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FUNCTORS BETWEEN HOMOTOPY THEORIES

by Luciano STRAMACCIA[1]

RÉSUMÉ. Dans cet article, on considère des catégories ***C*** munies d'une notion d'homotopie, sous la forme d'une structure de I-catégorie au sens de Baues, engendrée par un foncteur cylindre I, et on étudie la préservation des propriétés d'homotopie relativement à un foncteur $S: \mathbf{C} \to \mathbf{A}$, en particulier lorsque S est un réflecteur. Le cas d'un proréflecteur est aussi examiné.

INTRODUCTION.

There are various ways to introduce a homotopy notion in a category ***C***, all related to the concept of model category of Quillen [9]. Most notably, those due to Brown [2] and, more recently, to Baues [1], seem to be very interesting and more manageable than the original one. However there exists, up to author's knowledge, a certain lack in the literature concerning subcategories and comparison of homotopy structures.

In this paper we are concerned with categories endowed with the structure of an I-category in the sense of [1], which is generated by a cylinder functor I on it [1,6]. We study the preservation of homotopy properties by means of a functor S from ***C*** to ***A***. In particular, we are interested in the case where S is a reflector, which means that ***A*** is a full subcategory of ***C*** and S is left adjoint to the embedding functor $T: \mathbf{A} \to \mathbf{C}$. Also the case of a proreflector P is considered.

1. PRELIMINARIES.

Let ***C*** be a category and let Σ be a class of morphisms of ***C*** which we call "weak equivalences". A new category $\mathbf{C}[\Sigma^{-1}]$ can be constructed by formally inverting weak equivalences. $\mathbf{C}[\Sigma^{-1}]$ has the same objects as ***C*** and is defined by the following properties:

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(i) There is a functor $P_\Sigma : \boldsymbol{C} \to \boldsymbol{C}[\Sigma^{-1}]$ which is the identity on objects and which inverts all weak equivalences, that is, $P_\Sigma(s)$ is an isomorphism in $\boldsymbol{C}[\Sigma^{-1}]$, for every $s \in \Sigma$.

(ii) If $G: \boldsymbol{C} \to \boldsymbol{D}$ is a functor which inverts all weak equivalences, then there is a unique functor $G^*: \boldsymbol{C}[\Sigma^{-1}] \to \boldsymbol{D}$ such that $G^* \cdot P_\Sigma = G$.

$\boldsymbol{C}[\Sigma^{-1}]$ always exists, but its description is particularly nice whenever Σ admits a "calculus of left fractions" in $\boldsymbol{C}$ [3].

Let $\boldsymbol{A}$ be another category endowed with a notion of weak equivalence and let Λ be the class of such weak equivalences. A functor $F: \boldsymbol{C} \to \boldsymbol{A}$ can be extended to a functor $F^*: \boldsymbol{C}[\Sigma^{-1}] \to \boldsymbol{A}[\Lambda^{-1}]$ iff F *preserves weak equivalences*, that is $F(\Sigma) \subset \Lambda$. In such a case F^* is the unique functor with $F^* \cdot P_\Lambda = P_\Sigma \cdot F$. F^* acts on objects as F does.

PROPOSITION 1.1 (cf. [2], p. 426). *Let* $T: \boldsymbol{A} \to \boldsymbol{C}$ *and* $S: \boldsymbol{C} \to \boldsymbol{A}$ *be functors which preserve weak equivalences. If* S *is left adjoint to* T, *then* S^* *is left adjoint to* T^*.

DEFINITION 1.2. a) A *cylinder functor* for a category $\boldsymbol{C}$ is a functor $I: \boldsymbol{C} \to \boldsymbol{C}$ together with natural transformations

$$e_0, e_1: 1_C \to I \text{ and } \sigma: I \to 1_C$$

such that $\sigma \cdot e_0 = \sigma \cdot e_1 =$ identity.

Two morphisms $f, g \in \boldsymbol{C}(X, Y)$ are *homotopic*, written $f \simeq g$, whenever there is a "homotopy" $H: I(X) \to Y$ with $H \cdot e_0(X) = f$ and $H \cdot e_1(X) = g$. Shortly $H: f \simeq g$.

b) Once a cylinder functor is given for $\boldsymbol{C}$, one can define a morphism $t \in \boldsymbol{C}(X, Y)$ to be a *weak equivalence* when it has a homotopy inverse, that is there exists an

$$s \in \boldsymbol{C}(X, Y) \text{ such that } s \cdot t \simeq 1_X \text{ and } t \cdot s \simeq 1_Y.$$

Let Σ be the class of such weak equivalences in $\boldsymbol{C}$.

The cylinder functor I is said to be *generating* for $\boldsymbol{C}$ (compare [6]) whenever $(\boldsymbol{C}, I, \phi)$ is an I-*category* in the sense of Baues [1], with respect to the classes Σ of weak equivalences above and the class Γ of *cofibrations*, defined by the usual homotopy extension property. Let us denote by ϕ the initial object of $\boldsymbol{C}$.

Whenever I is generating, the class Σ of weak equivalences admits a calculus of left fractions in $\boldsymbol{C}$ and the category

$\boldsymbol{C}[\Sigma^{-1}] = \mathrm{Ho}\boldsymbol{C}$ is called the *homotopy category* of $\boldsymbol{C}$ with respect to I. For every pair of objects X, Y of $\boldsymbol{C}$, $\mathrm{Ho}\boldsymbol{C}(X,Y) = [X,Y]$ is the set of homotopy classes of morphisms X→Y in $\boldsymbol{C}$.

c) Let $J = (J, d_0, d_1, \delta)$ and $I = (I, e_0, e_1, \sigma)$ be cylinder functors for the categories $\boldsymbol{A}$ and $\boldsymbol{C}$, respectively. We say that $F: \boldsymbol{C} \to \boldsymbol{A}$ *respects the cylinder functors* whenever the following hold:

(i) $F \cdot I = J \cdot F$,

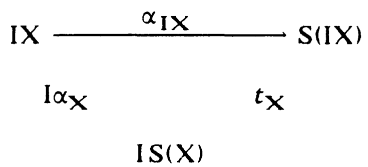
(ii) a) $F \cdot e_i = d_i \cdot F$, $i = 0,1$; b) $F \cdot \sigma = \delta \cdot F$.

2. FUNCTORS PRESERVING CYLINDERS.

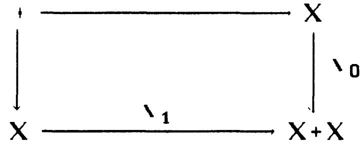
It is easily seen that a functor $F: \boldsymbol{C} \to \boldsymbol{A}$ which respects the cylinder functors preserves homotopies: in particular F preserves weak equivalences and induces a uniquely determined functor $\mathrm{Ho}F: \mathrm{Ho}\boldsymbol{C} \to \mathrm{Ho}\boldsymbol{A}$ between the homotopy categories.

The converse is not true in general: the simplest example is perhaps a constant functor TOP→TOP which induces a constant functor between the homotopy categories, but does not preserve homotopies. We wish to study this situation in detail, in the case where $\boldsymbol{A}$ is a reflective subcategory of $\boldsymbol{C}$ with inclusion T such that $T \cdot I = I \cdot T$ and reflector S which preserves weak equivalences.

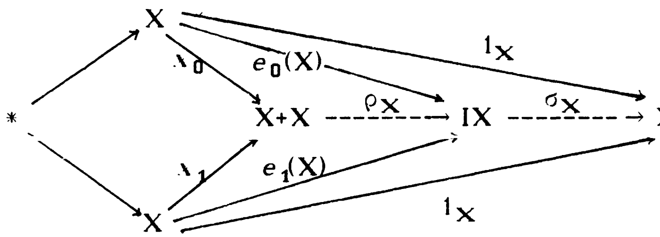
Let us denote by $\alpha: 1 \to T \cdot S$ the unit of this adjunction: then by the universal property of the reflection there exists a unique morphism t_X which renders the following diagram commutative:



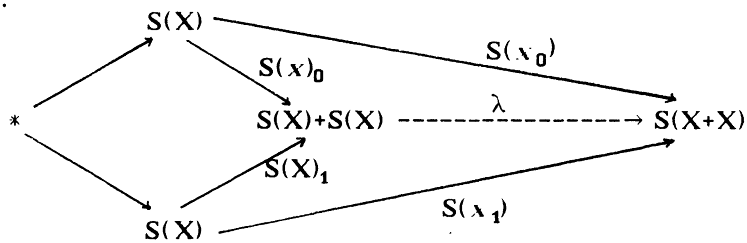
Assume now that I is a generating cylinder functor for $\boldsymbol{C}$. For every object $X \in \boldsymbol{C}$, there exists the pushout



As for notations, let us also consider the following commutative diagrams



from which it follows that $\sigma_X \cdot \rho_X = (1_X, 1_X)$ is the folding map for X.



LEMMA 2.1. *For every* $X \in \mathbf{C}$ *the following holds:*

(i) $t_X \cdot S(e_0(X)) = e_0(S(X))$; *in particular* t_X *is a weak equivalence.*

(ii) $S(1_X \cdot 1_X) \cdot \lambda = (1_{S(X)}, 1_{S(X)})$.

(iii) $t_X \cdot S(\rho_X) \cdot \lambda = \rho_{S(X)}$.

(iv) $\sigma_{S(X)} \cdot t_X = S(\sigma_X)$.

PROOF. Consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\alpha_X} & S(X) & = & S(X) \\ \downarrow{\scriptstyle e_0(X)} & & \downarrow{\scriptstyle S(e_0(X))} & \searrow{\scriptstyle h_X} & \downarrow{\scriptstyle e_0(S(X))} \\ IX & \xrightarrow[\alpha_{IX}]{} & S(IX) & \xrightarrow[t_X]{} & IS(X) \end{array}$$

where the left square is commutative. By the universal property of the reflection. there exists a unique morphism h_X such that

$$h_X \cdot \alpha_X = t_X \cdot \alpha_{IX} \cdot e_0(X) = I\alpha_X \cdot e_0(X).$$

Hence the following diagram is also commutative:

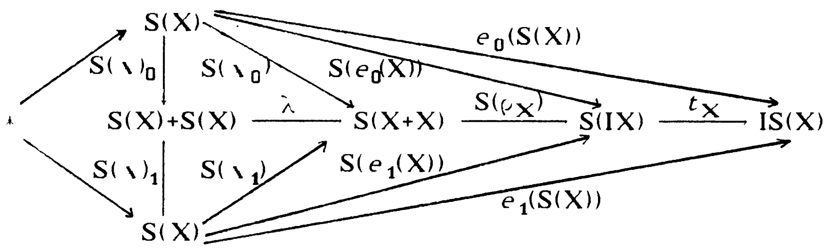
$$\begin{array}{ccc} X & \xrightarrow{\alpha_X} & S(X) \\ \downarrow{\scriptstyle e_0(X)} & & \downarrow{\scriptstyle h_X} \\ IX & \xrightarrow[I\alpha_X]{} & S(IX) \end{array}$$

and this forces $h_X = e_0(S(X))$. Finally

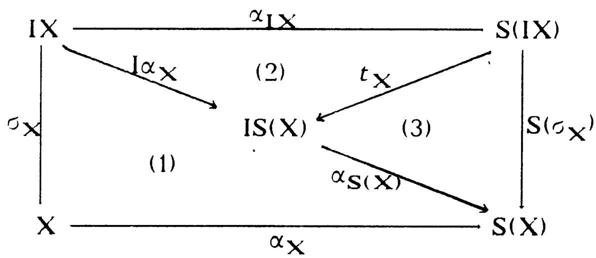
$t_X \cdot \alpha_{IX} \cdot e_0(X) = e_0(S(X)) \cdot \alpha_X$ and $t_X \cdot S(e_0(X)) \cdot \alpha_X = e_0(S(X)) \cdot \alpha_X$.

Hence $t_X \cdot S(e_0(X)) = e_0(S(X))$. t_X is a weak equivalence since $e_0(X)$ is, for every X, and S preserves weak equivalences.

Part (ii) follows from the second diagram, applying the functor S. Part (iii) follows from (i) considering the diagram:



(iv) Consider again a diagram:

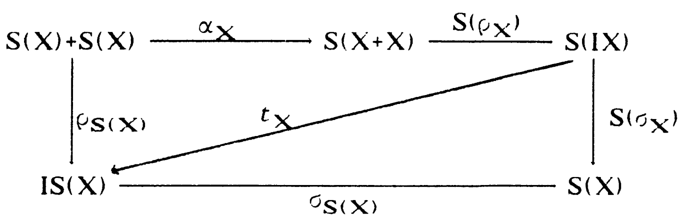


The outer square is commutative since α is a natural transformation. Square (1) commutes since σ is a natural transformation. Triangle (2) commutes by assumption. Let us prove that triangle (3) is also commutative:

$$(\sigma_{S(X)} \cdot t_X) \cdot \alpha_{IX} = \sigma_{S(X)} \cdot (t_X \cdot \alpha_{IX}) = \sigma_{S(X)} \cdot I\alpha_X = S(\sigma_X) \cdot \alpha_{IX}.$$

By the universal property of the reflection it follows that $\sigma_{S(X)} \cdot t_X = S(\sigma_X)$.

Let us observe that the previous lemma implies that the following diagram is commutative:



In ([1], §5a, p. 112) a diagram similar to the above is constructed in order to prove via a "weak lifting" $L = (h \cdot j)$ of it, that the correspondence

$$[IX, Y] \xrightarrow{\quad S \quad} [S(IX), S(Y)] \xrightarrow{\quad L^* = h^* (j^*)^{-1} \quad} [IS(X), S(Y)]$$

given by $(L^* \cdot S)(H) = L^*(S(H))$, carries (homotopy classes of) homotopies to (homotopy classes of) homotopies. A condition on a general functor S, for L^* to be a bijection is that S be compatible with pushouts of the form X+X (see [1]). In case S is a reflector, as we do assume, the work above allows us to obtain the following

THEOREM 2.2. $L^* = (t_X^*)^{-1}$.

In other words L^* is a bijection, which may be restated by saying that the reflector S "respects the cylinder functor up to homotopy".

Let us observe that the phrase "S respects the cylinder functor" above is not correct since $\boldsymbol{A}$ has not its own cylinder functor as well. To be precise we put the following definitions.

DEFINITION 2.3. a) A full subcategory $\boldsymbol{A}$ of $\boldsymbol{C}$ is called a *homotopy subcategory (h-subcategory, for short)* whenever $I(A) \in \boldsymbol{A}$, for every $A \in \boldsymbol{A}$.

In other words, $\boldsymbol{A}$ is a h-subcategory of $\boldsymbol{C}$ when the restriction of I to $\boldsymbol{A}$ is a cylinder functor for $\boldsymbol{A}$ itself.

Let us denote by Ho$\boldsymbol{A}$ the category obtained by formally inverting the weak equivalences of $\boldsymbol{C}$ that are contained in $\boldsymbol{A}$. Ho$\boldsymbol{A}$ is the full subcategory of Ho$\boldsymbol{C}$ having the same objects as $\boldsymbol{A}$.

b) Let now I and J be cylinder functors for $\boldsymbol{C}$ and $\boldsymbol{A}$, respectively, let again $S: \boldsymbol{C} \to \boldsymbol{A}$ be left adjoint to $T: \boldsymbol{A} \to \boldsymbol{C}$ and assume that T respects the cylinder functors.

We say that the functor $S: \boldsymbol{C} \to \boldsymbol{A}$ *respects homotopies* whenever the correspondence

$$\boldsymbol{A}(JS(X), A) \to \boldsymbol{A}(SI(X), A), \text{ given by } K \vdash K \cdot t_X,$$

is onto, for every $A \in \boldsymbol{A}$. Then S respects homotopies iff t_X is a section, as one easily verifies.

PROPOSITION 2.4. *Let* $\boldsymbol{A}$ *be an epireflective h-subcategory of* $\boldsymbol{C}$

with reflector S: $\boldsymbol{C} \to \boldsymbol{A}$ *which respects homotopies. If* I *preserves epimorphisms then* S *respects the cylinder functor and* HoS: $\mathrm{Ho}\boldsymbol{A} \to \mathrm{Ho}\boldsymbol{C}$ *is a reflector.*

We note that, since the epimorphisms in $\boldsymbol{C}$ = TOP are the onto continuous maps, then, whenever S is the epireflector of a full subcategory $\boldsymbol{A}$ in TOP, which contains the unit interval I, the following statements are equivalent (cf. [10] Th. 1.2):

i) $S(X \times I) = S(X) \times I$, for every space X.

ii) S respects homotopies.

iii) S takes homotopic maps to homotopic maps.

HoS: HoTOP $\to$ Ho$\boldsymbol{A}$ is a reflector whenever these hold.

It is shown in [10] that every quotient reflective subcategory $\boldsymbol{A}$ of TOP such that $I \in \boldsymbol{A}$ satisfies the conditions above. This can be obtained from the following more general result, using Theorem 3.5 of Schwarz [9].

PROPOSITION 2.5. *Let* $\boldsymbol{C}$ *be a monotopological category with a cylinder functor* $- \times I$ *where* I *is an exponential object of* $\boldsymbol{C}$*. Let* $\boldsymbol{A}$ *be a quotient reflective subcategory of* $\boldsymbol{C}$ *such that* $I \in \boldsymbol{A}$*. Then the reflector* S *respects the cylinder functor and* HoS *is still a reflector.*

3. THE HOMOTOPY STRUCTURE.

Let us recall from [4,5] that the cylinder functor I induces on $\boldsymbol{C}$ a *semicubical homotopy system* $Q_I: \boldsymbol{C} \cdot \boldsymbol{C} \to \boldsymbol{K}$. For every $X, Y \in \boldsymbol{C}$, $Q_I(X,Y)$ is the semicubical complex having $\boldsymbol{C}(I^n(X), Y)$ as the set of n-cubes, where $I^0(X) = X$ and, for every $n \geq 1$, $I^n(X) = I(I^{n-1}(X))$. Face and degeneracy operators are defined, respectively, by the following:

$$\varepsilon^i_n = \boldsymbol{C}(I^{i-1}(e_\varepsilon(I^{n-1}(X), 1_Y) \text{ and } \xi^j_n = \boldsymbol{C}(I^{j-1}(\sigma_\varepsilon(I^{n+1-j}(X),), 1_Y)$$

$\varepsilon = 0,1$. The *edge* of a $\varphi \in \boldsymbol{C}(I^n X, Y)$ is defined to be

$$D\varphi = (0^1_n\varphi, 1^1_n\varphi, \dots, 0^n_n\varphi, 1^n_n\varphi).$$

For every pair $X, Y \in \boldsymbol{C}$, we can construct the *fundamental groupoid* $\Pi_I(X,Y)$. Its objects are the 0-cubes of $Q_I(X,Y)$, while a morphism $f \to g$ in $\Pi_I(X,Y)$ is an equivalence class $[\alpha]$ of 1-cubes with $D\alpha = (f,g)$, with respect to the following relation:

if $\alpha,\beta \in \boldsymbol{C}(\mathrm{IX},\mathrm{Y})$, then $\alpha \simeq \beta$ whenever a $\varphi \in \boldsymbol{C}(\mathrm{I}^2\mathrm{X},\mathrm{Y})$ exists, in such a way that

$$D\varphi = (\alpha,\beta,\xi_0^1 0_1^1\alpha,\xi_0^1 1_1^1\alpha).$$

The fundamental groupoid may also be considered as a functor $\Pi_{\mathrm{I}}: \boldsymbol{C}\times\boldsymbol{C}\to\mathrm{Grd}$.

If $f: \mathrm{X}\to\mathrm{Y}$ is a homotopy equivalence in $\boldsymbol{C}$, there are induced natural transformations

$$f^*: \Pi_{\mathrm{I}}(\mathrm{X},\mathrm{Z}) \to \Pi_{\mathrm{I}}(\mathrm{Y},\mathrm{Z}),\quad f_*: \Pi_{\mathrm{I}}(\mathrm{Z},\mathrm{X}) \to \Pi_{\mathrm{I}}(\mathrm{Z},\mathrm{Y})$$

which are natural equivalences of groupoids, for every $\mathrm{Z}\in\boldsymbol{C}$.

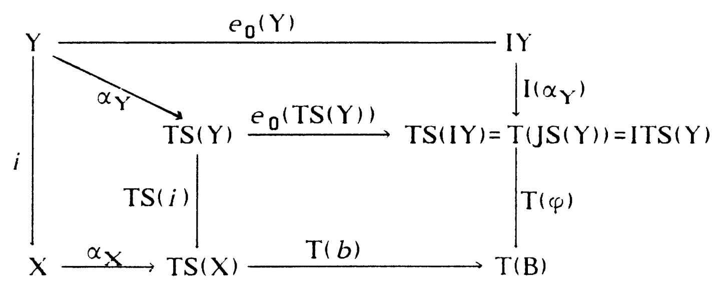
Moreover, any functor $\mathrm{F}: \boldsymbol{C}\to\boldsymbol{A}$ which respects the cylinder functors (I for $\boldsymbol{C}$ and J for $\boldsymbol{A}$) induces a natural transformation $\Pi_{\mathrm{I}}(\mathrm{X},\mathrm{Y})\to\Pi_{\mathrm{J}}(\mathrm{F}(\mathrm{X}),\mathrm{F}(\mathrm{Y}))$. In fact, F preserves homotopies, hence it takes n-cubes to n-cubes, and also it preserves the equivalence of 1-cubes, as one verifies with a short calculation.

THEOREM 3.1. *Let* $\mathrm{S}: \boldsymbol{C}\to\boldsymbol{A}$ *be left adjoint to* $\mathrm{T}: \boldsymbol{A}\to\boldsymbol{C}$ *and assume that they respect the cylinder functors. Then*

(i) S preserves weak equivalences and cofibrations;

(ii) For every $\mathrm{A}\in\boldsymbol{A}$ *and* $\mathrm{X}\in\boldsymbol{C}$, *there is an isomorphism of groupoids* $\Pi_{\mathrm{I}}(\mathrm{X},\mathrm{T}(\mathrm{A}))\approx\Pi_{\mathrm{J}}(\mathrm{S}(\mathrm{X}),\mathrm{A})$.

PROOF. For (i) we have only to show that S preserves cofibrations. Let $i: \mathrm{Y}\to\mathrm{X}$ be a cofibration in $\boldsymbol{C}$ and consider a morphism $b: \mathrm{S}(\mathrm{X})\to\mathrm{B}$ and a homotopy $\psi: \mathrm{JS}(\mathrm{Y})\to\mathrm{B}$ in $\boldsymbol{A}$, such that $\psi\cdot d_0(\mathrm{S}(\mathrm{Y})) = b\cdot\mathrm{S}(i)$. Consider the commutative diagram.



There exists a homotopy $\Phi: \mathrm{I}(\mathrm{X})\to\mathrm{T}(\mathrm{B})$ in $\boldsymbol{C}$, such that

$$\Phi\cdot e_0(\mathrm{X}) = \mathrm{T}(b)\cdot\alpha_{\mathrm{X}} \text{ and } \Phi\cdot\mathrm{I}(i) = \mathrm{T}(\psi)\cdot\mathrm{I}(\alpha_{\mathrm{Y}}).$$

Then $\beta_{\mathrm{B}}\cdot\mathrm{S}(\Phi): \mathrm{SI}(\mathrm{X}) = \mathrm{IS}(\mathrm{X})\to\mathrm{B}$ is a homotopy in $\boldsymbol{A}$ such that

$$\beta_{\mathrm{B}}\cdot\mathrm{S}(\Phi)\cdot d_0(\mathrm{S}(\mathrm{X})) = \beta_{\mathrm{B}}\cdot\mathrm{S}(\Phi)\cdot\mathrm{S}(e_0(\mathrm{X})) = \beta_{\mathrm{B}}\cdot\mathrm{S}(\Phi\cdot e_0(\mathrm{X}))$$

$$= \beta_B \cdot ST(b) \cdot S(\alpha_X) = b \cdot \beta_{S(X)} \cdot S(\alpha_X) = b.$$

Moreover

$$\beta_B \cdot S(\Phi) \cdot J(S(i)) = \beta_B \cdot S(\Phi) \cdot S(I(i)) = \beta_B \cdot S(\Phi I(i))$$
$$= \beta_B \cdot ST(\psi) \cdot SI(\alpha_Y) = \beta_B \cdot S(T(\psi)) \cdot JS(\alpha_Y) = \psi \cdot \beta_{JS(Y)} \cdot JS(\alpha_Y)$$
$$= \psi \cdot \beta_{SI(Y)} \cdot S(\alpha_{IY}) = \psi \cdot S(\text{identity}) = \psi.$$

It follows that $S(i): S(Y) \to S(X)$ is a cofibration in $\boldsymbol{A}$. Part (ii) follows from the discussion above and Proposition 2.3.

4. A GENERALIZATION.

We wish to consider now the case of *proreflectors*. Such functors arise in Shape Theory [7] and are a weakened form of reflectors. Here one deals with the procategory Pro$\boldsymbol{C}$ of the given category $\boldsymbol{C}$, whose objects are the inverse systems of objects of $\boldsymbol{C}$ of the form $\underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$. The cylinder functor I on $\boldsymbol{C}$ extends naturally to Pro$\boldsymbol{C}$ by taking $I\underline{X} = (IX_\lambda, Ip_{\lambda\lambda'}, \Lambda)$.

Let $\boldsymbol{A}$ be a full subcategory of $\boldsymbol{C}$, then a proreflector P: $\boldsymbol{C} \to$ Pro$\boldsymbol{A}$ is a functor which assigns to every $X \in \boldsymbol{C}$ an inverse system $\underline{X} \in$ Pro$\boldsymbol{A}$ and a morphism $X \to \underline{X}$ in Pro$\boldsymbol{C}$ which is initial with respect to every other morphism $X \to \underline{Y}$, $\underline{Y} \in$ Pro$\boldsymbol{A}$.

We refer to [7, 8, 11] for the definition of a procategory and related concepts.

Let us recall [11] that $\boldsymbol{A}$ is proreflective in $\boldsymbol{C}$ by means of P iff Pro$\boldsymbol{A}$ is reflective in Pro$\boldsymbol{C}$ by means of the functor P^* given by the composition of the extension of P to the procategories, ProP: Pro$\boldsymbol{C}$ $\to$ ProPro$\boldsymbol{A}$, with the inverse limit functor

$$\text{invlim}: \text{ProPro}\boldsymbol{A} \to \text{Pro}\boldsymbol{A}.$$

Recently Porter [8] has shown that the right homotopy category HoPro$\boldsymbol{C}$ of Pro$\boldsymbol{C}$ is that obtained by formally inverting the level homotopy equivalences in Pro$\boldsymbol{C}$. A level morphism in Pro$\boldsymbol{C}$ is a morphism between inverse systems indexed over the same directed set, which is actually a natural transformation.

THEOREM 4.1. *Let* $\boldsymbol{A}$ *be a proreflective h-subcategory of* $\boldsymbol{C}$, *with proreflector* P: $\boldsymbol{C} \to$ Pro$\boldsymbol{A}$. *If* P *respects the cylinder functors, then* HoPro$\boldsymbol{A}$ *is reflective in* HoPro$\boldsymbol{C}$ *by means of* HoP^*.

PROOF. We have only to show that P^* takes level homotopy equivalences in Pro$\boldsymbol{C}$ to level homotopy equivalences in Pro$\boldsymbol{A}$. Let $\underline{f}$: $\underline{X} \to \underline{Y}$ be a level homotopy equivalence in Pro$\boldsymbol{C}$; then ProP$(\underline{f})$ is an inverse system of homotopy equivalences in Pro$\boldsymbol{A}$. Finally P $(\underline{f})$ is a level homotopy equivalence in Pro$\boldsymbol{A}$. To see this one can look at the explicit description of the functor P, as given in ([11], 2.6) and making use of the reindexing theorem ([7], 3.3, Ch. 1).

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