CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

FERNANDO GONZALEZ RODRIGUEZ
ANTONIO BAHAMONDE

The topology of continuous partially-additive monoids

Cahiers de topologie et géométrie différentielle catégoriques, tome 31, nº 1 (1990), p. 13-19
<http://www.numdam.org/item?id=CTGDC_1990__31_1_13_0>

CAHIERS DE TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE CATÉGORIQUES

VOL. XXXI-1 (1990)

THE TOPOLOGY OF CONTINUOUS PARTIALLY-ADDITIVE MONOIDS

by Fernando Gonzalez RODRIGUEZ and Antonio BAHAMONDE

RÉSUMÉ. La Sémantique dénotationnelle peut être étudiée du point de vue ordonné (Scott) ou partiellement additif (Arbib & Manes); ici l'équivalence des deux est mise en relief par l'étude des propriétés topologiques du cadre naturel pour cette sémantique: l'ensemble des fonctions partiellement définies d'un ensemble de données A vers un ensemble de résultats B. Cet ensemble est muni d'une topologie naturelle induite par sa structure partiellement additive (pam); et on prouve que dans ce type de pam, l'ordre est équivalent à la topologie.

INTRODUCTION.

The question of providing a mathematical meaning to computer programs has got several approaches in last decades. The denotational semantics introduced by Dana Scott (see [9,6]) offers algebraic techniques for characterizing the denotation of a program; so, a partially-defined function transforming input data to output results can be computed from the syntax of the program. Therefore, the properties of a program can then be checked by direct comparison of the denotation with the specification.

Nowadays we have two viewpoints in denotational semantics: order semantics (Scott) and partially-additive semantics (Arbib and Manes [1,2,5]). In any case, when dealing with recursive programs, one has to compute a limit in the set of partially-defined functions from a set of input A to a set of outputs B, denoted Pfn(A,B).

In this paper we try to reinforce the equivalence of those approaches providing a study of the topological properties of a special kind of partially-additive monoid (pam) defined by Arbib and Manes in order to have an algebraic setting to build their semantic theory.

The topology of pams was introduced by Bahamonde in [3] and here we prove that in continuous pams (Pfn(A,B) is one of those pams) the topology induced by its additive structure is equivalent to their ordering.

1. THE ALGEBRAIC STRUCTURE OF PARTIALLY-ADDITIVE MONOIDS (PAMS).

1.1. DEFINITION [1]. A *partially-additive monoid* (pam) is a pair (A,Σ^{A}), where A is a non empty set, Σ^{A} is a partial operation on countable (i.e. finite or denumerable) families in A subject to the following axioms:

(Pam 1) Partition-associativity axiom: If the countable set I is partitioned into $(I_j : j \in J)$ (i.e., $(I_j : j \in J)$ is a countable family of pairwise disjoint sets whose union is I), then for each family $(x_i : i \in I)$ in A,

$$\Sigma^{A}(x_i : i \in I) = \Sigma^{A}(\Sigma^{A}(x_i : i \in I_j) : j \in J),$$

in the sense that the left hand side is defined iff the right is defined, and then the values are equal.

(Pam 2) Unary sum axiom: For the one-element families the sum is defined and $\Sigma^{A}(a) = a$.

(Pam 3) Limit axiom: If $(x_i : i \in I)$ is a countable family in A and if $\Sigma^{A}(x_i : i \in F)$ is defined for every finite subset F of I, then $\Sigma^{A}(x_i : i \in I)$ is defined.

The families for which Σ^{A} is defined are called A-*summable.*

1.2. From (Pam 1) we infer that any subfamily of a summable family is summable, so given that (Pam 2) guarantees the existence of some sums, the sum of the empty family is defined. This sum will be denoted by $\perp_A$ and can be easily proved to be the countable zero element of (A,Σ^{A}).

We usuallly drop the A from Σ^{A}, $\perp_A$ and A-summable if no confusion arises. To ease the reading we eventually write

$$\Sigma(x_i : i \in \{0,\dots,n\}) = \Sigma(x_i : i := 0,\dots,n) = x_0 + \dots + x_n.$$

1.3. It can be easily seen that a pam is countable-commutative. In fact, if $(x_i : i \in I)$ and $(y_j : j \in J)$ are countable families in A, and $\psi : I \to J$ is a bijection such that $y_{\psi(i)} = x_i$ for all $i \in I$, then from (Pam 1) and (Pam 2) axioms we have that

$$\Sigma(x_i : i \in I) = \Sigma(\Sigma(x_i : i \in I_j = \{\psi^{-1}(j)\}) : j \in J) = \Sigma(y_j : j \in J)$$

in the sense that the left side is defined iff the right side is defined, and then they are equal.

1.4. A pam satisfies the following "positivity property":

$$(\Sigma(x_i : i \in I) = \perp) \Rightarrow (\forall\ i \in I,\ x_i = \perp).$$

That is why pams are called positive partial monoids in [8]. To prove this, let $i \in I$ and set y to be $\Sigma(x_j : j \in I - \{i\})$. Then

$$x_i = x_i + (y + x_i) + (y + x_i) + \ldots = (x_i + y) + (x_i + y) + \ldots = \perp .$$

1.5. If (A,Σ) is a pair satisfying (Pam 1) and (Pam 2), then (Pam 3) is equivalent to the following axiom:

(Pam 3') If $(x_i : i \in N)$ is a family in A such that $(x_i : i := 0, \ldots, n)$ is summable for all $n \in N$, then $(x_i : i \in N)$ is summable.

1.6. Types of Pams.

DEFINITION. Let (A,Σ) be a pam and $a, b \in A$, we say that *a is smaller than b* (denoted, as usual, by $a \leq b$) iff there exists $c \in A$ such that $a + c = b$. In symbols

$$a \leq b \Leftrightarrow (\exists\ c \in A \mid a + c = b).$$

This is a reflexive and transitive relation but, in general, it fails to be antisymmetric; we will give some counterexamples below.

DEFINITION. (A,Σ) is an *ordered pam* iff the relation "$\leq$" defined above is an ordering in A; that is to say, iff "$\leq$" is antisymmetric.

DEFINITION [3]. A pam (A,Σ) is said to be a *continuous pam* provided that whenever $\Sigma(x_i : i := 0, \ldots, n) \leq a$ for all $n \in N$, then $\Sigma(x_i : i \in N) \leq a$.

1.7. EXAMPLES.

The basic example in denotational semantics of computer programs is the set of partially defined functions from a set A to a set B: Pfn(A,B). Here the sum is defined for families of functions $(f_i : i \in I)$ whose domains of definition DDf_i are pairwise disjoint, and we define

$$\Sigma^{di}(f_i : i \in I)(a) = \begin{cases} f_i(a) & \text{if } \exists\ i \in I \mid a \in DDf_i \\ \text{undefined else} & \end{cases}$$

This definition can be extended to allow sums of coherent overlapping families of functions, i.e., $f_i(a) = f_j(a)$ whenever a is in $DDf_i \cap DDf_j$. We will denote this extended sum by Σ^{ov}.

Obviously, the "$\leq$" relation defined in 1.6 is the same for these sums and it is an order relation in Pfn(A,B). Moreover, it is the usual extension ordering of partial functions; that is,

$$f \leq g \Leftrightarrow (DDf \subset DDg \text{ and } \forall\ x \in DDf,\ f(x) = g(x)).$$

Let $(L,<)$ be an ordered set with least upper bound (lub) of any countable subset, then (L,lub) is a pam where the order relation induced by its pam structure is the given one at the beginning.

Let W be a set and let S be a nonempty collection of subsets of W closed for countable unions. Then, (S, $\cup$) is a pam. Moreover, S endowed with the disjoint union is also a pam, and in any case we have the natural order of subsets.

In the unit interval [0,1], we define the sum $\Sigma(x_i: i \in \mathrm{I})$ to be: the usual series sum whenever it is defined and belongs to [0,1], $\Sigma(x_i: i \in \emptyset) := 0$; and $\Sigma(x_i: i \in \mathrm{I})$ is undefined otherwise. So defined, the pair ([0,1],Σ) is an ordered pam which induces the usual order again.

Let (G,+) be a commutative monoid and define G^ (cf. [7]) to be the disjoint union of G and $\{\perp,\mathrm{T}\}$, where $\perp$ and T do not belong to G. Then it is possible to provide G^ with a pam structure such that $\perp$ is the countable zero defining the sum for families in G$\cup$\{T\} as follows:

$$\Sigma(a_i: i \in \mathrm{I}) := \begin{cases} \mathrm{T} & \text{if I is infinite or finite but } a_i = \mathrm{T} \text{ for some } i \in \mathrm{I} \\ a_1 + \ldots + a_n & \text{if } \mathrm{I} = \{1,\ldots,n\} \text{ and } \forall\, i \in \mathrm{I},\ a_i \in \mathrm{G} \end{cases}$$

where $a_1 + \ldots + a_n$ is the sum in (G,+). It can easily be shown that in general the "smaller than" relation here is not an ordering.

2. PAM TOPOLOGY.

A pam (A,Σ) can be provided with a natural topology as was shown in [3]. In this section we recall this topology and some immediate properties.

DEFINITION. Let (A,Σ) be a pam. A subset $\mathrm{U} \subset \mathrm{A}$ is said to be in $\tau\Sigma$ iff it satisfies the following two axioms:

(1) $y \in \mathrm{U}$ whenever $x \in \mathrm{U}$ and $x \leq y$ (1.6);

(2) if $\Sigma(x_i: i \in \mathrm{N}) \in \mathrm{U}$, then there exists $n \in \mathrm{N}$ such that $\Sigma(x_i: i := 0,\ldots,n) \in \mathrm{U}$.

The collection $\tau\Sigma$ is a topology (the additive topology) in A.

Given that the countable zero, $\perp$, is "smaller than" any other element in A, from the first axiom we infer that the only open set that contains $\perp$ is A. Therefore $\tau\Sigma$ is trivially compact and connected.

3. MAIN THEOREM.

This section is devoted to prove the central result of this paper which stresses the relation between the topology and the "smaller than" ($\leq$) relation for continuous pams.

3.1. To prove this theorem we will need to state the following:

PROPOSITION. *Let* (A,Σ) *be a continuous pam and let* $a, b \in A$. *Then the next statements are equivalent:*

(i) $a \leq b$;
(ii) $a \in \{b\}^\wedge$ *(the topological closure of* $\{b\}$ *in* $(A,\tau\Sigma)$;
(iii) $\forall\, U \in \tau\Sigma,\ a \in U \Rightarrow\ b \in U$.

PROOF. The equivalence between (ii) and (iii) is standard. Moreover, (i) implies (iii) from the defintion of $\tau\Sigma$. To prove that (iii) implies (i) let us assume that $a \neg\leq b$; i.e.,

$$a \in Rb := \{c \in A:\ c \neg\leq\ b\}.$$

So it will be enough if we show that Rb is an open set since we have $b \not\in Rb$; but this is trivial because if

$$\Sigma(x_i:\ i \in N) \in Rb \Leftrightarrow \Sigma(x_i:\ i \in N) \neg\leq\ b,$$

then there exists an integer n such that $\Sigma(x_i:\ i := 0,\dots,n) \in Rb$ since otherwise,

$$\forall\, n\ \Sigma(x_i:\ i := 0,\dots,n) \leq\ b \text{ implies that } \Sigma(x_i:\ i \in N) \leq\ b$$

given that (A,Σ) is continuous

3.2. THEOREM. *Let* (A,Σ) *and* (A,Σ') *be continuous pams. then the associated topologies are equivalent iff the induced "smaller than" relations are. In symbols:*

$$\tau\Sigma = \tau'\Sigma \ \Leftrightarrow\ \text{"}\leq\text{"} = \text{"}\leq'\text{"}.$$

PROOF. The induced "$\leq$" relations coincide if $\tau\Sigma = \tau\Sigma'$ due to the last proposition. To see the opposite it is enough to establish that $\tau\Sigma \subset \tau\Sigma'$. To do this let $U \in \tau\Sigma$ be such that $\Sigma'(x_i:\ i \in N) \in U$ for some family Σ'-summable in A. We must find an integer n for which $\Sigma'(x_i:\ i := 0,\dots,n) \in U$. First of all, we are going to build up by induction a family $(y_i:\ i \in N)$ such that

$$\forall\, n \in N,\ \Sigma(y_i:\ i := 0,\dots,n) = \Sigma'(x_i:\ i := 0,\dots,n).$$

Define $y_0 := x_0$ and having built y_n notice that

$$\Sigma(y_i:\ i := 0,\dots,n) = \Sigma'(x_i:\ i := 0,\dots,n) \leq \Sigma'(x_i:\ i := 0,\dots,n+1),$$

therefore

$$\exists y_{n+1} \in A \mid \Sigma(y_i : i := 0, \dots, n) + y_{n+1} = \Sigma'(x_i : i := 0, \dots, n+1).$$

Now we have that

$$\forall n \; \Sigma'(x_i : i := 0, \dots, n) = \Sigma(y_i : i := 0, \dots, n) \le \Sigma(y_i : i \in N),$$

and taking account that (A, Σ') is continuous, we have that

$$\Sigma'(x_i : i \in N) \le \Sigma(y_i : i \in N).$$

Since $U \in \tau\Sigma$ we have that $\Sigma(y_i : i \in N) \in U$ and therefore there exists an integer n such that

$$\Sigma'(x_i : i := 0, \dots, n) = \Sigma(y_i : i := 0, \dots, n) \in U$$

as we wanted to show.

3.3. Now it is trivial to show that the pam topologies induced by $(\mathrm{Pfn}(A,B), \Sigma^{di})$ and $(\mathrm{Pfn}(A,B), \Sigma^{ov})$ (1.7) are the same since they have the same ordering associated. Moreover, the pam topology of Pfn(A,B) is the Scott countable topology. In fact, let $\sup(f_i : i \in I)$ be the least upper bound of $\{f_i : i \in I\}$ whenever this is a countable directed family. The pair $(\mathrm{Pfn}(A,B), \sup)$ is a pam that induces the natural ordering of partially-defined functions. Therefore its pam topology is the pam topology of $(\mathrm{Pfn}(A,B), \Sigma^{di})$ or $(\mathrm{Pfn}(A.B). \Sigma^{ov})$.

On the other hand, $U \subset \mathrm{Pfn}(A,B)$ is an open set (i.e., $U \in \tau \sup$) iff:

(1) $g \in U$ whenever $f \in U$ and $f \le g$:

(2) if D is countable directed, and $\sup(D) \in D$, then $D \cap U \neq \emptyset$.

But this is the Scott topology of Pfn(A,B) [9].

3.4. Finally, it is noteworthy that the previous theorem does not need the antisymmetric property of the "smaller than" ($\le$) relation. Moreover, due to the Proposition (3.1) a continuous pam (A, Σ) is ordered iff $(A, \tau\Sigma)$ is a T_0-space; and, in that case, the "smaller than" relation is the specialization order [6] of a T_0-space.

REFERENCES.

1. ARBIB, M.A. & MANES, E.G., Partially-additive monoids, graph-growing, and the algebraic semantics of recursive calls, *Lecture Notes in Computer Science* 73, Springer (1979), 127-138.
2. ARBIB, M.A. & MANES, E.G., The pattern-of-calls expansion is the canonical fixpoint for recursive definitions, *J. of the Association for Comput. Machinery*, ACM 29 (1982), 577-602.
3. BAHAMONDE, A., Partially-additive monoids, *Cahiers Top. et Géom. Diff. Cat.* XXVI-3 (1985), 221-244.
4. BAHAMONDE, A., Tensor product of partially-additive monoids, *Semigroup Forum* 32 (1985), 31-53.
5. MANES, E.G & ARBIB, M.A., *Algebraic approaches to program semantics*, Texts and monographs in Computer Science, Springer 1980.
6. GIERZ, G., HOFMANN, K.H., KEIMEL, K. LAWSON, J.D., MISLOVE, M. & SCOTT, D., *A compendium of continuous lattices*, Springer 1980.
7. GONZALEZ RODRIGUEZ, M.F., Monoides parcialmente aditivos, PhD dissertation, Univ. of Oviedo 1989.
8. MANES, E.G. & BENSON, D.B., The inverse semigroup of a sum-ordered semiring, *Semigroup Forum* 31 (1985), 129-152.
9. SCOTT, D., Continuous lattices, *Lecture Notes in Math.* 274, Springer (1972).

F. GONZALEZ RODRIGUEZ :
DEPARTAMENTO DE MATEMATICAS
UNIVERSIDAD DE LEON
SPAIN

A. BAHAMONDE:
ARTIFICIAL INTELLIGENCE CENTER
UNIVERSITY OF OVIEDO AT GIJON
C. RAMIRO DE MAETZU
E-33201 GIJON. SPAIN