A-source*. We say that $(g_j : X \to B_j)_J$ is an *extension* of $(f_i : X \to A_i)_I$ (and that $(f_i)_I$ is a *restriction* of $(g_j)_J$) provided I is a subclass of J and $(g_j)_I = (f_i)_I$.

Given a class E of U-morphisms and a collection M of A-sources, we say that (A,U) (or U) is *(E,M)-factorizable* provided for each U-source $(f_i : X \to A_i)_I$ there are

$$(e: X \to A) \in E \text{ and } (m_i : A \to A_i)_I \in M \text{ such that } (f_i)_I = (Um_i e)_I.$$

(A,U) is called an *(E,M)-functor* provided it is (E,M)-factorizable and each pair $(e: X \to A) \in E$, $(m_i : B \to A_i)_I \in M$ is *U-orthogonal*, i.e., whenever the outer rectangle of the diagram

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "f"'] & UA \arrow[d, "Ua_i"] \arrow[dl, dashed, "Ud"'] \\ UB \arrow[r, "Um_i"'] & UA_i \end{tikzcd}$$

commutes, i.e., $(Ua_i)e = (Um_i)f$ for all $i \in I$, then there exists exactly one A-morphism $d: A \to B$ (the *diagonal*) such that

$$f = (Ud)e \text{ and } (m_i d)_I = (a_i)_I.$$

We call any U-morphism $e: X \to A$ *M-orthogonal* provided for all $(m_i)_I \in M$, e and $(m_i)_I$ are U-orthogonal. We call U an *$(-,M)$-functor* provided there is a class E of U-morphisms such that U is an (E,M)-functor. A is called an *(E,M)-category* provided the identity functor on A is an (E,M)-functor.

An A-source $(m_i : A \to A_i)_I$ is called

— a *monosource* provided for each pair $(a,b): B \rightrightarrows A$ of A-morphisms $(m_i a)_I = (m_i b)_I$ implies $a = b$,

— *U-initial* provided whenever

$$(UB \xrightarrow{f} UA \xrightarrow{Um_i} UA_i)_I = (UB \xrightarrow{Ua_i} UA_i)_I$$

then there exists exactly one A-morphism $f^- : B \to A$ with $Uf^- = f$ (and with

$$(B \xrightarrow{f^-} A \xrightarrow{m_i} A_i)_I = (B \xrightarrow{a_i} A_i)_I \;),$$

— an *all-source* (on A) provided each A-morphism with domain A belongs to $(m_i)_I$.

(A,U) (or U) is called:

— *initially complete* provided U is an (identity, initial)-functor,

— *topologically algebraic* provided U is a (-, initial) functor ([7], 2.3),

— *essentially algebraic* provided U is (U-epi, monosource)-factorizable and U reflects isomorphisms [10] (cf. Proposition 4 (a), (b) below).

We use a *set-class-conglomerate* hierarchy. "Categories" with conglomerate-many objects are called *quasicategories*. A concrete quasicategory over X is called *legitimate* provided there exists an injection from the conglomerate of its objects into a class.

2. THE M-UNIVERSAL INITIAL COMPLETION.

Let (A,U) be a concrete category over X and let M be any conglomerate of A-sources. We generalize the well-known construction of a universal initial completion of (A,U) in an obvious manner, i.e., we construct a quasicategory (A_M,U_M) over X and a concrete *(comparison)* functor $\mathrm{E}:(A,\mathrm{U})\to(A_M,\mathrm{U}_M)$ which has the following properties:

(M1) (A_M,U_M) is initially complete and $\mathrm{E}:(A,\mathrm{U})\to(A_M,\mathrm{U}_M)$ carries over the sources in M into U_M-initial sources, and

$(M2)$ (*Universality of* E) whenever $\mathrm{F}:(A,\mathrm{U})\to(B,\mathrm{V})$ is a concrete functor into an initially complete concrete (quasi)category which carries over the sources in M into V-initial sources, then there exists exactly one initial sources preserving concrete functor $\bar{\mathrm{F}}:(A_M,\mathrm{U}_M)\to(B,\mathrm{V})$ such that the diagram

$$\begin{tikzcd}
(A,\mathrm{U}) \arrow[rr, "\mathrm{F}"] \arrow[d, "\mathrm{E}"'] & & (B,\mathrm{V}) \\
(A_M,\mathrm{U}_M) \arrow[urr, dashed, "\bar{\mathrm{F}}"'] & &
\end{tikzcd}$$

commutes. If $\bar{\mathrm{F}}:(A_M,\mathrm{U}_M)\to(B,\mathrm{V})$ is in particular an isomorphism, then $\mathrm{F}:(A,\mathrm{U})\to(B,\mathrm{V})$ is called an *M-universal initial completion.* (Note that we deviate from the normal usage of *completion*, since $\mathrm{E}:A\to A_M$ need not be full and A_M need not be legitimate.)

In case M is the conglomerate of all U-initial sources, $\mathrm{E}:(A,\mathrm{U})\to(A_M,\mathrm{U}_M)$ is just the universal initial completion (see [6,7,1]). In case M consists only of U-initial sources, $\mathrm{E}:(A,\mathrm{U})\to(A_M,\mathrm{U}_M)$ is a special case of Andrée C. Ehresmann's construction of a *universal* (Δ,Γ)-*completion* [4] if one puts $\Delta=M$ and Γ is the conglomerate of all X-sources. (Δ,Γ)-completions, where Δ is a conglomerate of U-initial sources and Γ a conglomerate of X-sources, were introduced to unify completions of concrete categories, so, for special choices of (Δ,Γ) one obtains the *universal initial completion* ([4], 3), or the *universal (concrete limit) completion*, due to Adamek and Koubek [2] (see also [9]). Although, for our purposes, we do not require that M consists only of U-initial sources (cf. Proposition 1), the construction of the M-universal initial completion and the proof of its properties (M1) and (M2) are essentially the same as for the (Δ,Γ)-completion or analogous to the universal initial completion. So, we give the construction of $E:(A,U)\to(A_M,U_M)$ and may omit the proofs of (M1) and (M2).

CONSTRUCTION OF $E:(A,U)\to(A_M,U_M)$.

Call a U-source $(f_i: X\to A_i)_I$ *M-enriched* provided it satisfies the following two conditions:

(C1) Whenever $a: A_i\to A$ is an A-morphism for some $i\in I$, then $(Ua)f_i: X\to A$ belongs to, $(f_i)_I$, and

(C2) Whenever $f: X\to A$ is a U-morphism and $(m_k: A\to A_k)_K \in M$ such that for each $k\in K$, $(Um_k)f: X\to A_k$ belongs to $(f_i)_I$, then $f: X\to A$ belongs to $(f_i)_I$.

Each U-source $S=(f_i: X\to A_i)_I$ has a least M-enriched extension $\bar{S}=(f_j: X\to A_j)_J$ called the *M-enrichment* of S.

A *source map* $f:(X,S)\to(Y,T)$ between M-enriched U-sources is an X-morphism $f: X\to Y$ such that for each member $g: Y\to A$ of T, $gf: X\to A$ is a member of S.

Now, let A_M be the quasicategory where its object conglomerate is the conglomerate of all M-enriched U-sources and its morphism class is the class of all source maps. Composition and identitites in A are adopted from X.

The concrete functor $U_M: A_M\to X$ is the projection functor

$$U_M(f:(X,S)\to(Y,T)) = f: X\to Y.$$

The object assignment of $E: A\to A_M$ is defined as follows: for any A-object A let S_A be the M-enrichment of the one-member U-source $U\mathrm{id}_A: UA\to A$, and put $EA=(UA,S_A)$. The morphism assignment of E is defined by

$$Ef = Uf:(U,S_A)\to(U,S_B) \text{ for } f: A\to B \text{ in } A.$$

(In fact, Ef is a source map, since the restriction of S_B to the U-source S of all members $g: UB\to C$ of S_B such that $gUf: UA\to C$ belongs to S_A contains $\mathrm{id}_{UB}: UB\to B$, and one easily checks that S is M-enriched, whence $S=\bar{S}=S_B$.)

For the main purpose of this paper, namely, the characterization of (E,M)-functors, we need only the M-universal initial completions, but it is worthwile to mention that given any conglomerate Δ of A-cones and any conglomerate Γ of X-cones with $U[\Delta] \subset V\Gamma$, one can construct a (possibly non-full and non-legitimate) universal (Δ,Γ)-completion of (A,U) in the sense of [4] (in [4], 3, one may drop the condition that Δ consists only of U-initial cones).

2. FULLNESS AND RIGHT ADJOINTNESS CRITERION FOR $E: (A,U) \to (A_M,U_M)$.

The completion $E: (A,U) \to (A_M,U_M)$ need not be a full embedding. The following Fullness Criterion shows that the fullness of the (Δ,Γ)-completion in the sense of [4] is not accidently implied by the assumption that Δ contains only U-initial sources:

PROPOSITION 1 (*Fullness Criterion*). *The following conditions are equivalent:*

(a) $E: (A,U) \to (A_M,U_M)$ *is a full embedding,*

(b) every member of M *is* U*-initial.*

PROOF. (a) $\Rightarrow$ (b): Let $(m_i: B \to A_i)_I \in M$ and consider

$$(UB \xrightarrow{f} UA \xrightarrow{Um_i} UA_i)_I = (UB \xrightarrow{Ua_i} UA_i)_I .$$

We show that f is a source map $f: (UA,S_A) \to (UB,S_B)$. Since S_A is M-enriched, each $(Um_i)f = Ua_i$ and hence $f: UA \to UB$ belongs to S_A. Thus the restriction of S_B to the U-source S of all members $g: UB \to C$ of S_B such that $gf: UA \to C$ belongs to S_A contains $id_{UB}: UB \to B$ and is M-enriched (as one easily checks), hence $S = S_B$, and

$$f: (UA,S_A) \to (UB,S) = (UB,S_B)$$

is a source map.

(b) $\Rightarrow$ (a): For each A-object A, let S be the restriction of S_A to the source of all members $g: UA \to B$ of S_A for which there exists an A-morphism $a: A \to B$ with $Ua = f$. S obviously satisfies (C1), and also (C2), since if $f: UA \to UB$ is a U-morphism, and $(m_i: UA \to A_i)_I$ belongs to S, then there exists an A-morphism $f^-: A \to B$ with $Uf^- = f$ (because $(m_i)_I$ is U-initial). Since each $(Um_i)f$ belongs to S_A, f belongs to S_A, hence also to S. So, S is M-enriched and obviously contains $id_{UA}: UA \to A$, hence $S = S_A$. Now, let $f: (UA,S_A) \to EB$ be a source map. Then $f = f id_{UB}$ belongs to S_A. Since $S_A = S$, there exists an A-morphism f: $A \to B$ with $f = Uf^- = Ef^-$. ∎

The main result of Herrlich & Strecker [11], namely that the universal initial completion of (A,U) is reflective iff every U-source is (U-epi, initial)-factorizable, is now extended to the M-universal initial completion.

We say that $\mathrm{E}: (A,\mathrm{U}) \to (B, V)$ is *a right adjoint* provided $\mathrm{E}: A \to A_M$ has a (not necessarily concrete) left adjoint.

PROPOSITION 2 *(Right Adjointness Criterion). The following conditions are equivalent:*

(a) $\mathrm{E}: (A,\mathrm{U}) \to (A_M, \mathrm{U}_M)$ *is a right adjoint,*

(b) every M-enriched U-source is (U-epi, all-source)-factorizable.

If (a) or (b) holds, then A_M is legitimate.

PROOF. (a)$\Rightarrow$(b): Let (X,S) be M-enriched, i.e., an object of A_M. There exists an E-universal morphism $r: (\mathrm{X,S}) \to \mathrm{EA}$. Since r is an E-epi, $r: \mathrm{X} \to \mathrm{UA}$ is also a U-epi. Since $r: (\mathrm{X,S}) \to \mathrm{EA}$ is a source map, $r: \mathrm{X} \to \mathrm{UA}$ belongs to S. Since S satisfies (C1), for each member $a: \mathrm{A} \to \mathrm{B}$ of the all-source on A, $(\mathrm{U}a)r: \mathrm{X} \to \mathrm{B}$ belongs to S, which, together with the universality of $r: (\mathrm{X,S}) \to \mathrm{EA}$ implies that S is (U-epi, all-source)-factorizable.

(b)$\Rightarrow$(a): Let (X,S), $\mathrm{S} = (f_i: \mathrm{X} \to \mathrm{A}_i)_\mathrm{I}$ be an A_M-object. There exists a (U-epi, all-source)-factorization

$$(\mathrm{X} \xrightarrow{e} \mathrm{UA} \xrightarrow{\mathrm{U}a_i} \mathrm{UA}_i)_\mathrm{I}$$

of S. Since $e: \mathrm{X} \to \mathrm{UA}$ belongs to S and is a U-epi, $e: (\mathrm{X,S}) \to \mathrm{EA}$ is an E-epi. For each E-morphism $f: (\mathrm{X,S}) \to \mathrm{UB}$, $f: \mathrm{X} \to \mathrm{UB}$ belongs to S, so $e: (\mathrm{X,S}) \to \mathrm{EA}$ is E-universal.

Now, if (a) holds, choose for any M-enriched U-source (X,S) a (U-epi, all-source)-factorization

$$(\mathrm{X} \xrightarrow{e_\mathrm{S}} \mathrm{UA} \xrightarrow{\mathrm{U}a_i} \mathrm{UA}_i)_\mathrm{I}$$

of S. The M-enrichment of $(e_\mathrm{S}: \mathrm{X} \to A)$ obviously coincides with S. So, the assignment $(\mathrm{X,S}) \mapsto e_\mathrm{S}$ is an injection from the conglomerate of all M-enriched U-sources into the class of all U-morphisms, thus A_M is legitimate. ∎

LEMMA. *If*

$$(\mathrm{X} \xrightarrow{e} \mathrm{UA} \xrightarrow{\mathrm{U}a_i} \mathrm{UA}_i)_\mathrm{I}$$

is a (U-epi, all-source)-factorization of an M-enriched U-source, then e is M-orthogonal.

PROOF. Consider

$$(X \xrightarrow{e} UA \xrightarrow{U b_j} UB_j)_J = (X \xrightarrow{f} UB \xrightarrow{U m_j} UA_j)_J$$

where $(m_j: B \to B_j)_J \in M$. Since $((U b_j) e: X \to B_j)_J$ is a restriction of the M-enriched source $((U a_i) e: X \to A_i)_I$, $f: X \to B$ belongs to it, i. e., there is a $k \in I$ with

$$(f: X \to B) = (U a_k) e: X \to A_k.$$

We have

$$U(m_j a_k) e = (U m_j)(U a_k) e = (U m_j) f = (U b_j) e,$$

hence $m_j a_k = b_j$ for all $j \in J$ (since e is U-epi). So, $a_k: A \to A_k = B$ functions as a diagonal. ∎

REMARK 1. If (A,U) satisfies (a) and (b) of Proposition 2, then the M-universal initial completion of (A,U) can be given in a more convenient form, namely, up to equivalence, as the quasi-category B of all M-orthogonal U-epis (e,A) as the B-objects; a B-morphism $f: (e,A) \to (e',A')$ between M-orthogonal $e: X \to A$ and $e': X' \to A'$ is an X-morphism $f: X \to X'$ for which there exists an A-morphism $a: A \to A'$ such that $e'f = (Ua)e$. This is clear, since the object assignment $(X,S) \mapsto e_S$ given in the last part of the proof of Proposition 2 can easily be extended to a full embedding from A_M into B which is an equivalence. (By the above lemma, e_S is M-orthogonal.) This observation generalizes Herrlich & Strecker's construction of a universal initial completion of a topologically-algebraic (A,U) ([11], 2.5).

3. (E,M)-FUNCTORS AND $E: (A,U) \to (A_M, U_M)$.

If (A,U) has a right adjoint M-universal initial completion then every U-source has a (M-orthogonal, source)-factorization. This follows from Proposition 2 and the lemma. Now, we look for a condition for M guaranteeing that every U-source is (M-orthogonal, M)-factorizable, i.e., that U is an $(-, M)$-functor, namely:

DEFINITION. M is called U-*restrictive* provided that for each (U-epi, all-source)-factorization

$$(X \xrightarrow{f} UB \xrightarrow{U m_j} UA_j)_J$$

of the M-enrichment of a U-source $(f_i: X \to UA_i)_I$ the restriction $(m_j)_I$ belongs to M.

Now we state our main result. There we assume the trivial condition that M is *isomorphism closed*, i.e., whenever $(m_i : A \to A_i)_I \in M$ and $f: B \to A$ is an A-isomorphism, then we have $(m_i f : B \to A_i)_I \in M$.

THEOREM. *The following conditions are equivalent, for any isomorphism closed M:*

(a) $U: A \to M$ *is an (E,M)-functor for some E,*

(b) $U: A \to M$ *is* (U-*epi*, M)-*factorizable and M is* U-*restrictive,*

(c) $E: (A,U) \to (A_M, U_M)$ *is a right adjoint and M is* U-*restrictive.*

PROOF. (a)$\Rightarrow$(b): By [11], 2.1, every (E,M)-functor U is (U-epi, M)-factorizable. Now we prove that M is U-restrictive: let

$$(f_j : X \longrightarrow UA_j)_J = (X \xrightarrow{\ e\ } UA \xrightarrow{\ Um_j\ } UA_j)_J$$

be a (U-epi, all-source)-factorization of the M-enrichment $(f_j)_J$ of a U-source $(f_i)_I$. There exists an (E,M)-factorization

$$(f_i : X \longrightarrow UA_i)_I = (X \xrightarrow{\ \bar{e}\ } U\bar{A} \xrightarrow{\ U\bar{m}_i\ } UA_i)_I .$$

Let $(f_k : X \to A_k)_K$ be the restriction of $(f_j)_J$ to the U-source of all members $g: X \to B$ of $(f_j)_J$ for which there exists (exactly one) A-morphism $a: \bar{A} \to B$ such that

$$(g : X \longrightarrow B) = (X \xrightarrow{\ \bar{e}\ } U\bar{A} \xrightarrow{\ Ua\ } UB) .$$

Since $\bar{e}: X \to \bar{A}$ is a U-epi (see [11], 2.1), $(f_k)_K$ is an extension of $(f_i)_I$, and it is M-enriched, since it obviously satisfies (C1), and if $f: X \to B$ is any U-morphism and $(n_l : B \to B_l)_L \in M$ such that each $(Un_l) f : X \to B_l$ belongs to $(f_k)_K$, then for each $l \in L$ there is an A-morphism $a_l : \bar{A} \to B_l$ such that the outer rectangle of the diagram

$$\begin{tikzcd} X \arrow[r, "\bar{e}"] \arrow[d, "f"'] & U\bar{A} \arrow[d, "Ua_l"] \arrow[dl, dashed, "Ud"] \\ UB \arrow[r, "Un_l"'] & UA_l \end{tikzcd}$$

commutes for each $l \in L$. Since $\bar{e}$ is M-orthogonal, there exists a diagonal $d: \bar{A} \to B$ in A. Since each $(Un_l) f$ belongs to the M-enriched $(f_j)_J$, f belongs to $(f_j)_J$, hence to $(f_k)_K$. Thus, $(f_k)_K$ is an M-enriched extension of $(f_i)_I$, so $(f_k)_K = (f_j)_J$ and $\bar{e}: X \to \bar{A}$ belongs to $(f_k)_K$. Now, $e: X \to A$ belongs to $(f_j)_J = (f_k)_K$, so there are A-morphisms

$$a: A \to \bar{A} \text{ and } \bar{a}: \bar{A} \to A \text{ with } \bar{e} = (Ua)e \text{ and } e = (U\bar{a})\bar{e}.$$

Since e and $\bar{e}$ are U-epis, $a: \mathrm{A} \to \bar{\mathrm{A}}$ is an A-isomorphism. Because $(f_i)_\mathrm{I}$ is a restriction of $(f_j)_\mathrm{J}$, the diagram

$$\begin{tikzcd}
\mathrm{X} \arrow[rr, "e"] \arrow[d, "\bar{e}"'] & & \mathrm{UA} \arrow[d, "\mathrm{U}m_i"] \arrow[dll, "\mathrm{U}a"] \\
\mathrm{U}\bar{\mathrm{A}} \arrow[rr, "\mathrm{U}\bar{m}_i"'] & & \mathrm{UA}_i
\end{tikzcd}$$

commutes, for each $i \in \mathrm{I}$. Since $(\bar{m}_i)_\mathrm{I} \in M$ and M is isomorphism closed, $(m_i)_\mathrm{I} \in M$.

(b) $\Rightarrow$ (c): Let

$$(f_j : \mathrm{X} \longrightarrow \mathrm{UA}_j)_\mathrm{J} = (\mathrm{X} \xrightarrow{\ e\ } \mathrm{UA} \xrightarrow{\mathrm{U}m_j} \mathrm{UA}_j)_\mathrm{J}$$

be a (U-epi, M)-factorization of an M-enriched $(f_j)_\mathrm{J}$. Then e: $\mathrm{X} \to \mathrm{A}$ belongs to $(f_j)_\mathrm{J}$, and the U-epi property of e implies that $(m_j)_\mathrm{J}$ is an all-source on A. (c) follows now from Proposition 2.

(c) $\Rightarrow$ (a): Let $\mathrm{S} = (f_i : \mathrm{X} \to \mathrm{UA}_i)_\mathrm{I}$ be a U-source and $\bar{\mathrm{S}} = (f_j : \mathrm{X} \to \mathrm{UA}_j)_\mathrm{J}$ be its M-enrichment. By Proposition 2, there exists a (U-epi, all-source)-factorization

$$(\mathrm{X} \xrightarrow{\ e\ } \mathrm{UA} \xrightarrow{\mathrm{U}m_j} \mathrm{UA}_j)_\mathrm{J}$$

of $\bar{\mathrm{S}}$. By the lemma, $e: \mathrm{X} \to \mathrm{A}$ is M-orthogonal. Since M is U-restrictive, $(m_i)_\mathrm{I} \in M$. So

$$(\mathrm{X} \xrightarrow{\ e\ } \mathrm{UA} \xrightarrow{\mathrm{U}m_i} \mathrm{UA}_i)_\mathrm{I}$$

is an (M-orthogonal, M)-factorization of S. ∎

REMARK 2. We mention the following fact (and omit its proof):
For each (U-epi, all-source)-factorization

$$(\mathrm{X} \xrightarrow{\ e\ } \mathrm{UA} \xrightarrow{\mathrm{U}m_j} \mathrm{UA}_j)_\mathrm{J}$$

of the M-enrichment of a U-source $(f_i : \mathrm{X} \to \mathrm{UA}_i)_\mathrm{I}$, the restriction $(m_i)_\mathrm{I}$ belongs to M_U, i.e., the conglomerate of all A-sources $(n_k)_\mathrm{K}$ such that for each M-orthogonal U-epi $f: \mathrm{X} \to \mathrm{B}$ the pair $f, (n_k)_\mathrm{K}$ is M-orthogonal.

From this observation, we obtain the following consequence:

If U is (U-epi, M)-factorizable and E is the class of all M-orthogonal U-epis, then M_U is the largest among all A-sources N such that U is an (E, N)-functor, and for all these pairs (E, N), the N- and the M_U-universal initial completions of (A, U) coincide (cf. Remark 1). (In [14], there is an example of an (E, M)-functor with $M \neq M_\mathrm{U}$.)

4. APPLICATIONS. MONO-UNIVERSAL INITIAL COMPLETIONS.

Now we apply the theorem of §3 to special U-restrictive M's, obtaining that U is an $(-,M)$-functor iff $E:(A,U)\to(A_M,U_M)$ is a right adjoint.

(a) $M=\varnothing$:

$E:(A,U)\to(A_\varnothing,U_\varnothing)$ coincides with the largest initially dense extension of (A,U) (see [6,7]). E is only reflective when the base category X is empty.

(b) M = conglomerate of all A-sources:

Considering empty A-sources $(A,\varnothing)$, one has that for each X-object X the U-source of all U-morphisms $f:X\to A$ is the only source-enriched U-source on X, i.e., $U_M:A_M\to X$ is an isomorphism, so $U:(A,U)\to(X,\mathrm{id}_X)$ is a source-universal initial completion, which is a right adjoint iff U: $A\to X$ is a (-, source)-functor.

(c) M = the conglomerate of all U-initial sources:

Here, the full embedding $E:(A,U)\to(A_M,U_M)$ is the universal initial completion (see [6,7]). E is reflective iff U is topologically-algebraic. This is the main result of Herrlich & Strecker ([11], 2.7).

(d) M = the conglomerate of all monosources in A:

Here, we substitute the prefix M by "mono" and call $E:(A,U)\to(A_M,U_M)$ a *mono-universal initial completion.*

PROPOSITION 3. *The conglomerate of all monosources in A is U-restrictive.*

PROOF. Let

$$(f_j:X\longrightarrow UA_j)_J=(X\xrightarrow{\ e\ }UA\xrightarrow{\ Um_j\ }UA_j)_J$$

be a (U-epi, all-source)-factorization of the mono-enrichment $(f_j)_J$ of a U-source $(f_i)_I$. Let $(x,y):B\rightrightarrows A$ be a pair of A-morphisms such that $m_ix=m_iy$ for all $i\in I$. Let K be the class of all $j\in J$ with $m_jx=m_jy$. $(f_k:X\to UA_k)_K$ is an extension of $(f_i)_I$ and we prove that $(f_k)_K$ is mono-enriched: for any $k\in K$, let $a:A_k\to B$ be an A-morphism. Since $(f_j)_J$ is mono-enriched, there is some $j\in J$ such that $(Ua_k)f_k$: $X\to B$ equals $f_j:X\to A_j$.

$$\begin{tikzcd}
X \arrow[r, "f_k"] \arrow[dr, "e"'] & UA_k \arrow[r, "Ua"] & UB = UA_j \\
& UA \arrow[u, "Um_k"'] \arrow[ur, "Um_j"'] &
\end{tikzcd}$$

We have

$$(\mathrm{U}m_j)e = f_j = (\mathrm{U}a)f_k = (\mathrm{U}a\,\mathrm{U}m_k)e = \mathrm{U}(a m_k)e,$$

hence $m_j = a m_k$ (since e is a U-epi), so

$$m_j x = a m_k x = a m_k y = m_j y,$$

i.e., $j \in \mathrm{K}$ and $f_j = (\mathrm{U}a)f_k$ belongs to $(f_k)_\mathrm{K}$. Now, let $f\colon \mathrm{X} \to \mathrm{B}$ be a U-morphism and $(n_l\colon \mathrm{B} \to \mathrm{B}_l)_\mathrm{L}$ a mono-source in A such that $(\mathrm{U}n_l)f\colon \mathrm{X} \to \mathrm{B}_l$ belongs to $(f_k)_\mathrm{K}$ for all $l \in \mathrm{L}$, i.e., for each $l \in \mathrm{L}$ there exists a $k_l \in \mathrm{K}$ such that $(\mathrm{U}n_l)f$ equals $f_{k_l}\colon \mathrm{X} \to \mathrm{A}_{k_l}$. Since $(f_j)_\mathrm{J}$ is mono-enriched, there is some $j \in \mathrm{J}$ such that $f\colon \mathrm{X} \to \mathrm{B}$ equals $f_j\colon \mathrm{X} \to \mathrm{A}_j$. So, the diagram

$$\begin{tikzcd}
\mathrm{X} \arrow[r, "f_{k_l}"] \arrow[d, "e"'] \arrow[dr, "f_j = f" description] & \mathrm{UA}_{k_l} = \mathrm{UB}_l \\
\mathrm{UB} \arrow[r, "\mathrm{U}m_j"'] & \mathrm{UA}_j = \mathrm{UB} \arrow[u, "\mathrm{U}n_l"']
\end{tikzcd}$$

commutes for each $l \in \mathrm{L}$, hence

$$\mathrm{U}(n_l m_j)e = f_{k_l} = (\mathrm{U}m_{k_l})e, \text{ so } n_l m_j = m_{k_l}$$

for all $l \in \mathrm{L}$. Therefore

$$n_l m_j x = m_{k_l} x = m_{k_l} y = n_l m_j y$$

for each $l \in \mathrm{L}$, hence $m_j x = m_j y$ (since $(n_l)_\mathrm{L}$ is a monosource), which shows that $j \in \mathrm{K}$, i.e., $f = f_j$ belongs to $(f_k)_\mathrm{K}$. Thus, $(f_k)_\mathrm{K}$ is a mono-enriched extension of $(f_i)_\mathrm{I}$, hence $\mathrm{K} = \mathrm{J}$, i.e., $m_j x = m_j y$ for all $j \in \mathrm{J}$, which implies $x = y$ (since id_A is a member of the all-source $(m_j)_\mathrm{J}$ on A). ∎

As for topologically-algebraic functors, we are now able to characterize essentially algebraic concrete categories (for definition, see §1). They were introduced by Herrlich [10] as a generalization of the concept of an "algebraic" category. From Propositions 1 and 3 (for an essentially algebraic (A,U), A-monosources are U-initial [10], VI) and from the theorem of §3 we obtain:

PROPOSITION 4. *The following conditions are equivalent:*

(a) (A,U) *is essentially algebraic,*

(b) $\mathrm{U}\colon A \to X$ *is a (–, monosource)-functor and reflects isomorphisms.*

(c) (A,U) *has a full and reflective mono-universal legitimate initial completion.*

5. EXAMPLES OF MONO-UNIVERSAL INITIAL COMPLETIONS.

As initial sources for topological categories, monosources play a basic role for algebraic categories, which is also emphasized by Proposition 4, stating that the essential algebraicity of (A,U) and the reflectivity of its *mono*-universal initial completion are equivalent. So it is a natural objective to determine the mono-universal initial completions of (essentially) algebraic categories.

In general, any mono-universal initial completion of an essentially algebraic category (A,U) contains its universal initial completion as a full concrete subcategory, and the two completions coincide iff all U-initial sources are monosources. By Remark 1, the mono-universal initial completion of (A,U) is (up to equivalence) the category of all mono-orthogonal U-epis, which, as one easily proves, equals the category of all extremal U-epis e: X→A, i.e., A is *generated* by e in the usual algebraic sense (cf. [11], 3.4, who show that the universal initial completion of an (essentially) algebraic (A,U) coincides with the category of all extremal U-epis under the *restrictive* condition whereby all U-initial sources are monosources).

We give some examples:

(a) Consider the trivially concrete category (**Set**,id) over **Set** (= category of sets and maps) and the concrete category (**Top**,U) over **Set** of all topological spaces and continuous maps. Both are initially complete, so they are their own universal initial completions. The extremal (U-)epis in (**Set**,id) (resp. (**Top**,U)) are just the surjective maps (resp. surjective maps into discrete spaces). Thus, the mono-universal initial completion of (**Set**,id) as well as of (**Top**,U) is the category of all pairs (X,R), where R is an equivalence relation on the set X, and of all equivalence relation preserving maps.

(b) More general: For algebraic (= regular in the sense of [5], 2.1) categories (A,U), the mono-universal initial completion of (A,U) is (up to equivalence) the category of all pairs (X,r) where X is an X-object and r: FX→A is a regular epimorphism in A with FX the U-free object with base X, and the obvious morphisms. For example, the mono-universal initial completion of the concrete category (over **Set**) of all groups and homomorphisms is the category of all pairs (X,N), where X is a set and N is a normal subgroup of the free group with base X, together with the obvious morphisms.

(c) The mono-universal initial completion of the essentially algebraic (but non-algebraic) concrete category **Cat** over **Set** of all small categories and functors between them (cf. [10], IV) is (up to equivalence) the category of all maps $e: X \to A$, where $e[X]$ *generates* the small category A, i.e., every identity in A is a domain- or codomain-identity of some member of $e[X]$ and every non-identity member of A belongs to the compositive hull of $e[X]$ in A. This completion cannot be obtained by the category of all pairs (X,r) defined in (b).

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