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J. BARJA
C. RODRIGUEZ

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HOMOLOGY GROUPS $H_n^q(-)$ AND EIGHT-TERM EXACT SEQUENCES

by *J. BARJA and C. RODRIGUEZ*

RÉSUMÉ. Etant donné deux sous-groupes normaux N et M d'un groupe G tels que $N \cdot M = G$, on obtient dans cet article une longue suite exacte d'homologie à coefficients dans $\mathbb{Z}_q$:

$$\cdots \to H_{n+1}^q(G) \to H_{n+1}^q(G/N) \oplus H_{n+1}^q(G/M) \to L_{n-1}\vartheta_2(\alpha,\gamma) \to H_n^q(G) \to \cdots \to L_0\vartheta_2(\alpha,\gamma) \to H_1^q(G) \to H_1^q(G/N) \oplus H_1^q(G/M) \to 0.$$

On donne une description explicite des 8 derniers termes à l'aide de présentations libres. En particulier, si $q = 0$, on obtient $H_2(G)$ et $H_3(G)$.

Pour $M = G$, cette suite se réduit à la longue suite exacte d'homologie associée à un homomorphisme surjectif de groupes $G \to G/N$.

1.

Several authors have obtained an eight-term exact sequence of homology

$$H_3(G) \to H_3(Q) \to V \to H_2(G) \to H_2(Q) \to N/[N,G] \to H_1(G) \to H_1(Q) \to 0$$

from a short exact sequence of groups $1 \to N \to G \to Q \to 1$, the term V varying from author to author.

In this way, Eckmann & Hilton obtain an extension of this sequence to ten terms, associated to a central short sequence of groups. Eckmann, Hilton & Stambach in 1972 obtain an eight-term sequence for a central stem short sequence in [8], and drop the stem character in [9]. In the same direction we have the papers of Gut [14] and Gut & Stambach [15].

Brown & Loday [5] obtain an eight-term sequence, for integer homology, associated to two normal subgroups M and N of a group G. This sequence, for $M = G$, reduces to the one given above. Ellis [11] gives an algebraic proof of the sequence of [5].

The main purpose of this paper is to get two long exact sequences of homology with coefficients in $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$, one associated with a surjective group homomorphism $\alpha: G \rightarrow Q$ (i.e., a group G and a normal subgroup $N = \text{Ker}\,\alpha$), and the other one associated to a group G and two normal subgroups N and M such that $N \cdot M = G$. Furthermore, interpretations of the last eight terms of these sequences by means of free presentations are given.

The method consists, basically, in reducing the degree of the derived functor by changing the functor or even the domain category, and using simplicial techniques for deriving functors. The basic results employed can be found in [1,18,21,22].

Similar results have been obtained by Brown [2] and Brown & Loday [5], for integral homology, using algebraic and topological techniques, which do not apply here and as yet their methods have not yielded results on homology with non-integer coefficients. It would be interesting to establish the relationship between our results (for $q = 0$) and those of [5,6]. It seems a "very interesting and challenging technical problem". In the last section we give a partial approximation ($M = G$).

2.

In this section we recall several concepts and results about the homotopy of Kan complexes and the homology of their associated Moore complexes. A detailed exposition can be found in [20] or [18].

DEFINITION 2.1. A simplicial set **X** is said to be a *Kan complex* if for every collection of n+1 n-simplices $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1}$ that satisfies $d_i x_j = d_{j-1} x_i$ if $k \neq i < j \neq k$ ($x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1}$ are called compatible), there exists an n+1-simplex $x \in X_{n+1}$ such that $d_i x = x_i$ if $i \neq k$.

DEFINITION 2.2. Let **X** be a Kan complex and $x, x' \in X_n$ ($n \geq 0$): x and x' will be *homotopic* if there exists a $y \in X_{n+1}$ such that

$$d_i(y) = s_{n-1} d_i(x) = s_{n-1} d_i(x') \text{ (for } i = 0,1,\ldots,n-1),$$
$$d_n(y) = x \text{ and } d_{n+1}(y) = x'.$$

We will say that y is a homotopy from x to x' and will denote this as $y: x \sim x'$.

Note that from this definition it follows that two homotopic simplices x and x' have the same boundary $d(x) = d(x')$. "$\sim$" is an equivalence relation in X_n, for $n \geq 0$.

DEFINITION 2.3. A vertex $* \in X_0$ generates a subcomplex of $\mathbf{X}$ which has in each dimension exactly one simplex $s_{n-1} \cdots s_1 \cdot s_0(*)$ which will also be called $*$ as well as the subcomplex that it generates in $\mathbf{X}$. A pointed Kan complex $(\mathbf{X},*)$ is a Kan complex $\mathbf{X}$ with a chosen vertex $*$.

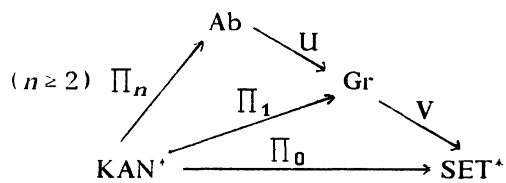
DEFINITION 2.4. For a pointed Kan complex $(\mathbf{X},*)$ we define:

$$\overline{X}_n = \{x \in X_n \mid dx = (*,\ldots,*)\} \text{ and}$$

$$\prod_n(\mathbf{X},*) = \overline{X}_n/\sim \ (n \geq 1).\ \prod_0(\mathbf{X},*) = X_0/\sim.$$

$\prod_n(\mathbf{X},*)$ is a group if $n \geq 1$. which is abelian if $n \geq 2$. $\prod_0(\mathbf{X},*)$ will be referred to as the *component set of* $\mathbf{X}$.

In an obvious manner the $\prod_n$ are functors from the category of pointed Kan complexes. KAN^*. to the category of pointed sets SET^*. More precisely:



DEFINITION 2.5. A simplicial map p: $\mathbf{X} \to \mathbf{Y}$ will be called a *fibration* if for each compatible collection $(x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1})$ of n+1 n-simplices of $\mathbf{X}$ and for each $y \in Y_{n+1}$ such that $d_i y = p x_i$ $(i \neq k)$, there exists an $x \in X_{n+1}$ such that

$$d_i x = x_i \ (i \neq k) \text{ and } px = y\,.$$

Note that the case $\mathbf{Y} = *$ yields the definition of a Kan complex $\mathbf{X}$. When we choose a base point $*$ in $\mathbf{Y}$, then $\mathbf{F} = p^{-1}(*)$ will be called the fiber of the fibration. The simplicial set $\mathbf{Y}$ is called the base complex. $\mathbf{X}$ the total complex.

PROPOSITION 2.6. *(1) The fiber of a fibration is a Kan complex.*

(2) The base complex of a surjective fibration is a Kan complex if the total complex is.

(3) If the base complex of a fibration is a Kan complex. then so is the total space.

PROPOSITION 2.7. *(Long exact sequence of a fibration). Let p:* $\mathbf{X} \to \mathbf{Y}$ *be a fibration.* $*$ *a vertex of* $\mathbf{Y}$, $\mathbf{F} = p^{-1}(*)$ *the fiber.* $* \in \mathbf{F}$

$\subset$ **X** *a vertex of* **F**. *Then the following sequence is exact*

$$\cdots \to \Pi_n(\mathbf{F},*) \to \Pi_n(\mathbf{X},*) \to \Pi_n(\mathbf{Y},*) \to \Pi_{n-1}(\mathbf{F},*) \to$$
$$\cdots \to \Pi_1(\mathbf{Y},*) \to \Pi_0(\mathbf{F},*) \to \Pi_0(\mathbf{X},*) \to \Pi_0(\mathbf{Y},*) \to *.$$

PROPOSITION 2.8. *Every simplicial group is a Kan complex and every surjective simplicial group homomorphism is a fibration.*

DEFINITION 2.9. From a simplicial group **G** we can derive a chain complex **MG**, the Moore chain complex, as follows:

$$(\mathbf{MG})_0 = G_0, \ (\mathbf{MG})_n = \bigcap_{i=1}^{n} \operatorname{Ker} d_i, \ n \geq 1.$$

Since for $x \in (\mathbf{MG})_{n+1}$ we have

$$d_i d_{n+1} x = d_n d_i x = 1, \ i = 0, \dots, n,$$

the map d_0 can be restricted to a map $d: (\mathbf{MG})_{n+1} \to (\mathbf{MG})_n$ which we take as the boundary map of the complex.

PROPOSITION 2.10. *The homotopy groups of a simplicial group coincide with the homology groups of its Moore chain complex* $\Pi_n(\mathbf{G}) = H_n(\mathbf{MG})$.

REMARK 2.11. Keune in [18] introduces the Moore complex **M'G** associated to a simplicial group as follows:

$$(\mathbf{M'G})_0 = G_0, \ (\mathbf{M'G})_n = \bigcap_{i=0}^{n-1} \operatorname{Ker} d_i, \ n \geq 1,$$

the boundary map $(\mathbf{M'G})_{n+1} \to (\mathbf{M'G})_n$ being the restriction of d_{n+1}.

In [23] it has been proved that $H_n(\mathbf{M'G}) = H_n(\mathbf{MG})$.

3.

This section is dedicated to the introduction of the categories of Rinehart and of the simplicial method to derive functors from categories of Rinehart to the category of groups. An exhaustive study of these topics can be found in [21, 22].

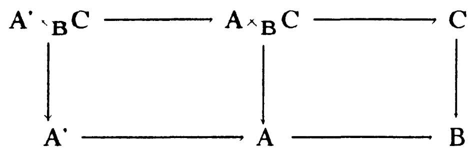
DEFINITION 3.1. Let **C** be a category and ε a projective class of epimorphisms of **C** (**C** is assumed to possess sufficient ε-projectives). The pair $(\mathbf{C}, \varepsilon)$ is called a *base category* if the following statements hold:

B.1. If $A \to B \in \varepsilon$ and $C \to B$ is any morphism in **C**, then **C** contains the fiber product $A \times_B C$.

B.2. ε is closed under composition.

B.3. If the composition $A \to B \to C \in \varepsilon$, then $B \to C \in \varepsilon$.

B.4. If $A' \to A$, $A \to B$ and $C \to B$ are morphisms of **C** such that $A' \to A \to B \in \varepsilon$, then $A' \times_B C \to A \times_B C \in \varepsilon$ iff $A' \to A \in \varepsilon$.

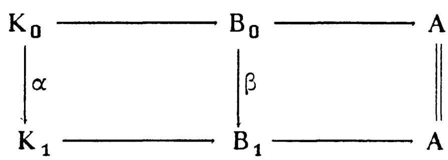


DEFINITION 3.2. A *Rinehart category* is a base category $(\mathbf{C}, \varepsilon)$ in which the following statements hold:

R.1. If **P** is the class of ε-projectives, a finite number of elements of **P** have a coproduct in **C**.

R.2. The morphisms of ε have their kernels in **C**.

R.3. For any commutative diagram



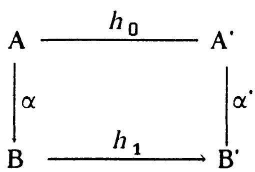
with $B_i \to A \in \varepsilon$ and K_i the kernel of $B_i \to A$, $\alpha \in \varepsilon$ iff $\beta \in \varepsilon$.

REMARK 3.3. Let **C** be an algebraic category with a zero object, and let ε be the class of surjective morphisms. Then $(\mathbf{C}, \varepsilon)$ is a category of Rinehart.

DEFINITION 3.4. Let $(\mathbf{C}, \varepsilon)$ be a base category. The category $\mathbf{C}_1$ is defined as follows:

The class of objects of $\mathbf{C}_1$ is the class ε.

Let $\alpha: A \to B$ and $\alpha': A' \to B'$ be objects in $\mathbf{C}_1$. A $\mathbf{C}_1$-morphism $h: \alpha \to \alpha'$ is a pair (h_0, h_1) of **C**-morphisms such that the square



is commutative.

Let ε_1 be the class of $\mathbf{C}_1$-morphisms such that h_1 and $(h_0, \alpha): A \to A' \times_B B$ are morphisms of ε. Inductively we define $(\mathbf{C}_n, \varepsilon_n)$.

PROPOSITION 3.5. *If* $(\mathbf{C}, \varepsilon)$ *is a base category, then so is* $(\mathbf{C}_n, \varepsilon_n)$ *and if* $(\mathbf{C}, \varepsilon)$ *is a Rinehart category, then* $(\mathbf{C}_n, \varepsilon_n)$ *is a Rinehart category.*

REMARK 3.6. We denote by $\mathbf{Gr}_n$, $n \geq 0$, the category of Rinehart $(\mathbf{Gr}_n, \varepsilon_n)$ introduced in Definition 3.4, starting with the Rinehart category $(\mathbf{Gr}, \varepsilon)$ (see Remark 3.3).

PROPOSITION 3.7. *Let* $(\mathbf{C}, \varepsilon)$ *be a Rinehart category. Then* $\mathbf{P}_n$, *the class of* ε_n*-projectives in* $(\mathbf{C}_n, \varepsilon_n)$ *is given by*

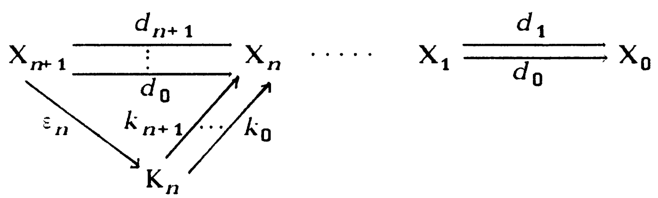
$$\mathbf{P}_n = \{P_0 \to P_1 \in \varepsilon_{n-1} \text{ such that } P_0, P_1 \in \mathbf{P}_{n-1}\}.$$

DEFINITION 3.8. Let $(\mathbf{C}, \varepsilon)$ be a Rinehart category, $\mathbf{D}$ a category with kernels and $F: \mathbf{C} \to \mathbf{D}$ a functor. For $n \geq 0$, $F_n: \mathbf{C}_n \to \mathbf{D}$ is defined as follows:

$$F_0 = F, \quad F_1(\alpha) = \operatorname{Ker} F_0, \quad F_1(h_0, h_1) = F_0(h_0)|_{F_1(\alpha)},$$
$$F_n = (F_{n-1})_1 \quad (n \geq 2)$$

where α is an object of $\mathbf{C}_1$ and $h = (h_0, h_1): \alpha \to \alpha'$ is a $\mathbf{C}_1$-morphism.

DEFINITION 3.9. Let ε be a class of epimorphic maps in a category $\mathbf{C}$ (with simplicial kernels) and let $\mathbf{X}$ be a (semi)-simplicial object. If in the diagram below



all the ε_n are in ε, where ε_n denotes the factorization through K_n, the simplicial kernel of $d_i: X_n \to X_{n-1}$, $i = 0, \ldots, n$, then we say that the (semi-)simplicial object $\mathbf{X}$ is *exact.*

DEFINITION 3.10. Let $(\mathbf{C}, \varepsilon)$ be a category of Rinehart, and A an object of $\mathbf{C}$. An ε-exact (semi-)simplicial object $(\mathbf{X}, d)$, with the X_n ε-projective $(n \geq 0)$ together with a morphism

$$d_0: X_0 \to A, \quad d_0 \in \varepsilon, \text{ such that } d_0 d_1 = d_0 d_0$$

will be called a *(semi-)simplicial* ε*-resolution of* A.

If X_n is not ε-projective, then we will say that $(\mathbf{X}, d)$ is an ε-exact augmented (semi-)simplicial object of A.

PROPOSITION 3.11. *Let* $(\mathbf{C},\varepsilon)$ *be a Rinehart category. Then every object* A *of* $\mathbf{C}$ *has a simplicial* ε*-resolution.*

PROOF. We give here an explicit description of the first two terms of this resolution, which will be used later. Modi [21, page 64] provides a complete description of a modified form of the procedure developed by André [1].

Let $d_0: X_0 \to A$ be an ε-projective presentation of A, $R_1 = \operatorname{Ker} d_0$ and $\omega: X_1 \to R_1$ an ε-projective presentation of R_1. Then the diagram

$$X_0 * X_1 \underset{d_1 = \binom{1}{0}}{\overset{d_0 = \binom{1}{\omega}}{\rightrightarrows}} X_0 \xrightarrow{\ d_0\ } A$$

holds, where $X_0 * X_1$ denotes the free product (coproduct) of X_0 and X_1.

DEFINITION 3.12. Let $(\mathbf{C},\varepsilon)$ be a Rinehart category and $F:\mathbf{C}\to \mathrm{Gr}$ a functor such that $F(0) = 0$. If A and A' are objects in $\mathbf{C}$, d: $\mathbf{X}\to A$ is a simplicial ε-resolution and $f: A\to A'$ is a morphism in $\mathbf{C}$, then the derived functors of F, $L_n^{\varepsilon}F$, are defined for $n \geq 0$ by

$$L_n^{\varepsilon}F(A) = \Pi_n(F\mathbf{X}),\ L_n^{\varepsilon}F(f) = \Pi_n(F\bar{f})$$

where $(F\mathbf{X})$ is the simplicial image of $\mathbf{X}$ in Gr, Π_n is the n^{th} homotopy group and $\bar{f}$ a lifting of f to the simplicial resolutions of A and A'. The simplicial comparison theorem [18, page 44] ensures the validity of this definition.

REMARK 3.13. From Proposition 2.10, $\Pi_n(F\mathbf{X}) = H_n(\mathbf{M}F\mathbf{X})$.

PROPOSITION 3.14. *Let* $(\mathbf{C},\varepsilon)$ *be a category of Rinehart, and* $F' \to F \to F''$ *a sequence of zero preserving functors from* $\mathbf{C}$ *to* Gr, *exact on* ε*-projectives of* $\mathbf{C}$, *i.e., we have an exact sequence*

$$0 \to F'(P) \to F(P) \to F''(P) \to 0.$$

in Gr, *for each* ε*-projective object* P *of* $\mathbf{C}$. *Then, for each object* A *of* $\mathbf{C}$, *there is a natural exact sequence of groups*

$$\cdots \to L_n^{\varepsilon}F'(A) \to L_n^{\varepsilon}F(A) \to L_n^{\varepsilon}F''(A) \to L_{n-1}^{\varepsilon}F'(A) \to \cdots \to 0.$$

PROOF. Apply Proposition 2.7.

PROPOSITION 3.15. *Let* $(\mathbf{C},\varepsilon)$ *be a Rinehart category,* $F:\mathbf{C}\to\mathrm{Gr}$ *a* *functor such that* $F(0)=0$. *Then the derived functors* $L_n^{\varepsilon}F_m\colon \mathbf{C}_m \to \mathrm{Gr}$ $(n, m \geq 0)$ *are characterized by the following properties:*

i) $L_n^{\varepsilon} F_m(P) = F_m(P)$, *for every* ε_m*-projective* P.

ii) $L_0^{\varepsilon}(L_0^{\varepsilon} F_m) = L_0^{\varepsilon} F_m$ $(m \geq 0)$.

iii) $L_n^{\varepsilon} F_m(P) = 0$ *for every* ε_m*-projective* P $(n \geq 1,\ m \geq 0)$.

iv) If $\alpha\colon A \to B \in \varepsilon_m$ $(m \geq 0)$, *then there exists a long and natural exact sequence*

$$\cdots \to L_n^{\varepsilon} F_{m+1}(\alpha) \to L_n^{\varepsilon} F_m(A) \to L_n^{\varepsilon} F_m(B) \to L_{n-1}^{\varepsilon} F_{m+1}(\alpha) \to \cdots \to L_0^{\varepsilon} F_m(B) \to 0.$$

4.

In this section we obtain the exact sequence for homology with coefficients in $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$, associated with two normal subgroups M and N of G such that $M \cdot N = G$.

Let $\mathfrak{V}$ be a variety of groups. We denote by V(G) the verbal subgroup of a group G with respect to $\mathfrak{V}$ and consider the functors $V\colon \mathrm{Gr} \to \mathrm{Gr}$ which take G to V(G), and $\mathfrak{V}\colon \mathrm{Gr} \to \mathrm{Gr}$ taking G to G/V(G). With this notation, the derived functors $L_n V_m$ and $L_n \mathfrak{V}_m$ are also defined for $n, m \geq 0$. (Definition 3.12).

PROPOSITION 4.1. *Let* M *and* N *be two normal subgroups of a group* G, $L = M \cap N$ *and* $\mathfrak{V}$ *a variety of groups. Now consider the following object in the Rinehart category* Gr_2

$$\begin{array}{ccc} G & \xrightarrow{\ \alpha\ } & G/N \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\delta} \\ G/M & \xrightarrow{\ \gamma\ } & G/MN \end{array}$$

Then we obtain a commutative diagram with exact rows and columns (Figure 1).

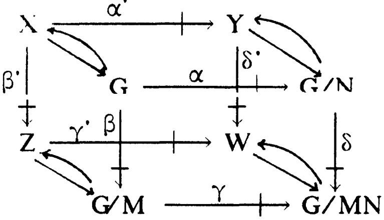
PROOF. It is a consequence of Proposition 3.15.

PROPOSITION 4.2. *On the hypothesis of Proposition 4.1, we have: For each* $x \in L_n\mathfrak{V}(G)$ *such that* $\gamma_n'\beta_n'(x) = 1$, *there exist elements* $y \in L_n\mathfrak{V}_1(\delta)$ *and* $z \in L_n\mathfrak{V}_1(\gamma)$ *such that*

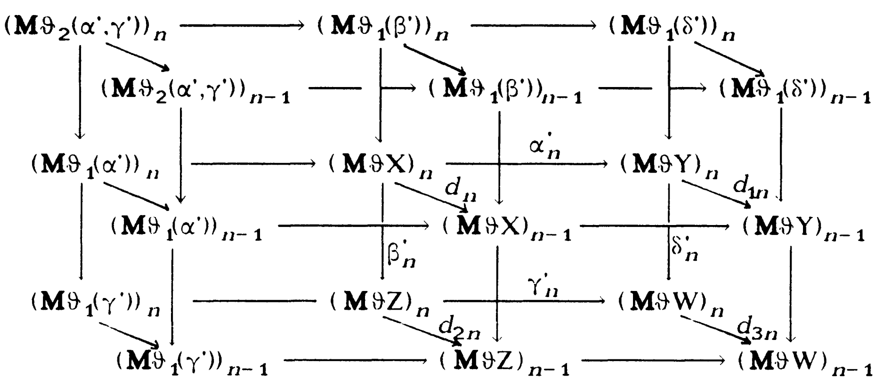
$$\lambda_n(y) = \alpha_n'(x),\ \mu_n(z) = \beta_n'(x),\ \Theta_{n-1}(y)\eta_{n-1}(z) = 1.$$

FIGURE 1

PROOF. Let

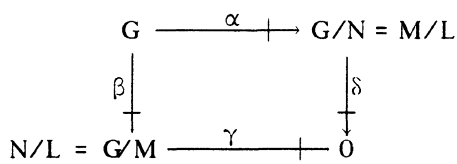


be a simplicial ε_2-resolution of (α,γ) in Gr_2 (Definition 3.10). If $\mathbf{M}\vartheta X$ denotes the Moore chain complex associated with the simplicial group ϑX (similarly $\mathbf{M}\vartheta_1$ and $\mathbf{M}\vartheta_2$), then the proof is a simple diagram chase using connecting homomorphism in the following exact diagram of Moore chain complex:



REMARK 4.3. In the particular situation that $M\cdot N = G$, since $L_n\vartheta(G/M\cdot N) = 1$, μ_n and λ_n are isomorphisms and we can assure that, for every $x \in L_n\vartheta(G)$, $\Theta_{n-1}\alpha_n'(x)\cdot\eta_{n-1}\beta_n'(x) = 1$, $n \geq 1$ (where λ_n^{-1} and μ_n^{-1} have been omitted for simplicity).

PROPOSITION 4.4. *Let* M *and* N *be two normal subgroups of a group* G *such that* $M\cdot N = G$. *Let* $L = M\cap N$, (α,γ) *the object of* Gr_2



and H_n^q *denotes the n-th homology group* G *with coefficients in* $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$. *Then, if* ϑ *is a variety of Abelian groups of exponent* q, *there exist a long exact sequence:*

$$\cdots \to H^q_{n+1}(G) \to H^q_{n+1}(M/L)\oplus H^q_{n+1}(N/L) \to L_{n-1}\vartheta_2(\alpha,\gamma) \to H^q_n(G) \to \cdots \to H^q_3(G) \to H^q_3(M/L)\oplus H^q_3(N/L) \to L_1\vartheta_2(\alpha,\gamma) \to H^q_2(G) \to H^q_2(M/L)\oplus H^q_2(N/L) \to L_0\vartheta_2(\alpha,\gamma) \to H^q_1(G) \to H^q_1(M/L)\oplus H^q_1(N/L) \to 0.$$

PROOF. Diagram chase on Figure 1 and application of Remark 4.3 yield a long exact sequence:

$$\cdots \to L_n\vartheta(G) \xrightarrow{(\alpha'_n,\beta'_n)} L_n\vartheta(M/L)\oplus L_n\vartheta(N/L) \xrightarrow{\binom{\Theta_{n-1}}{\eta_{n-1}}} L_{n-1}\vartheta_2(\alpha,\gamma) \xrightarrow{\Delta_{n-1}} L_{n-1}\vartheta(G) \to L_0\vartheta(M/L)\oplus L_0\vartheta(N/L) \to 0.$$

The isomorphisms $H^q_{n+1}(G) = L_n\vartheta(G)$, for $n \geq 0$, give the result.

5.

This section is dedicated to the computation of the groups $L_1\vartheta_2(-,-)$ and $L_0\vartheta_2(-,-)$ to give an interpretation of the last eight terms of the exact sequence of homology of Section 4 in terms of a free presentation of G.

PROPOSITION 5.1. *Let ϑ be the variety of Abelian groups of exponent q, V the verbal subgroup functor, I the identity functor of Gr and (α,γ) the object in the Rinehart category Gr_2. Then, since $1 \to V_n \to I_n \to \vartheta_n \to 1$ is a sequence of functors, $Gr_n \to Gr$, exact over ε_n-projectives, $n \geq 0$, we get an exact sequence of derived functors*

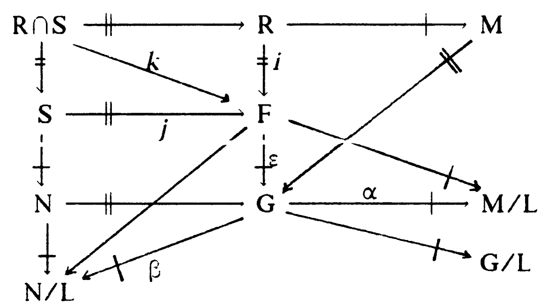
$$\cdots \to L_m\vartheta_n \to L_{m-1}V_n \to L_{m-1}I_n \to L_{m-1}\vartheta_n \to \cdots$$

Since $L_m I_n = \begin{cases} I_n & \text{if } m = 0 \\ 1 & \text{if } m \geq 1 \end{cases}$ we obtain, for $n = 2$,

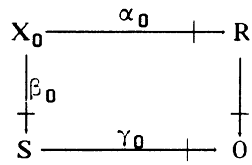
$$L_1\vartheta_2(\alpha,\gamma) = \operatorname{Ker}(L_0V_2(\alpha,\gamma)\to I_2(\alpha,\gamma)),$$
$$L_0\vartheta_2(\alpha,\gamma) = \operatorname{Coker}(L_0V_2(\alpha,\gamma)\to I_2(\alpha,\gamma)).$$

Consequently the calculation of $L_0V_2(\alpha,\gamma)$ leads to $L_1\vartheta_2(\alpha,\gamma)$ and $L_0\vartheta_2(\alpha,\gamma)$.

Next, consider M, N two normal subgroups of a group G such that $M\cdot N = G$. Put $L = M\cap N$. Let $\varepsilon: F \twoheadrightarrow G$ be a free presentation of G, R the kernel of the composition morphism $\beta\varepsilon: F\to G\to N/L$ and S the kernel of $\alpha\varepsilon: F\to G\to M/L$. Thus we have a diagram



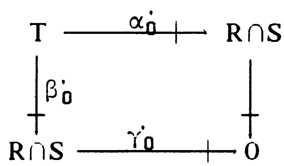
Given two groups A and B. A∗B will denote the free product (coproduct) of A and B. Now consider the following object in Gr_2:

$(\alpha_0;\gamma_0)=$ 

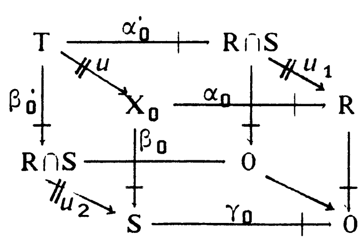
where $X_0 = (R\cap S)' * R * S$, $(R\cap S)'$ being an isomorphic copy of $(R\cap S)$ and α_0 and β_0 are the projections killing the first and last, and the first and second cofactors, respectively, so

$$\alpha_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \beta_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

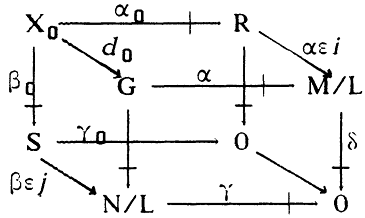
If we write d_0 for the morphism $\begin{pmatrix} \varepsilon k \\ \varepsilon i \\ \varepsilon j \end{pmatrix}: X_0 \to G$ and T its kernel then there is an object

$(\alpha_0';\gamma_0') =$ 

obtained by restricting a_0 and β_0 to T and an obvious map $\underline{u}$: $(\alpha_0';\gamma_0')\to(\alpha_0;\gamma_0)$ in Gr_2 that is

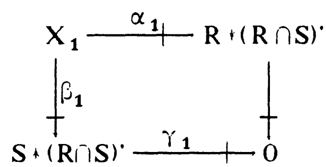


and an epimorphism $(\alpha_0 . \gamma_0) \to (\alpha . \gamma)$ that is

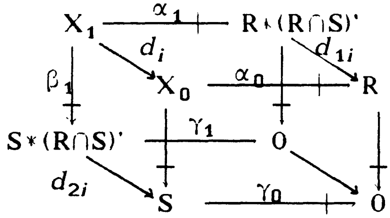


Similarly we have the object in Gr_2:

$(\alpha_1 \cdot \gamma_1) =$



where $X_1 = X_0 + T'$. T' being an isomorphic copy of T. $\alpha_1 = \alpha_0 + \alpha'_0$ and $\beta_1 = \beta_0 + \beta'_0$ and maps $\underline{d}_i$: $(\alpha_1 . \gamma_1) \to (\alpha_0 . \gamma_0)$ $(i = 0.1)$: this means that



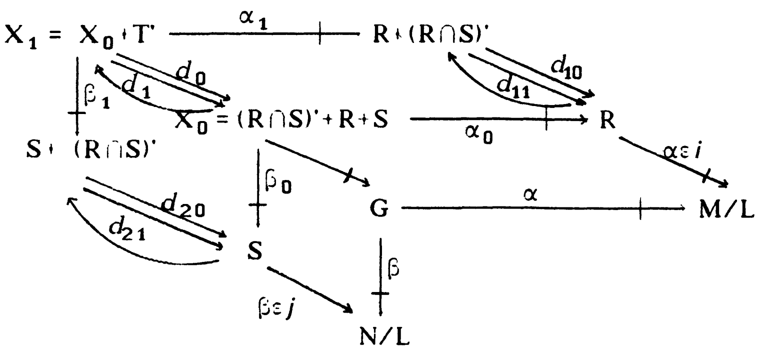
where

$$d_0 = \binom{1}{u}. \quad d_1 = \binom{1}{0}. \quad d_{10} = \binom{1}{u_1}. \quad d_{11} = \binom{1}{0}. \quad d_{20} = \binom{1}{u_2}. \quad d_{21} = \binom{1}{0}$$

Then we have the diagram

$$(\alpha_1 . \gamma_1) \rightrightarrows (\alpha_0 . \gamma_0) \longrightarrow (\alpha . \gamma)$$

that is



PROPOSITION 5.2. *With the above notation:*

i) $(\alpha_0', \gamma_0') \to (\alpha_0, \gamma_0) \to (\alpha, \gamma)$ *is a projective presentation in* Gr_2.

ii)

$$(\alpha_1, \gamma_1) \rightrightarrows (\alpha_0, \gamma_0) \longrightarrow (\alpha, \gamma)$$

are the first two terms of a simplicial resolution of (α, γ) *in the Rinehart category* Gr_2.

PROOF. i) (α_0, γ_0) is an ε-projective object (Proposition 3.7). Since $(\alpha_0, g_0) \to (\alpha, \gamma) \in \varepsilon_2$ (Definition 3.4) and $(\alpha_0', \gamma_0') \to (\alpha_0, \gamma_0)$ is the kernel, the result follows.

ii) It follows from Proposition 3.11.

DEFINITION 5.3. Let M and N be two subgroups of a group G and $q \in \mathbb{N}$. We use the symbol $M \#_q N$ to denote the subgroup of G generated by

$$[m, n] t^q = m n m^{-1} n^{-1} t^q, \quad m \in M, \ n \in N, \ t \in M \cap N.$$

Note that if M and N are normal subgroups of G, so is $M \#_q N$.

If N is a subgroup of G, then $N \#_q G$ is the subgroup of G defined by Stammbach in [25, page 2]: and clearly $G \#_q G$ is the verbal subgroup of G for the variety of Abelian groups of exponent q.

LEMMA 5.4. *Let* A *and* B *two groups and* $q \in \mathbb{N}$. *Then, if* B^{A*B} *denotes the smallest normal subgroup of* $A*B$ *containing* B *then*

a) $[B^{A*B}, A*B]$ *is the subgroup of* $A*B$ *generated by*

$$\{[b,a],[b_1,b_2] \mid b, b_1, b_2 \in B, \ a \in A\}.$$

b) $(A*B) \#_q B^{A*B}$ *is the subgroup of* $A*B$ *generated by*

$$\{[b,a],[b_1,b_2], \ b^q \mid b, b_1, b_2 \in B, \ a \in A\}.$$

PROOF. a) The result follows from $[B^{A*B}, A*B] = [B, A*B]$,

$$[b, \prod_1^n a_i b_i] = [b, \prod_1^{n-1} a_i \, b_i] \cdot [b, a_n b_n]^{\prod_1^{n-1} a_i b_i} =$$

$$[b, \prod_1^{n-1}{}_i a_i b\,] \cdot [b, a_n]^{\prod_1^{n-1} a_i b_i} \cdot [b, b_n]^{(\prod_1^{n-1} a_i b_i) \cdot a_n}$$

and

$$[b,a]^{b_1} = [b_1 b, a] \cdot [a, b_1], \qquad [b,a]^{a_1} = [a_1, b] \cdot [b, a_1 a],$$

$$[b, b_1]^a = [a, [b, b_1]] \cdot [b, b_1] \text{ and } [b_1, b_2]^b = [b_1^b, b_2^b].$$

b) $(A*B)\#_q B^{A*B}$ is the smallest subgroup of $A*B$ containing $[A*B, B^{A*B}]$ and the set $\{x^q \mid x \in B^{A*B}\}$. The result follows from (a) and the equalities

$$(b^a)^q = (b^q)^a = [a, b^q]\cdot b^q.$$

LEMMA 5.5. *Let* A *and* B *be two groups and* $q \in \mathbb{N}$. *Then:*

$$A^{A*B} \cap ((A*B)\#_q(A*B)) = (A*B)\#_q A^{A*B}.$$

PROOF. Clearly, the left side contains the right side. Conversely, let

$$\beta: A*B \xrightarrow{\binom{0}{1}} B \xrightarrow{\mu} A*B$$

μ being the inclusion homomorphism of B in the free product. Let $x_i, y_i, z_i \in A*B$, with $\prod_1^n [x_i, y_i] z_i^q \in A^{A*B}$; then

$$\beta(\prod_1^n [x_i, y_i] z_i^q) = 1$$

and

$$\prod [x_i, y_i] z_i^q = \prod [x_i, y_i] z_i^q \cdot (\prod [\beta x_i, \beta y_i] \beta z_i^q)^{-1} =$$

$$\prod [\beta x_i (\beta x_i)^{-1} x_i, \beta y_i (\beta y_i)^{-1} y_i] (\beta z_i (\beta z_i)^{-1} z_i)^q \cdot (\prod [\beta x_i, \beta y_i] \beta z_i^q)^{-1}$$

$\equiv 1 \bmod (A*B)\#_q A^{A*B}$, because $(\beta x_i)^{-1} x_i \in \operatorname{Ker}\beta = A^{A*B}$.

Note that if $q = 0$, we get

$$A^{A*B} \cap [A*B, A*B] = [A^{A*B}, A*B].$$

It should be emphasized that these results are true for any variety (not only Abelian groups and Abelian groups of exponent q) as shown by Modi [21, page 109], under the unneeded hypothesis of A and B being free groups.

PROPOSITION 5.6. *With the notations of 5.2, if* $A = (R\cap S)'*S$, $B = (R\cap S)'*R$, $\mathfrak{B}$ *is the variety of Abelian groups of exponent* q *and* V *is its verbal subgroup functor, then*

$$L_0V_2(\alpha,\gamma) = \frac{(X_0\#_q A^{X_0}) \cap (X_0\#_q B^{X_0})}{T\#_q(A^{X_0}\cap B^{X_0}))((T\cap A^{X_0})\#_q B^{X_0})((T\cap B^{X_0})\#_q A^{X_0})}$$

$$= \frac{(X_0\#_q A^{X_0}) \cap B^{X_0}}{(T\#_q(A^{X_0}\cap B^{X_0}))[(T\cap A^{X_0}), B^{X_0}][(T\cap B^{X_0}), A^{X_0}]}.$$

PROOF. Application of the functor $V_2: Gr_2 \to Gr$ to the simplicial resolution of (α,γ) yields the simplicial group :

$$\cdots \; V_2\left(\begin{array}{c}\xrightarrow{\alpha_1}\\ \downarrow\beta_1 \quad \downarrow\\ \longrightarrow 0\end{array}\right) \; \underset{V_2(d_1,d_{11},d_{21},0)}{\overset{V_2(d_0,d_{10},d_{20},0)}{\rightrightarrows}} \; V_2\left(\begin{array}{c}\xrightarrow{\alpha_0}\\ \downarrow\beta_0 \quad \downarrow\\ \longrightarrow 0\end{array}\right)$$

The Moore chain complex associated with this simplicial group is

$$\cdots \to \mathbf{M}_1 = \operatorname{Ker}\left(V_2(d_1,d_{11},d_{21},0) \to \mathbf{M}_0 = V_2\left(\begin{array}{c}\longrightarrow\\ \downarrow\beta_0 \quad \downarrow\\ \longrightarrow 0\end{array}\right)\right),$$

and the homology group of this complex in $\mathbf{M}_0$ is the group $L_0V_2(\alpha,\gamma)$. Calculation of the groups $\mathbf{M}_0$ and $\mathbf{M}_1$ shows that

$$\begin{aligned}\mathbf{M}_0 &= \operatorname{Ker}(V\alpha_0)\cap\operatorname{Ker}(V\beta_0) = VX_0\cap\operatorname{Ker}\alpha_0\cap\operatorname{Ker}\beta_0\\ &= VX_0\cap((R\cap S)'+S)^{X_0}\cap((R\cap S)'+R)^{X_0}.\end{aligned}$$

$$\begin{aligned}\mathbf{M}_1 &= VX_1\cap\operatorname{Ker}\alpha_1\cap\operatorname{Ker}\beta_1\cap\operatorname{Ker}d_1\\ &= V(((R\cap S)'+R+S)+T')\cap\operatorname{Ker}\alpha_1\cap\operatorname{Ker}\beta_1\cap\operatorname{Ker}d_1.\end{aligned}$$

Thus

$$L_0V_2(\alpha,\gamma) = \frac{VX_0\cap((R\cap S)'+S)^{X_0}\cap((R\cap S)'+R)^{X_0}}{d_0(V(((R\cap S)'+R+S)+T')\cap\operatorname{Ker}\alpha_1\cap\operatorname{Ker}\beta_1\cap\operatorname{Ker}d_1)}$$

Furthermore,

$$\begin{aligned}&d_0(V(((R\cap S)'+R+S)+T')\cap\operatorname{Ker}\alpha_1\cap\operatorname{Ker}\beta_1\cap\operatorname{Ker}d_1)\\ &= [T,\operatorname{Ker}\alpha_0\cap\operatorname{Ker}\beta_0]\cdot[T\cap\operatorname{Ker}\alpha_0,\operatorname{Ker}\beta_0]\cdot[T\cap\operatorname{Ker}\beta_0,\operatorname{Ker}\alpha_0]\cdot\\ &\quad\cdot((T\cap\operatorname{Ker}\alpha_0\cap\operatorname{Ker}\beta_0)\#_q(T\cap\operatorname{Ker}\alpha_0\cap\operatorname{Ker}\beta_0)).\end{aligned}$$

In fact, let $[t,x]$ be a generator of any of the first three factors of the right hand side, then

$$d_0[t',x] = [t,x], \text{ where } t'\in T' \text{ and } x\in X_0 \to X_0+T' = X_1.$$

Furthermore, $t\in T\cap\operatorname{Ker}\alpha_0\cap\operatorname{Ker}\beta_0$: then $d_0(t'^q) = t^q$. We thus have an inclusion.

Conversely, since $\operatorname{Ker}d_1 = T^{X_1}$, from Lemma 5.5 we obtain

$$V(((R\cap S)'+R+S)+T')\cap\operatorname{Ker}\alpha_1\cap\operatorname{Ker}\beta_1\cap\operatorname{Ker}d_1) =$$

$$\left(((R\cap S)'+R+S)+T')\#_qT'^{((R\cap S)'+R+S)+T'}\right)\cap\operatorname{Ker}\alpha_1\cap\operatorname{Ker}\beta_1 .$$

If

$$y\in\left(((R\cap S)'+R+S)+T')\#_qT'^{((R\cap S)'+R+S)+T'}\right)\cap\operatorname{Ker}\alpha_1\cap\operatorname{Ker}\beta_1 .$$

then, from Lemma 5.4,

$$y = \prod_{i=1}^{n}[t'_i,x_i]^{\sigma_i}[t'_{1i},t'_{2i}](t'_{3i})^q$$

with

$t_i', t_{1i}', t_{2i}', t_{3i}' \in T', \; \lambda_i \in (R \cap S)' * R * S, \; \sigma_i = \pm 1, \; \alpha_1(y) = 1, \; \beta_1(y) = 1.$

But

$$\alpha_1(y) = \prod [(\alpha_0(t_i))', \alpha_0(\lambda_i)]^{\sigma_i} \cdot [(\alpha_0(t_{1i}))', (\alpha_0(t_{2i}))'] \cdot (\alpha_0(t_{3i}))'^{q}$$

where $(\alpha_0(t))'$ and $\alpha_0(\lambda)$ denote the elements

$$\begin{array}{ccccccc} T & \xrightarrow{\alpha_0} & R \cap S & \twoheadrightarrow & (R \cap S)' & \rightarrowtail & R * (R \cap S)' \\ t & \text{-----} & & & (\alpha_0(t))' & \text{-----} & (\alpha_0(t))' \end{array}$$

and

$$\begin{array}{ccccc} (R \cap S)' * R * S & \longrightarrow & R & \rightarrowtail & R * (R \cap S)' \\ \lambda & \text{-----} & \alpha_0(\lambda) & \text{-----} & \alpha_0(\lambda). \end{array}$$

also

$$\beta_1(y) = \prod [(\beta_0(t_i))', \beta_0(\lambda_i)]^{\sigma_i} \cdot [(\beta_0(t_{1i}))', (\beta_0(t_{2i}))'] \cdot (\beta_0(t_{3i}))'^{q}$$

with $(\beta_0(t))'$ and $\beta_0(\lambda)$ as above.

If we consider the canonical injection

$$(R \cap S)' * R \rightarrowtail (R \cap S)' * R * S$$

we have

$$d_0(y) = d_0(y) \cdot \alpha_1(y) = \prod [t_i, \lambda_i]^{\sigma_i} [t_{1i}, t_{2i}] (t_{3i})^{q} \cdot$$
$$\cdot \prod [(\alpha_0(t_i))', \alpha_0(\lambda_i)]^{\sigma_i} \cdot [(\alpha_0(t_{1i}))', (\alpha_0(t_{2i}))'] \cdot (\alpha_0(t_{3i}))'^{q}.$$

Since $[t_i, \lambda_i]^{\sigma_i}, [t_{1i}, t_{2i}], t_{3i} \in T$ and

$[\alpha_0(t_i))', \alpha_0(\lambda_i)]^{\sigma_i}, [\alpha_0(t_{1i}))', (\alpha_0(t_{2i}))'], \alpha_0(t_{3i}))' \in \mathrm{Ker}\alpha_0 \cap \mathrm{Ker}\beta_0$,

the following congruences mod. $[T, \mathrm{Ker}\alpha_0 \cap \mathrm{Ker}\beta_0]$ hold:

$$\begin{aligned} d_0(y) &\equiv \prod [t_i, \lambda_i]^{\sigma_i} \cdot [(\alpha_0(t_i))', \alpha_0(\lambda_i)]^{\sigma_i} \cdot [t_{1i}, t_{2i}] \cdot \\ &\quad \cdot [(\alpha_0(t_{1i}))', (\alpha_0(t_{2i}))'] \cdot (t_{3i})^{q} \cdot (\alpha_0(t_{3i}))'^{q} \\ &\equiv \prod [t_i, \lambda_i]^{\sigma_i} \cdot [(\alpha_0(t_i))', \alpha_0(\lambda_i)]^{\sigma_i} \cdot [t_{1i}, t_{2i}] \cdot \\ &\quad \cdot [(\alpha_0(t_{1i}))', (\alpha_0(t_{2i}))'] \cdot (t_3 \, (\alpha_0(t_{3i}))')^{q}. \end{aligned}$$

If we consider the homomorphism

$$\begin{array}{ccccccc} T & \xrightarrow{\alpha_0} & R \cap S & \twoheadrightarrow & R & \rightarrowtail & R * (R \cap S)' \\ t & \text{-----} & & & & \text{-----} & \alpha_0(t) \end{array}$$

then, for $t \in T$, $\lambda \in (R \cap S)' * R * S$, we get

$$(\alpha_0(t))', \; \lambda^{-1} \alpha_0(\lambda) \beta_0(\lambda) \in \mathrm{Ker}\alpha_0 \cap \mathrm{Ker}\beta_0,$$

$[t, \beta_0(\lambda)], \; \alpha_0(t)^{-1} (\alpha_0(t))' t \in T \cap \mathrm{Ker}\alpha_0$ and $\alpha_0(\lambda) \in \mathrm{Ker}\beta_0$;

thus, the following congruences

$$\text{mod. } [T, \mathrm{Ker}\alpha_0 \cap \mathrm{Ker}\beta_0] \; [T \cap \mathrm{Ker}\alpha_0, \mathrm{Ker}\beta_0]$$

hold:

$$[t, \lambda] = [t, \alpha_0(\lambda) \beta_0(\lambda) \beta_0(\lambda)^{-1} \alpha_0(\lambda)^{-1} \lambda] \equiv [t, \alpha_0(\lambda) \beta_0(\lambda)],$$

$$[t, \lambda] \cdot [\alpha_0(t))', \alpha_0(\lambda)] \equiv [t, \alpha_0(\lambda) \beta_0(\lambda)] \cdot [(\alpha_0(t))', \alpha_0(\lambda)] =$$

$$= [t.\alpha_0(x)]\cdot[t,\beta_0(x)]^{\alpha_0(x)}\cdot[(\alpha_0(t))', \alpha_0(x)] \equiv$$

$$\equiv [t.\alpha_0(x)]\cdot[\alpha_0(t))'.\alpha_0(x)]\cdot[t.\beta_0(x)] \equiv [(\alpha_0(t))'t.\alpha_0(x)]\cdot[t.\beta_0(x)] =$$

$$= [\alpha_0(t)(\alpha_0(t)^{-1}(\alpha_0(t))'t.\alpha_0(x)]\cdot[t,\beta_0(x)]$$

$$\equiv [\alpha_0(t).\alpha_0(x)]\cdot[t.\beta_0(x)].$$

$$[t_1.t_2]\cdot[(\alpha_0(t_1)'.(\alpha_0(t_2))'] =$$

$$= [t_1,t_2]\cdot[(\alpha_0(t_1)'.\alpha_0(t_2)(\alpha_0(t_2))^{-1}(\alpha_0(t_2))'] =$$

$$= [t_1,t_2]\cdot[(\alpha_0(t_1)'.\alpha_0(t_2)]\cdot[(\alpha_0(t_1))',(\alpha_0(t_2))^{-1}(\alpha_0(t_2))']^{\alpha_0(t_2)} =$$

$$= [(\alpha_0(t_1).\alpha_0(t_2)]\cdot[t_1,\beta_0(t_2)] \equiv [t_1.\beta_0(t_2)]\cdot[\alpha_0(t_1),\alpha_0(t_2)],$$

$$(t(\alpha_0(t))')^q = (t(\alpha_0(t))'\alpha_0(t)^{-1}\alpha_0(t))^q \equiv (t(\alpha_0(t))'\alpha_0(t)^{-1})^q(\alpha_0(t))^q$$

Thus

$$d_0(y) \equiv \prod[\alpha_0(t_i),\alpha_0(x_i)]^{\sigma_i}\cdot[t_i.\beta_0(x_i)]^{\sigma_i}\cdot[t_{1i}.\beta_0(t_{2i})]\cdot$$
$$\cdot[\alpha_0(t_{1i}).\alpha_0(t_{2i})]\cdot(t_{3i}(\alpha_0(t_{3i}))'\alpha_0(t_{3i})^{-1})^q(\alpha_0(t_{3i}))'^q.$$

Since $[t.\beta_0(x)]$, $[t_1,\beta_0(t_2)]$, $t_3(\alpha_0(t_3))'\alpha_0(t_3)^{-1}$) $\in T\cap \mathrm{Ker}\alpha_0$, and $[\alpha_0(t),\alpha_0(x)]$, $[\alpha_0(t_1),\alpha_0(t_2)]$, $\alpha_0(t_3) \in \mathrm{Ker}\beta_0$, we get

$$d_0(y) \equiv \prod[\alpha_0(t_i),\alpha_0(x_i)]^{\sigma_i}\cdot[\alpha_0(t_{1i}),\alpha_0(t_{2i})]\cdot(\alpha_0(t_{3i}))^q\cdot$$
$$\cdot[t_i.\beta_0(x_i)]^{\sigma_i}\cdot[t_{1i}.\beta_0(t_{2i})]\cdot(t_{3i}(\alpha_0(t_{3i}))'\alpha_0(t_{3i})^{-1})^q \equiv$$
$$\equiv \prod[\alpha_0(t_i).\alpha_0(x_i)]^{\sigma_i}\cdot[\alpha_0(t_{1i}),\alpha_0(t_{2i})]\cdot(\alpha_0(t_{3i}))^q\cdot$$
$$\cdot\prod[t_i.\beta_0(x_i)]^{\sigma_i}\cdot[t_{1i}.\beta_0(t_{2i})]\cdot(t_{3i}(\alpha_0(t_{3i}))'\alpha_0(t_{3i})^{-1})^q.$$

But considering the homomorphism

$$\tau_1: R*(R\cap S)' \xrightarrow{\binom{1}{u_1}} R \rightarrowtail (R\cap S)'*R*S.$$

then

$$\prod[\alpha_0(t_i).\alpha_0(x_i)]^{\sigma_i}\cdot[\alpha_0(t_{1i}).\alpha_0(t_{2i})]\cdot(\alpha_0(t_{3i}))^q = \tau_1(\alpha_1(y)) = 1$$

and consequently

$$d_0(y) \equiv \prod[t_i.\beta_0(x_i)]^{\sigma_i}\cdot[t_{1i}.\beta_0(t_{2i})]\cdot(t_{3i}(\alpha_0(t_{3i}))'\alpha_0(t_{3i})^{-1})^q$$
$$\mathrm{mod.}\ [T.\mathrm{Ker}\alpha_0\cap\mathrm{Ker}\beta_0]\cdot[T\cap\mathrm{Ker}\alpha_0.\mathrm{Ker}\beta_0].$$

Furthermore, since $\beta_1(y) = 1$. we get:

$$d_0(y) \equiv d_0(y)\cdot\prod[(\beta_0(t_i))'.\beta_0(x_i)]^{\sigma_i}\cdot[(\beta_0(t_{1i}))'.(\beta_0(t_{2i}))']\cdot(\beta_0(t_{3i}))')^q.$$

but $[t.\beta_0(x)]^\sigma$. $[t_1.\beta_0(t_2)]^\sigma$. $t_3(\alpha_0(t_3))'\alpha_0(t_3)^{-1} \in T\cap\mathrm{Ker}\alpha_0$.

$[(\beta_0(t))'.\beta_0(x)]^\sigma$. $[(\beta_0(t_1))'.(\beta_0(t_2))']$. $(\beta_0(t_3))' \in \mathrm{Ker}\beta_0$,

and then

$$d_0(y) \equiv \prod[t_i.\beta_0(x_i)]^{\sigma_i}\cdot[(\beta_0(t_i))'.\beta_0(x_i)]^{\sigma_i}\cdot[t_{1i},\beta_0(t_{2i})]\cdot$$
$$\cdot[(\beta_0(t_{1i}))'.(\beta_0(t_{2i}))']\cdot(t_{3i}(\alpha_0(t_{3i}))'\cdot\alpha_0(t_{3i})^{-1}\cdot(\beta_0(t_{3i}))')^q$$
$$\mathrm{mod.}\ [T.\mathrm{Ker}\alpha_0\cap\mathrm{Ker}\beta_0]\cdot[T\cap\mathrm{Ker}\,\alpha_0,\mathrm{Ker}\beta_0].$$

Similarly,

$$[t,\beta_0(x)]^\sigma\cdot [(\beta_0(t))',\beta_0(x)]^\sigma \equiv$$
$$\equiv [\beta_0(t),\beta_0(x)]^\sigma\cdot[t,\alpha_0(\beta_0(x))]^\sigma = [\beta_0(t),\beta_0(x)]^\sigma .$$

$$[t_1,\beta_0(t_2)]\cdot[(\beta_0(t_1))',(\beta_0(t_2))'] \equiv [t_1,\beta_0(t_2)]\cdot[(\beta_0(t_1))',\beta_0(t_2)] \equiv$$
$$\equiv [t_1,\alpha_0(\beta_0(t_2))]\cdot[\beta_0(t_1), \beta_0(t_2)] = [\beta_0(t_1),\beta_0(t_2)]$$
$$\text{mod. } [T,\mathrm{Ker}\alpha_0\cap\mathrm{Ker}\beta_0]\cdot[T\cap\mathrm{Ker}\,\beta_0,\mathrm{Ker}\,\alpha_0].$$

Finally, since

$t(\alpha_0(t))'\alpha_0(t)^{-1}(\beta_0(t))'\beta_0(t)^{-1}\in T\cap\mathrm{Ker}\alpha_0\cap\mathrm{Ker}\beta_0$, $\beta_0(t)\in\mathrm{Ker}\alpha_0$,

we obtain the following congruences:

$$(t\cdot(\alpha_0(t))'\cdot\alpha_0(t)^{-1}\cdot(\beta_0(t))')^q$$
$$= (t\cdot(\alpha_0(t))'\cdot\alpha_0(t)^{-1}\cdot(\beta_0(t))'\cdot\beta_0(t)^{-1}\cdot\beta_0(t))^q \equiv$$
$$\equiv (t\cdot(\alpha_0(t))'\cdot\alpha_0(t)^{-1}\cdot(\beta_0(t))'\cdot\beta_0(t)^{-1})^q\cdot(\beta_0(t))^q \equiv (\beta_0(t))^q$$

mod. $[T\cap\mathrm{Ker}\,\beta_0,\mathrm{Ker}\,\alpha_0].((T\cap\mathrm{Ker}\alpha_0\cap\mathrm{Ker}\,\beta_0)\#_q(T\cap\mathrm{Ker}\,\beta_0\cap\mathrm{Ker}\,\alpha_0))$,
and therefore

$$d_0(y)\equiv\prod[(\beta_0(t_i))',\beta_0(x_i)]^\sigma\cdot[\beta_0(t_{1i}),\beta_0(t_{2i})]\cdot(\beta_0(t_{3i}))^q$$
$$\text{mod. } [T,\mathrm{Ker}\alpha_0\cap\mathrm{Ker}\beta_0]\cdot[T\cap\mathrm{Ker}\alpha_0,\mathrm{Ker}\beta_0]\cdot[T\cap\mathrm{Ker}\,\beta_0,\mathrm{Ker}\,\alpha_0]\cdot$$
$$\cdot((T\cap\mathrm{Ker}\alpha_0\cap\mathrm{Ker}\beta_0)\#_q(T\cap\mathrm{Ker}\,\beta_0\cap\mathrm{Ker}\,\alpha_0)).$$

If now we consider the homomorphism

$$\tau_2:\ S*(R\cap S)'\xrightarrow{\binom{1}{u_1}} S \longmapsto (R\cap S)'*R*S,$$

we have $d_0(y)\equiv\tau_2(\beta_1(y)) = 1$, and consequently,

$$d_0(y)\in[T,\mathrm{Ker}\alpha_0\cap\mathrm{Ker}\beta_0]\cdot[T\cap\mathrm{Ker}\alpha_0,\mathrm{Ker}\beta_0]\cdot[T\cap\mathrm{Ker}\,\beta_0,\mathrm{Ker}\,\alpha_0]\cdot$$
$$\cdot((T\cap\mathrm{Ker}\alpha_0\cap\mathrm{Ker}\beta_0)\#_q(T\cap\mathrm{Ker}\,\alpha_0\cap\mathrm{Ker}\,\beta_0)).$$

The result now follows from Lemma 5.5 and

$$\mathrm{Ker}\,\alpha_0 = A^{X_0},\ \mathrm{Ker}\,\beta_0 = B^{X_0}.$$

REMARK 5.7. Note that

$$[T,\mathrm{Ker}\alpha_0\cap\mathrm{Ker}\beta_0]\cdot[T\cap\mathrm{Ker}\alpha_0,\mathrm{Ker}\beta_0]\cdot[T\cap\mathrm{Ker}\,\beta_0,\mathrm{Ker}\,\alpha_0]\cdot$$
$$\cdot((T\cap\mathrm{Ker}\alpha_0\cap\mathrm{Ker}\beta_0)\#_q(T\cap\mathrm{Ker}\,\beta_0\cap\mathrm{Ker}\,\alpha_0)) =$$
$$= (T\#_q(\mathrm{Ker}\alpha_0\cap\mathrm{Ker}\beta_0))\cdot((T\cap\mathrm{Ker}\alpha_0)\#_q\mathrm{Ker}\beta_0)\cdot((T\cap\mathrm{Ker}\,\beta_0\#_q\mathrm{Ker}\,\alpha_0)$$
$$= (T\#_q(\mathrm{Ker}\alpha_0\cap\mathrm{Ker}\beta_0))\cdot[T\cap\mathrm{Ker}\alpha_0,\mathrm{Ker}\beta_0]\cdot[T\cap\mathrm{Ker}\,\beta_0,\mathrm{Ker}\,\alpha_0];$$

and it is the smallest subgroup containing

$$[T,\mathrm{Ker}\alpha_0\cap\mathrm{Ker}\beta_0],\ [T\cap\mathrm{Ker}\alpha_0,\mathrm{Ker}\beta_0],\ [T\cap\mathrm{Ker}\,\beta_0,\mathrm{Ker}\,\alpha_0]$$

and the elements t^q, with $q\in\mathbb{N}$ and $t\in T\cap\mathrm{Ker}\alpha_0\cap\mathrm{Ker}\beta_0$.

PROPOSITION 5.8. *Under the hypothesis of Proposition 5.6:*

i) $L_0\vartheta_2(\alpha.\gamma) = (M\cap N)/(M\#_q N)$.

ii)

$$L_1\vartheta_2(\alpha.\gamma) = \frac{T\cap(X_0\#_q A^{X_0})\cap B^{X_0}}{(T\#_q(A^{X_0}\cap B^{X_0}))[T\cap A^{X_0}.\ B^{X_0}][T\cap B^{X_0}.\ A^{X_0}]}.$$

PROOF. ii) It is a consequence of Propositions 5.1 and 5.6.

i) From Proposition 5.1. since

$$L_0 I_2(\alpha.\gamma) = I_2(\alpha.\gamma) = \mathrm{Ker}\,\alpha\cap\mathrm{Ker}\,\beta = M\cap N,$$

and $L_0\vartheta_2(\alpha.\gamma)$ is a quotient of $(X_0\#_q A^{X_0})\cap B^{X_0}$. it is sufficient to show that

$$d_0(X_0\#_q A^{X_0})\cap B^{X_0}) = M\#_q N.$$

Let $m\in M$. $n\in N$. $t\in M\cap N$. Then there exist $r\in R$. $s\in S$ and $k\in R\cap S$. such that $\varepsilon(r)=m$. $\varepsilon(s)=n$ and $\varepsilon(k) = t$. Then

$$d_0([r.s]k'^q) = [m.n]t^q;$$

therefore $[r.s]\cdot k'^q\in(X_0\#_q A^{X_0})\cap B^{X_0}$ since $r\in B$. $s\in A$. $k'\in(A\cap B)'$.

Let $y\in(X_0\#_q A^{X_0})\cap B^{X_0}$. Then. from Lemma 5.4.

$$y = \prod_{i=1}^{n}[a_i.r_i]\cdot[a_{1i}.a_{2i}]\cdot(a_{3i})^q \text{ with } \beta_0(y)=1.$$

If we consider $\varepsilon j: S\to G$. we get

$$1 = \varepsilon j\beta_0(y) = \prod[\varepsilon j\beta_0(a_{1i}).\varepsilon j\beta_0(a_{2i})]\cdot(\varepsilon j\beta_0(a_{3i}))^q$$

so that

$$d_0(y) = d_0(y)\cdot(\varepsilon j\beta_0(y))^{-1} = \prod[d_0(a_i).d_0(r_i)]\cdot$$

$$\cdot[\varepsilon j\beta_0(a_{1i})\varepsilon j\beta_0(a_{1i})^{-1}d_0(a_{1i}).\varepsilon j\beta_0(a_{2i})\varepsilon j\beta_0(a_{2i})^{-1}d_0(a_{2i})]\cdot$$

$$(\varepsilon j\beta_0(a_{3i})\varepsilon j\beta_0(a_{3i})^{-1}d_0(a_{3i}))^q\cdot(\prod[\varepsilon j\beta_0(a_{1i}).\varepsilon j\beta_0(a_{2i})]\cdot(\varepsilon j\beta_0(a_{3i}))^q)^{-1}.$$

Then. since $d_0(a)$. $\varepsilon j\beta_0(a)\in N$. $d_0(r)\in M$. $\varepsilon j\beta_0(a)^{-1}d_0(a)\in M\cap N$. we have $d_0(y)\in M\#_q N$. ·

6.

In this section we obtain the long exact sequence of homology with coefficients in $\mathbb{Z}_q$. associated to a surjective group homomorphism.

PROPOSITION 6.1. *Let* N *be a normal subgroup of a group* G *and we consider* M = G. *Then we have the exact sequence:*

$$\cdots \to H^q_{n+1}(G) \to H^q_{n+1}(G/N) \to L_{n-1}\vartheta_2(\alpha.\gamma) \to H^q_n(G) \to \cdots$$
$$\to H^q_3(G) \to H^q_3(G/N) \to L_1\vartheta_2(\alpha.\gamma) \to H^q_2(G) \to H^q_2(G/N) \to$$
$$L\vartheta_2(\alpha.\gamma) \to H^q_1(G) \to H^q_1(G/N) \to 0$$

where $(\alpha.\gamma)$ *denotes the object of the Rinehart category* Gr_2:

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & G/N \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\delta} \\ 0 & \xrightarrow{\gamma} & 0 \end{array}$$

PROOF. Since $G/M = 0$, the result follows from Proposition 4.4.

LEMMA 6.2. *Let* N *be a normal subgroup of a group* G, $\varepsilon: F \twoheadrightarrow G$ *a free presentation of* G, S *the kernel of*

$$\alpha\varepsilon:\ F \twoheadrightarrow G \twoheadrightarrow G/N$$

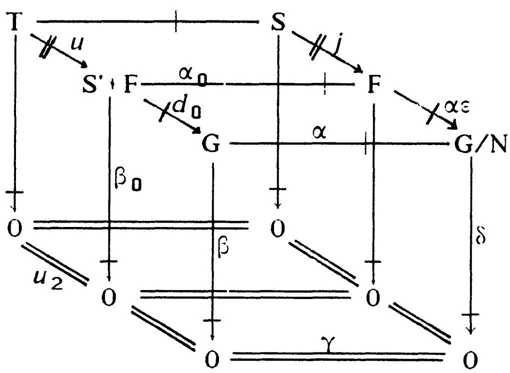
and T *the kernel of* $\binom{\varepsilon j}{\varepsilon}: S' * F \to G$. *Then*

$$(\alpha'_0.\gamma'_0) \longrightarrow (a_0.\gamma_0) \longrightarrow (\alpha.\gamma)$$

that is

$\alpha_0 = \binom{0}{1}$

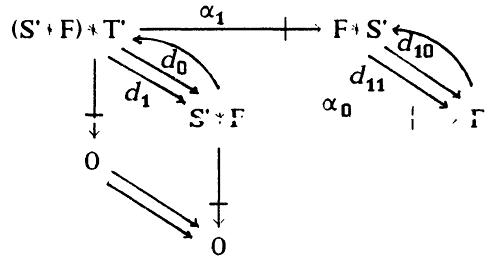
$d_0 = \binom{\varepsilon j}{\varepsilon}$



is an ε_2*-projective presentation of* $(\alpha.\gamma)$ *in the Rinehart category* Gr_2.

PROOF. α_0 and γ_0 are projective objects in Gr_1 and, since $(\beta_0.\delta_0): \alpha_0 \to \gamma_0$ is an epimorphism in Gr_1, we have that $(\alpha_0.\gamma_0)$ is a projective object in Gr_2. Furthermore, since $(\alpha_0.\gamma_0) \to (\alpha.\gamma)$ is an epimorphism in Gr_2 we get the result.

LEMMA 6.3. *With the same notation as above, the first two terms of a simplicial resolution of* $(\alpha.\gamma)$ *in the Rinehart category* Gr_2 *are*



where

$$\alpha_1 = \alpha_0 + \alpha_0', \quad d_0 = \binom{1}{u}, \quad d_1 = \binom{1}{0}, \quad d_{10} = \binom{1}{j}, \quad d_{11} = \binom{1}{0}.$$

PROOF. This is a particular case of Proposition 5.2, ii.

PROPOSITION 6.4. *With the same notation as above:*

$$L_0V_2(\alpha,\gamma) = \frac{((S'*F)\#_q S'^{S'*F})}{(T\#_q S'^{S'+F})((S'+F)\#_q(T\cap S'^{S'+F}))}.$$

PROOF. From Proposition 5.6, we have

$$L_0V_2(\alpha,\gamma) =$$

$$= \frac{(S'*F)\#_q \operatorname{Ker}\alpha_0)\cap \operatorname{Ker}\beta_0}{(T\#_q(\operatorname{Ker}\alpha_0\cap\operatorname{Ker}\beta_0))\cdot[T\cap\operatorname{Ker}\alpha_0,\ \operatorname{Ker}\beta_0]\cdot[T\cap\operatorname{Ker}\beta_0,\ \operatorname{Ker}\alpha_0]}.$$

The result follows from

$$\operatorname{Ker}\alpha_0 = S'^{S'*F} \text{ and } \operatorname{Ker}\beta_0 = S' * F.$$

PROPOSITION 6.5. *With the same notation as above:*

i) $L_0\vartheta_2(\alpha,\gamma) = N/(N\#_q G)$,

ii)

$$L_1\vartheta_2(\alpha,\gamma) = \frac{T\cap((S'*F)\#_q S'^{S'*F})}{(T\#_q S'^{S'*F})((S'*F)\#_q(T\cap S'^{S'+F}))}.$$

PROOF. This is a consequence of Propositions 5.8 and 6.4.

7.

In this section we obtain Hopf formula for H_2^q and a "kind of" Hopf formula for $H_3^q(G)$.

PROPOSITION 7.1. *Let* G *be a group.*

$$R \xrightarrow{i} F \longrightarrow G$$

a free presentation of G. *and* T *the kernel of* $\binom{i}{1}: R' * F \to F$. *Then*

i) $H_2^q(G) = (R \cap (F \#_q F))/(R \#_q F)$.

ii)

$$H_3^q(G) = \frac{T \cap ((R' * F) \#_q R^{\cdot R' * F})}{(T \#_q R^{\cdot R' * F})((R' * F) \#_q (T \cap R^{\cdot R' * F}))}.$$

PROOF. As in Section 6 we have the following diagram

$$\begin{array}{ccc} F & \xrightarrow{\ \alpha\ } & F/R = G \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \delta} \\ 0 & \xrightarrow{\ \gamma\ } & 0 \end{array}$$

From Proposition 6.1, since $H_n^q(F) = 0$, $n \geq 2$, we get

$$H_2^q(F/R) = \mathrm{Ker}(L_0 \vartheta_2(\alpha,\gamma) \to H_1^q(G)) \text{ and } H_3^q(G) = L_1 \vartheta_2(\alpha,\gamma).$$

Thus

i) $H_2^q(G) = \mathrm{Ker}(R/(R \#_q F) \to F/(F \#_q F)) = (R \cap (F \#_q F))/(R \#_q F)$.

ii) Since, in this case, $\mathrm{Ker}(F \to F/R) = R$ we have the result by Proposition 6.5.ii.

8.

In this section we prove that for the particular case of ($q = 0$) and only a subgroup $N \lhd G$ ($M = G$) we have that $L_0 V_2(\alpha,\gamma)$ coincides with the exterior product $N \wedge G$ (Proposition 8.3) and therefore $V = \mathrm{Ker}([\,,\,]: N \wedge G \to G)$ [6. Corollary 4.6] coincides with $L_1 V_2(\alpha,\gamma)$ (Corollary 8.15).

Moreover, in Remark 8.16 we show that the expression of $H_3(G)$ in Proposition 7.1 is a particular case of Theorem 1 of [3].

Let ϑ be a variety of Abelian groups. N a normal subgroup of G and $(\beta,\delta) = (\alpha,\gamma)$ the following object of the category of Rinehart Gr_2:

$$\begin{array}{ccc} G & \xrightarrow{\ \beta\ } & 0 \\ \downarrow{\scriptstyle \alpha} & & \Vert{\scriptstyle \gamma} \\ G/N & \xrightarrow{\ \delta\ } & 0 \end{array}$$

Then, by Proposition 3.15, we have

$$\cdots \to L_{n+1} V_1(\gamma) \to L_n V_2(\beta,\delta) \to L_n V_1(\alpha) \to L_n V_1(\gamma) \to \cdots$$

Since $L_n V_1(\gamma) = 0$, $n \geq 0$ and $(\beta,\delta) = (\alpha,\gamma)$, we have

$$L_n V_2(\alpha,\gamma) = L_n V_1(\alpha).$$

REMARK 8.1. If N is a normal subgroup of G and if we consider $\alpha: G \to G/N$ and $\varepsilon: F \to G$ a free presentation of G, letting S be the kernel of $\alpha\varepsilon: F \to G \to G/N$ and T the kernel of $\binom{\varepsilon i}{\varepsilon}: S' * F \to G$ (Proposition 6.4. for $q = 0$)

$$L_0 V_1(\alpha) = \frac{[S'^{S'*F}, S'*F]}{[S'^{S'*F}, T] \cdot [T \cap S'^{S'*F}, S'*F]} .$$

DEFINITION 8.2. Brown & Loday in [5,6] introduce the tensor product $M \otimes N$ and the exterior product $M \wedge N$ for two groups M and N equipped with an action of M on the left of N and an action of N on the left of M. It is always understood that a group acts on itself by conjugation: $y^x = xyx^{-1}$.

In the particular case of M and N being two normal subgroups of G we can consider that the actions of M on N and of N on M are by conjugation, and therefore: the tensor product $M \otimes N$ is the group generated by symbols $m \otimes n$, $m \in M$, $n \in N$, with relations

(a) $mm' \otimes n = (m'^m \otimes n^m)(m \otimes n)$,

(b) $m \otimes nn' = (m \otimes n)(m^n \otimes n'^n)$,

for all $m, m' \in M$, $n, n' \in N$.

The (non-Abelian) exterior product $M \wedge N$ is obtained from the tensor product $M \otimes N$ by imposing the additional relations $t \otimes t = 1$ for all $t \in M \cap N$. the image of a general element $m \otimes n$ in $M \wedge N$ is written $m \wedge n$.

PROPOSITION 8.3. *Let* N *be a normal subgroup of* G *and consider* $\alpha: G \to G/N$. *Then* $L_0V_1(\alpha) = N \wedge G$.

To prove this result we will construct two homomorphisms, inverse to each other, $\Psi: N \wedge G \to L_0V_1(\alpha)$ (Corollary 8.7) and $\tau: L_0V_1(\alpha) \to N \wedge G$ (Corollary 8.14).

LEMMA 8.4. *With the same notations as in 8.1, let* $\mu: G \to F$ *be a section of* ε *(i.e., μ is a map such that* $\varepsilon\mu = 1_G$*). For each* $n \in N$ *we have* $\mu(n) \in S$, *and we consider* $N \to S \to S'$ *given by* $n \to \mu(n)'$. *Then*

$$h: N \cdot G \longrightarrow L_0 V_1(\alpha) = \frac{[S'^{S'*F}, S'*F]}{[S'^{S'*F}, T] \cdot [T \cap S'^{S'*F}, S'*F]}$$

given by

$h(n,g) = [\mu(n)', \mu(g)] \cdot H$, *where* $H = [S'^{S'*F}, T] \cdot [T \cap S'^{S'*F}, S'*F]$

is a crossed pairing (or biderivation) [6, Definition 2.2].

PROOF. $h(n_1 n_2 . g) = [\mu(n_1 n_2)' . \mu(g)] \cdot H =$

$$= [\mu(n_1)'\mu(n_2)'(\mu(n_1)'\mu(n_2)')^{-1}\mu(n_1 n_2)', \mu(g)] \cdot H$$

$$= [\mu(n_1)'\mu(n_2)')^{-1} \mu(n_1 n_2)' . \mu(g)]^{\mu(n1)'\mu(n2)'} \cdot [\mu(n_1)'\mu(n_2)' . \mu(g)] \cdot H$$

$$= [\mu(n_1)'\mu(n_2)' . \mu(g)] \cdot H$$

because

$$\mu(n_1)'\mu(n_2)')^{-1} \mu(n_1 n_2)' \in T \cap S'^{S'*F}.$$

So

$$h(n_1 n_2 . g) = [\mu(n_2)'^{\mu(n1)'} . \mu(g)^{\mu(n1)'}] \cdot [\mu(n_1)' . \mu(g)] \cdot H$$

$$= [\mu(n_2{}^{n1})'\mu(n_2{}^{n1})'^{-1}\mu(n_2)'^{\mu(n1)'} . \mu(g^{n1})\mu(g^{n1})^{-1}\mu(g)^{\mu(n1)'}] \cdot [\mu(n_1)' . \mu(g)] \cdot H$$

$$= [\mu(n_2{}^{n1})' . \mu(g^{n1})] \cdot [\mu(n_1)' . \mu(g)] \cdot H$$

since

$$\mu(n_2{}^{n1})'^{-1} \mu(n_2)'^{\mu(n1)'} \in T \cap S'^{S'*F}, \ \mu(n_2{}^{n1})' \in S'^{S'+F}$$

$$\text{and } \mu(g^{n1})^{-1}\mu(g)^{\mu(n1)'} \in T$$

then

$$h(n_1 n_2 . g) = h(n_2^{n1} . g^{n1}) \cdot h(n_1 . g).$$

Analogously

$$h(n . g_1 g_2) = h(n . g_1) \cdot h(n^{g1} . g_2{}^{g1}).$$

taking into account that

$$\mu(n^{g1})'^{-1}\mu(n')^{\mu(g1)} \in T \cap S'^{S'+F}, \ \mu(n^{g1})' \in S'^{S'+F}$$

$$\text{and } \mu(g_2^{g1})^{-1}\mu(g_2)^{\mu(g1)} \in T.$$

REMARK 8.5. h is independent of the μ we choose, for if ν is also a section, then for each $g \in G$, $(\nu(g)^{-1}\mu(g)) \in T$; and for n in N, $(\nu(n)'^{-1}\mu(n)') \in T \cap S'^{S'*F}$. This way

$$[\mu(n)' . \mu(g)] \cdot H = [\nu(n)'\nu(n)'^{-1} \mu(n)' . \mu(g)] \cdot H =$$

$$[\nu(n)'^{-1}\mu(n)' . \mu(g)]^{\nu(n)'} \cdot [\nu(n)' . \nu(g)\nu(g)^{-1}\mu(g)] \cdot H = [\nu(n)' . \nu(g)] \cdot H$$

as $(\nu(n)'^{-1}\mu(n)') \in T \cap S'^{S'+F}$, $\nu(n)' \in S'^{S'+F}$, $\nu(g)^{-1}\mu(g) \in T$.

COROLLARY 8.6. *The biderivation* $h : N \times G - L_0V_1(\alpha)$ *determines a group homomorphism*

$h^* : N \otimes G - L_0V_1(\alpha)$ *such that* $h^*(n \otimes g) = [\mu(n)' . \mu(g)] \cdot H$. [6]

COROLLARY 8.7. $h^* : N \otimes G - L_0V_1(\alpha)$ *determines a homomorphism* $\Psi : N \wedge G - L_0V_1(\alpha)$ *such that* $\Psi(n \wedge g) = [\mu(n)' . \mu(g)] \cdot H$.

PROOF. It is enough to realize that

$$h^*(n \otimes n) = [\mu(n)' . \mu(n)] \cdot H =$$

$$= [\mu(n)' . \mu(n)'\mu(n)'^{-1}\mu(n)] \cdot H = [\mu(n)' . \mu(n)'] \cdot H,$$

as $\mu(n)'^{-1} \in S'^{S'*F}$, $\mu(n)'^{-1}\mu(n) \in T$ and therefore $h^*(n \otimes n) \in H$ and h^* induces Ψ.

REMARK 8.8. Let S' and F be groups and consider their free product $S' * F$. If

$$x = \prod_{i=1}^{n} s'_i f_i \in [S'^{S'*F}, S' * F],$$

then

$$\prod f_i = 1, \ \prod s'_i \in [S', S'].$$

PROPOSITION 8.9. *With the notations of 8.1, let*

$$d = \binom{\varepsilon i}{\varepsilon} : S' * F \to G, \ \binom{1}{0} : S' * F \to S', \ x \in S'^{S'*F}, \ y \in S' * F.$$

Then $[x, y] = \prod_{i=1}^{k} s'_i f_i$ *and we have*

$$d(x) \wedge d(y) =$$

$$= \prod_{i=1}^{k-1} \Big((d(\prod_{j=1}^{i} s'_j) \wedge d(\prod_{j=1}^{i} f_j))((d(\prod_{j=1}^{i} s'_j) \wedge d(\prod_{j=1}^{i+1} f_j))^{-1} \Big) \cdot \Big(d(\binom{1}{0}(x)) \wedge d(\binom{1}{0}(y)) \Big)$$

in $N \wedge G$.

PROPOSITION 8.10. *With the same notations, if* $\prod_{i=1}^{n} [s'_{1i}, s'_{2i}] = 1$, $s'_{1i}, s'_{2i} \in S'$, *we have*

$$\prod_{i=1}^{n} d(s'_{1i}) \wedge d(s'_{2i}) = 1 \ in \ N \wedge G.$$

PROOF. The map

$$[S', S'] \to S' \wedge S', \ [s'_1, s'_2] \to s'_1 \wedge s'_2$$

is an isomorphism [11, Proposition 2]. Moreover

$$S' \wedge S' \to N \wedge G, \ s'_1 \wedge s'_2 \to d(s'_1) \wedge d(s'_2)$$

is a group homomorphism.

COROLLARY 8.11. If

$$\prod_{i=1}^{n} [s'_{1i}, s'_{2i}] = \prod_{i=1}^{m} [s'_{3i}, s'_{4i}]$$

then we have

$$\prod_{i=1}^{n} (d(s'_{1i}) \wedge d(s'_{2i})) = \prod_{i=1}^{m} (d(s'_{3i}) \wedge d(s'_{4i})).$$

As a consequence, for $x = \prod_{i=1}^{n} [s'_{1i}, s'_{2i}]$, we denote by $\varphi(x)$ the element $\prod_{i=1}^{n} (d(s'_{1i}) \wedge d(s'_{2i}))$ of $N \wedge G$.

PROPOSITION 8.12. *With the above notations we have the following* $\varphi: [S'^{S'*F}, S'*F] \to N \wedge G$ *so that, for* $\prod s'_i f_i \in [S'^{S'*F}, S'*F]$,

$$\varphi(\prod_{i=1}^{k} s'_i f_i) = \prod_{i=1}^{k-1}\left((d(\prod_{j=1}^{i} s'_j) \wedge d(\prod_{j=1}^{i} f_j))((d(\prod_{j=1}^{i} s'_j) \wedge d(\prod_{j=1}^{i+1} f_j))^{-1}\right) \cdot \varphi(\prod_{i=1}^{k} s'_i)$$

is a group homomorphism.

PROOF. φ defines a mapping since $\varphi(\prod_{i=1}^{k} s'_i)$ is uniquely determined. Moreover, denoting $d(s'_i) = n_i$, $d(f_i) = g_i$, we have

$$\varphi(\prod_{i=1}^{k} s'_i f_i)\, \varphi(\prod_{i=k+1}^{t} s'_i f_i) =$$

$$= \prod_{i=1}^{k-1}\left(((\prod_{j=1}^{i} n_j) \wedge (\prod_{j=1}^{i} g_j)) \cdot ((\prod_{j=1}^{i+1} n_j) \wedge (\prod_{j=1}^{i} g_j))^{-1}\right) \cdot \left(\prod_{i=k+1}^{t} ((\prod_{j=k+1}^{i} n_j) \wedge (\prod_{j=k+1}^{i} g_j)) \cdot\right.$$

$$\left. \cdot ((\prod_{j=k+1}^{i+1} n_j) \wedge (\prod_{j=k+1}^{i} g_j))^{-1}\right)^{\prod_{}^{k} n_j} \cdot \varphi(\prod_{i=1}^{k} s'_i)\, \varphi(\prod_{i=k+1}^{t} s'_i),$$

as for $r, r' \in N \wedge G$ we have $r r' r^{-1} = r'^{\lambda r}$ [6, Proposition 2.3].

As a consequence, from $\prod_{i=1}^{k} f_i = 1$ it follows $\prod_{j=k+1}^{i} g_j = \prod_{j=1}^{i} g_j$, and using that for $n_1, n_2, n \in N$, $g \in G$,

$$((n_1 \wedge g)(n_2 \wedge g)^{-1})^n = (n n_1 \wedge g)(n n_2 \wedge g)^{-1},$$

we have

$$\varphi(\prod_{i=1}^{k} s'_i f_i)\, \varphi(\prod_{i=k+1}^{t} s'_i f_i) = \varphi((\prod_{i=1}^{k} s'_i f_i)(\prod_{i=k+1}^{t} s'_i f_i)).$$

REMARK 8.13. From Proposition 8.9 it follows that, for $x_i \in S'^{S'*F}$, $y_i \in S'*F$, $i = 1, \dots, k$, $\sigma_i = \pm 1$,

$$\varphi(\prod_{i=1}^{k} [x_i, y_i]^{\sigma_i}) = \prod_{i=1}^{k} (d(x_i) \wedge d(y_i))^{\sigma_i}.$$

COROLLARY 8.14. *The map* Ψ *of Cor. 8.7 is an isomorphism.*

PROOF. With the notation of 8.1 we have that

$$\varphi([S'^{S'*F}, T] \cdot [T \cap S'^{S'*F}, S'*F]) = 1,$$

because, if $x \in T$, then $d(x) = 1$ and $m \wedge 1 = 1$, $1 \wedge g = 1$. Therefore we have a $\tau: L_0 V_1(\alpha) \to N \wedge G$ induced by φ. It is clear now that τ is the required inverse of Ψ.

This result ends the proof of Proposition 8.3.

COROLLARY 8.15 ([5], Corollary 3). *For any given free presentation of* G, $\alpha: F \to G$, *if* $R = \mathrm{Ker}\,\alpha$ *and* T *the kernel of* $\binom{i}{1}: R * F \to F$, *then we have*

$$H_3(G) \approx \mathrm{Ker}([\,,\,]: R \wedge F \to F) \approx$$

$$L_1\vartheta_1(\alpha) = \frac{T\cap[R'^{R'*F}, R'*F]}{[R'^{R'*F}, T]\cdot[T\cap R'^{R'*F}, R'*F]} .$$

PROOF. We have the following commutative diagram

$$\begin{array}{ccccc} L_1\vartheta_1(\alpha) & \longrightarrow & L_0V_1(\alpha) & \longrightarrow & [N.G] \\ \downarrow & & \downarrow \tau & & \| \\ V & \longrightarrow & N\wedge G & \longrightarrow & [N.G] \end{array}$$

and since τ is an isomorphism, $V \approx L_1\vartheta_1(\alpha)$.

REMARK 8.16. Brown & Ellis in [3, Theorem 1] give the following expression for $H_r(G)$.

Let $R_1, \dots, R_n$ be normal subgroups of a group F such that $F/\prod_{1\le i\le n} R_i \approx G$, and for each proper subset A of $\langle n\rangle = \{1, \dots, n\}$ the groups

$$H_r(F/\prod_{i\in A} R_i), \quad r = \begin{cases} |A|+1, |A|+2 & \text{if } A \neq \emptyset \\ 2 & \text{if } A = \emptyset \end{cases}$$

are trivial (for example, these groups $F/\prod_{i\in A} R_i$ are free). Then there is an isomorphism

$$H_{n+1}(G) \approx \bigcap_{i=1}^{n} R_i \cap [F,F] / \prod_{\langle n\rangle\supset A} [\bigcap_{i\in A} R_i, \bigcap_{i\notin A} R_i].$$

Our expression for $H_3^q(G)$ in Proposition 7.1 (for $q=0$) is a particular case of Brown & Ellis's result because

$$\begin{array}{ccccc} T\cap R'^{R'*F} & \longrightarrow & T & \longrightarrow & R \\ \downarrow & & \downarrow & & \downarrow \\ R'^{R'*F} & \longrightarrow & R'*F & \longrightarrow & F \\ \downarrow & & \downarrow & & \downarrow \\ R & \longrightarrow & F & \longrightarrow & F/R \approx G \end{array}$$

is a commutative diagram whose rows and columns are short exact sequences, and therefore

$$G \approx R'*F/T\cdot(R'^{R'*F}) \text{ and } R'*F/T \approx F \approx R'*F/R'^{R'*F}$$

are free groups. Moreover

$$\frac{T\cap R'^{R'*F}\cap[R'*F, R'*F]}{[R'^{R'*F}, T]\cdot[T\cap R'^{R'*F}, R'*F]} \approx \frac{T\cap[R'^{R'*F}, R'*F]}{[R'^{R'*F}, T]\cdot[T\cap R'^{R'*F}, R'*F]} .$$

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MATHEMATICAL SYMBOLS USED:

$\longrightarrow$ Morphism or functor

$\rightarrowtail$ Monomorphism or injective function

$-\!\!\Vdash\!\!\longrightarrow$ Normal monomorphism

$\twoheadrightarrow$ Surjective function or element of a projective class of epimorphisms in a category

$\leftarrow$ Degeneracy operators of a simplicial object

$\binom{\alpha}{\beta}$ Induced morphism from a free product by α and β

DEPARTAMENTO DE ALGEBRA
UNIVERSIDAD DE SANTIAGO DE COMPOSTELA
SANTIAGO DE COMPOSTELA. SPAIN