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FINITARY FIBRATIONS

by Grzegorz JARZEMBSKI

RÉSUMÉ. Dans cet article, on caractérise les catégories concrètes représentables comme catégories de modèles de théories universelles finitaires, purement relationnelles, sans égalité. Le principal théorème assure que l'existence d'une représentation du premier ordre de cette sorte équivaut à l'existence de topologies convenables sur les fibres d'une certaine catégorie concrète.

The classical Beck-Linton Theorem ([5] p. 142) characterizing concrete categories representable by varieties of total algebras became the pattern for categorists investigating connections between classes of concrete categories and categories of models of first order theories of prescribed kinds. A general discussion summarizing results on this area has been presented in [8].

The present paper continues these investigations. We characterize here concrete categories concretely isomorphic to categories of models of universal $L_{\infty\infty}(\Sigma)$-, $L_{\infty\omega}(\Sigma)$- and $L_{\omega\omega}(\Sigma)$-theories without equality [8], where Σ is a finitary relational signature. A concept of fibration is introduced in order to get a convenient categorical framework for our considerations.

An important aspect of the investigated problem is that the considered categories are not assumed to be concretely complete. Thus the desired characterizations need completely new methods of analysis of the considered categories. The most important observation seems to be that the existence of an $L_{\omega\omega}(\Sigma)$-representation is equivalent to the existence of suitable topologizations of fibres of a given fibration. We hope that the results presented in this paper will be useful for more general investigations of representation problems in the inexplorable area of non-concretely complete concrete categories.

We would like to thank J. Rosický for the generosity with which he shared with us results contained in his preprint [9]. One of the examples of this paper, cited here in the last section, inspired us to consider the problem solved here. We also wish to thank J. Adamek and J. Rosický for an interesting discussion which helped us to improve the final version of the paper.

1. PRELIMINARIES.

For all unexplained concepts of category theory we refer the reader to [3]. By a concrete category we will understand a category **C** equipped with a forgetful functor $U: \mathbf{C} \to \mathbf{Set}$ into the category of sets such that the following two conditions are satisfied:

(1) U is transportable, that is if $A \in \mathbf{C}$, X is a set and $f: U(A) \to X$ is a bijection, then there is $B \in \mathbf{C}$ and an isomorphism $g: A \to B$ in **C** such that $U(B) = X$ and $U(g) = f$.

(2) U is amnestic, that is if $A, B \in \mathbf{C}$ and $f: A \to B$ is an isomorphism such that $U(A) = U(B)$ and $U(f) = id_{U(A)}$, then $A = B$ and $f = id_A$.

In what follows, we will often identify a morphism g: $A \to B$ with its underlying function $U(g)$. We say that the mapping $g: U(A) \to U(B)$ is a morphism from A to B.

A concrete category $(\mathbf{C}, U)$ will be often denoted briefly by **C**. Two concrete categories **C**, **D** are concretely isomorphic if there exists an isomorphism $F: \mathbf{C} \to \mathbf{D}$ which commutes with the corresponding underlying functors.

Each pair $(f: X \to U(A), A)$ is called a U-morphism. Each $g: B \to A$ in **C** such that $U(g) = f$ is said to be a *lift of* (f, A). A **C**-morphism $g: A \to B$ is called initial if a function $h: U(D) \to U(B)$ is a **C**-morphism from D to B whenever $U(g) \circ h$ is a morphism from D to A.

DEFINITION 1.1. A concrete category $(\mathbf{C}, U)$ is called a *fibration* iff the following holds:

(i) U is fibre small, i.e., for each set X the collection

$$F_{\mathbf{C}}X = \{A \in \mathbf{C} \mid U(A) = X\}$$

is a set.

(ii) Each U-morphism has an initial lift.

Fibrations in the sense defined above form a subclass of "fibrations with split cleavages" in the sense of Gray ([2] p. 32).

On the other hand, the introduced concept generalizes the Wyler's notion of a topological category [10]. Namely, a given fibration $(\mathbf{C},U)$ is a topological category iff it is concretely complete.

BASIC EXAMPLES. Let $\Sigma = (\Sigma_n | \in N)$ be an arbitrary but fixed finitary relational signature (we do not exclude 0-ary relation symbols). By MOD Σ we shall denote the category of all Σ-models. Empty models are admitted. Because of the presence of 0-ary relation symbols, there may be more than one empty (i.e., with the empty carrier) Σ-model.

MODΣ considered together with the obvious underlying functor U_Σ: MODΣ $\longrightarrow$**Set** is a topological category, hence a fibration. A Σ-homomorphism

$$h: (A,(r^A | r \in \Sigma)) \longrightarrow (B,(r^B | r \in \Sigma))$$

is a U_Σ-initial morphism if for every relation symbol $r \in \Sigma_n \in \Sigma$ and $\tilde{a} \in A^n$, then $\tilde{a} \in r^A \subset A^n$ provided that $h\tilde{a} \in r^B \subset B^n$; i.e., iff h is a strong Σ-homomorphism in the sense of [1].

A class $W \subset$ MODΣ is said to be initially closed iff for each U_Σ-initial homomorphism h: $\underline{A} \to \underline{B}$, $\underline{A} \in W$ provided that $\underline{B} \in W$.

In what follows, whenever we will discuss classes of relational systems of a given signature Σ, we will identify them with the corresponding full subcategories of MODΣ.

Clearly, for every initially closed class $W \subset$ MODΣ, the concrete category $(W, U = U_{\Sigma|W})$ is a fibration.

DEFINITION 1.2. A fibration $(\mathbf{C},U)$ is said to be *weakly finitary* iff it is concretely isomorphic to some initially closed full subcategory of finitary relational systems.

2. CLASSIFICATION OF WEAKLY FINITARY FIBRATIONS.

Initially closed classes of relational systems (and hence weakly finitary fibrations, too) have their natural classification related to their logical description. As before, let $\Sigma = (\Sigma_n | n \in N)$ be an arbitrary but fixed finitary relational signature. Let Var be a proper class of variables.

Following [8], we shall denote by $L_{\infty\infty}(\Sigma)$ the language such that atomic $L_{\infty\infty}(\Sigma)$-formulas are strings of the forms:

(i) $x = y$, where $x,y \in \mathrm{Var}$,

(ii) $r(\bar{x})$, with $r \in \Sigma_n$ and $\bar{x} = (x_1, x_2, \dots, x_n) \in \mathrm{Var}^n$

and with quantifier-free formulas defined as follows:

(iii) An atomic formula is a quantifier-free formula,

(iv) If φ, ψ are quantifier-free formulas, then $\varphi \Rightarrow \psi$, $\sim\varphi$ are quantifier-free formulas,

(v) If Φ is a non-empty set of quantifier-free formulas, then $\wedge\Phi$ and $\vee\Phi$ are quantifier-free formulas.

We restrict ourselves to quantifier-free formulas only, but this restricted version of $L_{\infty\infty}(\Sigma)$ is rich enough for our purposes. Moreover, the formulas we shall use will not contain non-trivial equalities. So, in the sequel, we shall simply write "formula" instead of "quantifier-free formula without non-trivial equalities".

Let $\underline{A} \in \mathrm{MOD}\,\Sigma$. The notion of satisfaction of a given formula φ with variables in a set $V \subset \mathrm{Var}$ at a valuation $h\colon V \to U_\Sigma(\underline{A})$ is defined as usual. We say that $\underline{A}$ satisfies φ if $\underline{A}$ satisfies at an arbitrary valuation of variables occurring in φ. Finally, for a given class Φ of formulas, by $\mathrm{MOD}\,\Phi$ we denote the class of all Σ-models satisfying each formula φ in Φ.

Notice that in the terminology of [8], $\mathrm{MOD}\,\Phi$ is precisely the class of models of the universal $L_{\infty\infty}(\Sigma)$-theory

$$\mathrm{Univ}\,\Phi = \{\forall(\bar{x})\varphi \mid \varphi \in \Phi\}$$

($\forall$ denotes the universal quantifier).

LEMMA 2.1. *For each class Φ of $L_{\infty\infty}(\Sigma)$-formulas, $\mathrm{MOD}\,\Phi$ is an initially closed class. Conversely, for each initially closed class $W \subset \mathrm{MOD}\,\Sigma$ there exists a class Φ of $L_{\infty\infty}(\Sigma)$-formulas such that $W = \mathrm{MOD}\,\Phi$.*

PROOF. Recall that all formulas we consider are without non-trivial equalities. We omit the routine verification of the first assertion. We prove the second. If W is empty, then

$$W = \mathrm{MOD}((x = x) \wedge \sim(x = x)).$$

Assume that W is a non-empty initially closed class. For each cardinal number α we choose a set $X_\alpha \subset \mathrm{Var}$ such that $\mathrm{card}\, X_\alpha = \alpha$. Next for each Σ-model with the carrier X_α,

$$\underline{X} = (X_\alpha, (r\underline{X} \mid r \in \Sigma)),$$

and such that $\underline{X} \in W$ let

$$\mathrm{Dg}\underline{X} = \mathrm{Dg}\underline{X}^+ \wedge \mathrm{Dg}\underline{X}^- = \bigwedge\{r(\overline{x}) \mid r \in \Sigma_n \in \Sigma,\ \overline{x} \in r\underline{X}\} \wedge$$
$$\bigwedge\{\sim r(\overline{x}) \mid r \in \Sigma_n \in \Sigma,\ \overline{x} \in X_\alpha^n,\ \overline{x} \notin r\underline{X}\}$$

and finally let

$$\varphi_\alpha = \bigvee\{\mathrm{Dg}\underline{X} \mid \underline{X} \in W,\ U_\Sigma(\underline{X}) = X_\alpha\}.$$

We claim that

$$W = \mathrm{M{\small OD}}\{\varphi_\alpha \mid \alpha \text{ is a cardinal}\}.$$

Let $\underline{A} \in W$. Let α an arbitrary but fixed cardinal number and let $h: X_\alpha \to U_\Sigma(\underline{A})$. Let $dom\,\tilde{h}$ be the domain of the initial lift of $(h,\underline{A})$. W is initially closed, hence $dom\,\tilde{h} \in W$ and consequently,

$$\varphi_\alpha = \mathrm{Dg}\ dom\,\tilde{h} \vee \varphi'_\alpha .$$

Clearly, $\underline{A}$ satisfies the formula $\mathrm{Dg}\,dom\,\tilde{h}$ under the valuation h. This proves

$$W \subset \mathrm{M{\small OD}}\{\varphi_\alpha \mid \alpha \text{ is a cardinal}\}.$$

If $\underline{B} \in \mathrm{M{\small OD}}\{\varphi_\alpha\}$, then for every $h: X \to U_\Sigma(\underline{B})$ the domain of the initial lift of $(h,\underline{B})$ is in W. It easily implies that $\underline{B} \in W$. ∎

The language $L_{\infty\infty}(\Sigma)$ contains among other sublanguages the language $L_{\infty\omega}(\Sigma)$ and the usual first language $L_{\omega\omega}(\Sigma)$. Namely, a formula φ is an $L_{\infty\omega}(\Sigma)$-formula if the set of variables occurring in φ is finite. φ is an $L_{\omega\omega}(\Sigma)$-formula if it does not contain neither infinite conjunctions nor infinite disjunctions.

Recall the following well known facts:

LEMMA 2.1. *Let* $W \subset \mathrm{M{\small OD}}\,\Sigma$ *be an arbitrary initially closed class.*

(i) $W = \mathrm{M{\small OD}}\,\Phi$, *where each* φ *in* Φ *is an* $L_{\infty\omega}(\Sigma)$*-formula if* W *has a finite character* [4], *that is, together with all finite submodels of a given* Σ*-model* $\underline{X}$, *it must contain* $\underline{X}$.

(ii) $W = \mathrm{M{\small OD}}\,\Phi$, *where each* φ *in* Φ *is an* $L_{\omega\omega}(\Sigma)$*-formula (that is* W *is a universally axiomatizable class in the usual sense) if* W *is closed under formation of ultraproducts.*

This leads to the following classification of weakly finitary fibrations:

DEFINITION 2.3. A weakly finitary fibration $(\mathbf{C},U)$ is called *finitary (strongly finitary)* iff it is concretely isomorphic to an initially closed class W of finitary relational systems such that W has finite character (W is universally axiomatizable).

3. WEAKLY FINITARY AND FINITARY FIBRATIONS.

In this section we characterize weakly finitary and finitary fibrations in categorical terms.

For each fibration $(\mathbf{C},U)$ we shall denote by $\mathrm{In}\,\mathbf{C}$ the subcategory of U-initial morphisms while

$$U' = U_{|\mathrm{In}\,\mathbf{C}}: \mathrm{In}\,\mathbf{C} \longrightarrow \mathbf{Set}.$$

By a direct system in $\mathbf{C}$ we mean every functor $D:(I,\leq)\to\mathbf{C}$ where $(I,\leq)$ is a directed poset considered to be a category in the usual way. By an initial subobject of a given $\mathbf{C}$-object A with the carrier $Y\subset U(A)$ we mean the $\mathbf{C}$-object B such that B is the domain of the initial lift of the identity embedding $(m: Y\to U(A), A)$.

For a given set X we shall denote by $\mathrm{Fin}\,X$ the direct system of all finite subsets of X. The direct system in $\mathrm{In}\,\mathbf{C}$ consisting of all initial finite subobjects of a given $\mathbf{C}$-object A is denoted by

$$(\underline{A}^f \mid A^f \in \mathrm{Fin}\,U(A)).$$

Notice that initial subobjects in $\mathrm{M{\small OD}}\,\Sigma$ are simply submodels of a given model.

Recall that $U:\mathbf{C}\to\mathbf{Set}$ is said to reflect direct colimits provided that whenever $D: (I,\leq)\to\mathbf{C}$ is a direct system and

$$(A,(w_i: D(i) \longrightarrow A \mid i \in I))$$

is a cocone over D such that (UA, Uw_i) is a colimit of UD in $\mathbf{Set}$, then $(A,(w_i))$ is a colimit of D.

THEOREM 3.1. *The following two conditions are equivalent for any fibration* $(\mathbf{C},U)$:

(i) $\mathbf{C}$ *is weakly finitary;*

(ii) $U':\mathrm{In}\,\mathbf{C}\to\mathbf{Set}$ *reflects direct colimits and the embedding* $\mathrm{In}\,\mathbf{C}\to\mathbf{C}$ *preserves them.*

PROOF. Obviously, $(\mathrm{M{\small OD}}\,\Sigma, U_\Sigma)$ satisfies (ii) for every finitary relational signature Σ. One can easily show that (ii) is also satisfied for each initially closed class $W\subset \mathrm{M{\small OD}}\,\Sigma$. This proves (i)$\Rightarrow$(ii).

(ii)$\Rightarrow$(i): Let $\mathbf{n} = \{0,1,\dots,n-1\}$ for each natural number n. For a given fibration $\mathbf{C}$ satisfying (ii) we consider the finitary signature

$$\Sigma(\mathbf{C}) = \{\Sigma(\mathbf{C})_n \mid n \in \mathrm{N}\}$$

where, for each $n \in \mathrm{N}$,

$$\Sigma(\mathbf{C})_n = \{r_{\mathrm{D}} \mid \mathrm{D} \in \mathbf{C} \text{ and } \mathrm{U}(\mathrm{D}) = \mathbf{n}\}.$$

Let $\mathrm{R}: \mathbf{C} \rightarrow \mathrm{M{\small OD}}\,\Sigma(\mathbf{C})$ be the concrete functor such that for each A in $\mathbf{C}$,

$$\mathrm{R}(\mathrm{A}) = (\mathrm{U}(\mathrm{A}), (r_{\mathrm{D}}^{\mathrm{A}} \mid r_{\mathrm{D}} \in \Sigma(\mathbf{C}))),$$

where, for each d: $\mathbf{n} \rightarrow \mathrm{U}(\mathrm{A})$ and $r_{\mathrm{D}} \in \Sigma(\mathbf{C})_n$,

$$d \in r_{\mathrm{D}}^{\mathrm{A}} \text{ iff } d: \mathrm{D} \longrightarrow \mathrm{A} \text{ in } \mathbf{C}.$$

Clearly, R is well defined. Now we show that R is full. Let h: $\mathrm{R}(\mathrm{A}) \rightarrow \mathrm{R}(\mathrm{B})$ and let $\underline{\mathrm{A}}^f$ be a finite initial subobject of A with the embedding m: $\underline{\mathrm{A}}^f \rightarrow \mathrm{A}$. Since U is transportable, there exists an isomorphism $i: \mathrm{D} \rightarrow \underline{\mathrm{A}}^f$ such that $\mathrm{U}(\mathrm{D}) = \mathbf{n}$ for some $n \in \mathrm{N}$. h is a $\Sigma(\mathbf{C})$-homomorphism, hence, in particular,

$$\text{if } mi \in r_{\mathrm{D}}^{\mathrm{A}}, \text{ then } hmi \in r_{\mathrm{D}}^{\mathrm{B}}.$$

It means that if $mi: \mathrm{D} \rightarrow \mathrm{A}$, then $hmi: \mathrm{D} \rightarrow \mathrm{B}$. This proves that $hm: \underline{\mathrm{A}}^f \rightarrow \mathrm{B}$ in $\mathbf{C}$ for an arbitrary $\mathrm{A}^f \in \mathrm{Fin}\,\mathrm{U}(\mathrm{A})$. By the assumption each A in $\mathbf{C}$ is a colimit of the direct system

$$(\underline{\mathrm{A}}^f \mid \mathrm{A}^f \in \mathrm{Fin}\,\mathrm{U}(\mathrm{A})).$$

Hence h: $\mathrm{A} \rightarrow \mathrm{B}$ in $\mathbf{C}$.

It remains to show that $\mathrm{R}(\mathbf{C})$ is an initially closed class. Let h: $(\mathrm{X}, (\bar{r}_{\mathrm{D}})) \rightarrow \mathrm{R}(\mathrm{A})$ be $\mathrm{U}_{\Sigma(\mathbf{C})}$-initial and let B be the domain of the initial lift of (h, A) in $\mathbf{C}$. We claim that $\mathrm{R}(\mathrm{B}) = (\mathrm{X}, (\bar{r}_{\mathrm{D}}))$. Indeed, for every $r_{\mathrm{D}} \in \Sigma(\mathbf{C})_n$ and $d: \mathbf{n} \rightarrow \mathrm{X} = \mathrm{U}(\mathrm{B})$, we have $d \in r_{\mathrm{D}}^{\mathrm{B}}$ iff $d: \mathrm{D} \rightarrow \mathrm{B}$, iff $hd: \mathrm{D} \rightarrow \mathrm{A}$ since $h: \mathrm{B} \rightarrow \mathrm{A}$ is U-initial. Next hd: $\mathrm{D} \rightarrow \mathrm{A}$ iff $hd \in r_{\mathrm{D}}^{\mathrm{A}}$. Finally, we obtain $hd \in r_{\mathrm{D}}^{\mathrm{A}}$ iff $d \in \bar{r}_{\mathrm{D}}$ because $h: (\mathrm{X}, (\bar{r}_{\mathrm{D}})) \rightarrow \mathrm{A}$ is $\mathrm{U}_{\Sigma(\mathbf{C})}$-initial. Thus $\bar{r}_{\mathrm{D}} = r_{\mathrm{D}}^{\mathrm{A}}$. ■

The functor $\mathrm{U}: \mathbf{C} \rightarrow \mathbf{Set}$ is said to create direct colimits if for each direct system $\mathrm{D}: (\mathrm{I}, \leq) \rightarrow \mathbf{C}$ with a colimit

$$(\mathrm{A}, (w_i: \mathrm{UD}(i) \longrightarrow \mathrm{A} \mid i \in \mathrm{I}))$$

of UD in **Set** there exists a unique family

$$(\underline{\mathrm{A}}, (\underline{w}_i: \mathrm{D}(i) \longrightarrow \underline{\mathrm{A}}))$$

such that $U(\underline{A}) = A$, $U(\underline{w_i}) = w_i$ for $i \in I$ and, moreover, $(\underline{A}, (\underline{w_i}))$ is a colimit of D in **C**.

THEOREM 3.2. *For each fibration* $(\mathbf{C}, U)$ *the following two conditions are equivalent:*

(i) **C** *is finitary:*

(ii) $U': \mathrm{In}\,\mathbf{C} \to \mathbf{Set}$ *creates direct colimits and the identity embedding* $\mathrm{In}\,\mathbf{C} \to \mathbf{C}$ *preserves them.*

PROOF. As in the proof of the preceding theorem note at first that $(\mathrm{Mod}\,\Sigma, U_\Sigma)$ satisfies (ii) above for an arbitrary finitary signature Σ. Next note that for an initially closed class $W \subset \mathrm{Mod}\,\Sigma$, W has finite character iff W is closed under formation of colimits of direct systems consisting of initial homomorphisms only. This immediately proves (i)$\Rightarrow$(ii).

To show the converse, consider again the functor R: $\mathbf{C} \to \mathrm{Mod}\,\Sigma(\mathbf{C})$ defined in Theorem 3.1. $R(\mathbf{C}) \subset \mathrm{Mod}\,\Sigma(\mathbf{C})$ is an initially closed class. Since $U': \mathrm{In}\,\mathbf{C} \to \mathbf{Set}$ creates direct colimits, $R(\mathbf{C})$ is closed under formation of colimits of direct systems consisting of initial homomorphisms only. Hence $R(\mathbf{C})$ has finite character. ∎

4. STRONGLY FINITARY FIBRATIONS.

As we have shown in the preceding section, in order to characterize finitary and weakly finitary fibrations, it is enough to examine elementary properties of the corresponding underlying functor. Our main problem — characterization of strongly finitary fibrations — needs more subtle methods. Namely, it needs an analysis of ordered structures of fibres of the considered fibration.

For the discussion we introduce the following notation. For a given fibration $(\mathbf{C}, U)$ and a set X, $F_{\mathbf{C}}X$ denotes the U-fibre over X, i.e.,

$$F_{\mathbf{C}}X = \{A \mid U(A) = X\},$$

considered as the poset such that, for $A, B \in F_{\mathbf{C}}X$,

$$A \le B \text{ iff } id_X: A \longrightarrow B \text{ in } \mathbf{C}.$$

For each function $f: X \to Y$, $F_{\mathbf{C}}f: F_{\mathbf{C}}Y \to FX$ is the function assigning to each $A \in F_{\mathbf{C}}Y$ the domain of the initial lift of (f, A).

We shall also use the concept of a Priestley space [7]: a Priestley space is an ordered compact topological space $(X,\leq,T)$ which is totoally order-disconnected, that is, given points $x,y \in X$ with $x \not\leq y$, there exists a T-clopen increasing set F such that $x \in F$ and $y \not\in F$ ($F \subset X$ is said to be increasing if

$$F = \uparrow F = \{x \in X \mid a \leq x \text{ for some } a \in F\}).$$

A morphism of Priestley spaces will mean a continuous and order preserving map.

THEOREM 4.1. *A finitary fibration* $(\mathbf{C},U)$ *is strongly finitary iff for every* $n \in N$, $F_{\mathbf{C}}\mathbf{n}$ *may be equipped with a structure of a Priestley space in such a way that for each* $h: \mathbf{m} \to \mathbf{n}$, $F_{\mathbf{C}}h$ *is a morphism of Priestley spaces.*

PROOF. *Necessity:* For an arbitrary finitary relation signature Σ and every $n \in N$, $F_{MOD\Sigma}\mathbf{n}$ is an algebraic lattice; a model with carrier $\mathbf{n}$ is a compact element of $F_{MOD\Sigma}\mathbf{n}$ if only finitely many of its relations are non-empty. Note also that for each function f between finite sets, $F_{MOD\Sigma}f$ preserves arbitrary infima and suprema of direct sets.

Every algebraic lattice L becomes a Priestley space if it is given the Lawson topology $\lambda(L)$ on L with the open subbase consisting of the sets $\uparrow k$ (k compact in L) and $L \setminus \uparrow x$ ($x \in L$) [7]. Then each function between algebraic lattices $h: L \to L_1$ preserving arbitrary infima and suprema of directed sets becomes a continuous map from $(L,\lambda(L))$ to $(L_1,\lambda(L_1))$.

Thus for every finitary relational signature Σ, MODΣ satisfies the required condition.

Let us consider an atomic formula $p(\overline{x})$, where

$$p \in \Sigma_n \text{ and } x = (x_1,x_2,\dots,x_n) \in \mathrm{Var}^n$$

and let $h: \mathrm{Var} \to \mathbf{m}$ for some $m \in N$, $h(x_i) = a_i$ for $i = 1,2,\dots,n$. Consider the model

$$\mathbf{m}_p = (\mathbf{m}, (\bar{r} \mid r \in \Sigma))$$

where

$$\bar{p} = \{(a_1,a_2,\dots,a_n)\} \text{ and } \bar{r} = \emptyset \text{ if } r \neg p.$$

Then, clearly

$$\uparrow \mathbf{m}_p = \{A \in F_{MOD\Sigma}\mathbf{m} \mid \mathbf{m}_p \leq A\}$$

is precisely the set of all models with carrier $\mathbf{m}$ satisfying $p(\overline{x})$ at the valuation h. $\mathbf{m}_p$ is compact, hence $\uparrow \mathbf{m}_p$ is clopen in the Lawson topology. This implies that the set

$$F_{\text{MOD}\Sigma}\mathbf{m} \cap \text{MOD}\{p(\overline{x})\}$$

is closed. In the same way one can show that

$$F_{\text{MOD}\Sigma}\mathbf{m} \cap \text{MOD}\{\sim p(\overline{x})\}$$

is closed for each relation symbol p. Consequently, for each $L_{\omega\omega}(\Sigma)$-formula φ of the form

$$\varphi = \bigvee_{i=1}^{n}\Big(\bigwedge_{j=1}^{n_i} r_j(\overline{x}) \wedge \bigwedge_{k=1}^{m_i} \sim r_k(\overline{x})\Big),$$

the set

$$F_{\text{MOD}\Sigma}\mathbf{m} \cap \text{MOD}\{\varphi\}$$

is a closed set. But every formula without equalities is equivalent to some formula of the form above. So finally, one can deduce that for every class $W = \text{MOD}\{\varphi_k \mid k \in K\}$, where every φ_k is an $L_{\omega\omega}(\Sigma)$-formula without equalities, the set

$$F_W\mathbf{m} = F_{\text{MOD}\Sigma}\mathbf{m} \cap W$$

is a closed subset of the Priestley space

$$(F_{\text{MOD}\Sigma}\mathbf{m},\ \lambda(F_{\text{MOD}\Sigma}\mathbf{m})).$$

Hence $F_W\mathbf{m}$ with the induced topology is a Priestley space. This proves necessity.

Sufficiency: Assume that a finitary fibration $(\mathbf{C}, U)$ satisfies the required condition. For every $n \in N$ let $Cl(n)$ denote the family of all clopen, increasing subsets of the Priestley space $F_\mathbf{C}\mathbf{n} = (F_\mathbf{C}\mathbf{n}, T_n)$. Let $\Sigma = (\Sigma_n \mid n \in N)$ be a finitary relational signature such that for every $n \in N$,

$$\Sigma_n = \{r_F \mid F \in Cl(n)\}.$$

We define a concrete functor $H: \mathbf{C} \to \text{MOD}\Sigma$ by the rule

$$H(A) = (U(A),\ (r_F^A \mid r_F \in \Sigma)),$$

where for every $F \in Cl(n)$ and $d: \mathbf{n} \to U(A)$, $d \in r_F^A$ provided that the domain of the initial lift of (d, A) is in F, that is, $F_C d(A) \in F$.

We show that H is full. If $h: H(A) \to H(B)$, then $hd \in r_F^B$ when $d \in r_F^A$, that is, $F_\mathbf{C} d(A) \in F$ implies that

$$F_\mathbf{C} hd(B) = F_\mathbf{C} d(F_\mathbf{C} h(B)) \in F.$$

By the total order disconnectedness,

$$F_\mathbf{C}\, d(A) \le F_\mathbf{C}\, d(F_\mathbf{C} h(B)).$$

In particular for every finite subobject A of U(A) with the identity embedding $m: A^f \to A$,

$$F_{\mathbf{C}} m(A) \le F_{\mathbf{C}} m(F_{\mathbf{C}} h(B)).$$

But it means precisely that $hm: \underline{A}^f \to B$ in $\mathbf{C}$, where $\underline{A}^f$ is the initial subobject of A with carrier A^f. But $\mathbf{C}$ is finitary, hence, by Theorem 3.2,

$$A = \operatorname{colim}(\underline{A}^f \mid A^f \in \operatorname{Fin} U(A)).$$

Thus $h: A \to B$ in $\mathbf{C}$.

In a similar way one can show that H is an embedding, that is for $A, B \in \mathbf{C}$, $A = B$ provided that $H(A) = H(B)$.

Let us consider the set Φ of all $L_{\omega\omega}(\Sigma)$-formulas of the following forms (for every $n \in N$, $\overline{x} \in \mathrm{Var}^n$):

(0) $$r_{F_{\mathbf{C}}\mathbf{n}}(\overline{x}),$$

(1) $$r_F(\overline{x}) \Rightarrow r_G(\overline{x}) \text{ for } F, G \in \mathrm{Cl}(n),\ F \subset G,$$

(2) $$\sim(r_{F_1}(\overline{x}) \wedge \ldots \wedge r_{F_k}(\overline{x})),$$

for $F_1, \ldots, F_k \in \mathrm{Cl}(n)$ such that $F_1 \cap \ldots \cap F_k = \emptyset$,

(3) $$(r_{F_1}(\overline{x}) \wedge \ldots \wedge r_{F_k}(\overline{x})) \Rightarrow r_F(\overline{x}),$$

for every $F_1, \ldots, F_k \in \mathrm{Cl}(n)$ and $F = F_1 \cap \ldots \cap F_k$,

(4) $$r_G(\overline{x} d) \Leftrightarrow (r_{F_1}(\overline{x}) \vee \ldots \vee r_{F_k}(\overline{x})),$$

for every $G \in \mathrm{Cl}(m)$, $F_1, \ldots, F_k \in \mathrm{Cl}(n)$ and $d: \mathbf{m} \to \mathbf{n}$ such that

$$(F_{\mathbf{C}} d)^{-1}(G) = F_1 \cup \ldots \cup F_k.$$

It is a routine to check that $H(\mathbf{C}) \subset \mathrm{Mod}\,\Phi$. Since $\mathbf{C}$ is finitary, $H(\mathbf{C})$ has finite character (Theorem 3.2). Hence in order to show the opposite inclusion it is enough to prove that every finite model from $\mathrm{Mod}\,\Phi$ is in $H(\mathbf{C})$. Let $A = (\mathbf{n}, (r_F^A)) \in \mathrm{Mod}\,\Phi$. Consider the set

$$I(A) = \{B \in F_{\mathbf{C}}\mathbf{n} \mid \mathrm{id}_{\mathbf{n}} \in r_F^A \text{ for every } F \in \mathrm{Cl}(n) \text{ such that } B \in F\}.$$

I(A) is non-empty. Otherwise,

$$F_{\mathbf{C}}\mathbf{n} = \bigcup(G(B) \mid B \in F_{\mathbf{C}}\mathbf{n}),$$

where every $G(B) \in \mathrm{Cl}(n)$ and $\mathrm{id}_{\mathbf{n}} \notin r_{G(B)}^A$. By compactness,

$$F_{\mathbf{C}}\mathbf{n} = G(B_1) \cup \ldots \cup G(B_k) \text{ for some } B_1, \ldots, B_k \in F_{\mathbf{C}}\mathbf{n}.$$

Then (0) and (4) lead us to a contradiction. Note that for every $F \in \mathrm{Cl}(n)$,

(+) $\mathrm{id}_{\mathbf{n}} \in r^{\mathbf{A}}_{\mathbf{F}}$ iff F contains some B such that $\mathrm{B} \in \mathrm{I(A)}$.

The right-to-left implication is trivial. For the converse, assume $\mathrm{F} \cap \mathrm{I(A)} = \varnothing$, that is for each $\mathrm{B} \in \mathrm{F}$ there is $\mathrm{E(B)} \in \mathrm{Cl}(n)$ such that $\mathrm{id}_{\mathbf{n}} \not\in r^{\mathbf{A}}_{\mathrm{E(B)}}$. Then

$$\mathrm{F} \subset \mathrm{E(B_1)} \cup \ldots \cup \mathrm{E(B_s)} \text{ for some } s \in \mathrm{N}.$$

This contradicts (1) and (4). This proves (+).

Assume that I(A) has a greatest element $\underline{\mathrm{B}}$. Then, by (+),

(++) $\mathrm{id}_{\mathbf{n}} \in r^{\mathbf{A}}_{\mathbf{F}}$ iff $\underline{\mathrm{B}} \in \mathrm{F}$

because F is increasing. Now, using (4), one can show that for each $m \in \mathrm{N}$, $\mathrm{G} \in \mathrm{Cl}(m)$ and $d: \mathbf{m} \to \mathbf{n}$,

$$d \in r^{\mathbf{A}}_{\mathbf{G}} \text{ iff } \underline{\mathrm{B}} \in (\mathrm{F}_{\mathbf{C}} d)^{-1}(\mathrm{G}).$$

But this means precisely that $\mathrm{H}(\underline{\mathrm{B}}) = \mathrm{A}$.

So it remains to show that I(A) has a greatest element. Let

$$\mathrm{D} = \bigcap\{\mathrm{F} \mid \mathrm{F} \in \mathrm{Cl}(n) \text{ and F contains some } \mathrm{B} \in \mathrm{I(A)}\}.$$

By compactness and (3), D is non-empty. We claim that $\mathrm{D} \cap \mathrm{I(A)}$ is non-empty. Otherwise, for every B in D there exists D(B) in $\mathrm{Cl}(n)$ such that $\mathrm{B} \in \mathrm{D(B)}$ and $\mathrm{id}_{\mathbf{n}} \not\in r^{\mathbf{A}}_{\mathrm{D(B)}}$. Then

$$\mathrm{D} \subset \bigcup\{\mathrm{D(B)} \mid \mathrm{B} \in \mathrm{D}\}.$$

By compactness, there exist $\mathrm{B}_1,\ldots,\mathrm{B}_k \in \mathrm{I(A)}$, $\mathrm{B}_{k+1},\ldots,\mathrm{B}_s \in \mathrm{D}$ and $\mathrm{F(B_1)},\ldots,\mathrm{F(B}_k) \in \mathrm{Cl}(n)$ such that $\mathrm{B}_i \in \mathrm{F(B}_i)$ for $i = 1,2,\ldots,k$ and

$$(\mathrm{F} =)\ \mathrm{F(B_1)} \cap \ldots \cap \mathrm{F(B}_k) \subset \mathrm{D(B}_{k+1}) \cup \ldots \cup \mathrm{D(B}_s).$$

$\mathrm{id}_{\mathbf{n}} \in r^{\mathbf{A}}_{\mathrm{F(B}_i)}$ for every $i = 1,2,\ldots,k$. By (3), $\mathrm{id}_{\mathbf{n}} \in r^{\mathbf{A}}_{\mathbf{F}}$, and, consequently, $\mathrm{id}_{\mathbf{n}} \in r^{\mathbf{A}}_{\mathrm{D(B}_j)}$ for some $j,k \leq s$. This is a contradiction. Thus $\mathrm{D} \cap \mathrm{I(A)} \neg \varnothing$. Now by total order disconnectedness, one can easily show that $\mathrm{D} \cap \mathrm{I(A)}$ is a singleton and its unique element is the greatest element of I(A). ■

5. REMARKS AND EXAMPLES.

1. For each fibration (**C**,U) the assignment

$$\mathrm{X} \longmapsto \mathrm{F}_{\mathbf{C}}\mathrm{X},\ f \longmapsto \mathrm{F}_{\mathbf{C}} f$$

determines a contravariant functor $\mathrm{F}_{\mathbf{C}}$: **Set**$^{\mathrm{op}}$→**Pos** into the category of posets and order preserving functions. We call this functor the theory of $(\mathbf{C},U)$. Using this concept, one can summarize the results of the present paper in the following short form:

PROPOSITION 5.1. *Let be an arbitrary fibration. Then*

(i) $\mathbf{C}$ *is weakly finitary iff for each set* X, $F_{\mathbf{C}}X$ *is an initial subobject of* $\lim(F_{\mathbf{C}}X^f \mid X^f \in \operatorname{Fin} X)$.

(ii) $\mathbf{C}$ *is finitary iff for each set* X,

$$F_{\mathbf{C}}X = \lim(F_{\mathbf{C}}X^f \mid X^f \in \operatorname{Fin} X).$$

(iii) $\mathbf{C}$ *is strongly finitary iff it is finitary and the restriction of its theory* $F_{\mathbf{C}}$ *to the subcategory of finite sets,* $F_{\mathbf{C}}$: $\mathbf{Set}_f^{op} \to \mathbf{Pos}$ *has a factorization*

$$F_{\mathbf{C}} = \tilde{U}\cdot\tilde{F}_{\mathbf{C}} : (\mathbf{Set}_f^{op} \longrightarrow \tilde{\mathbf{P}} \longrightarrow \mathbf{Pos})$$

where $\tilde{\mathbf{P}}$ *is the category of Priestley spaces and* $\tilde{U}$ *is the obvious forgetful functor.*

2. Each functor

$$S: \mathbf{Set}^{op} \longrightarrow \mathbf{Pos}$$

determines a fibration $(\mathrm{Fib}(S),U)$ such that objects of $\mathrm{Fib}(S)$ are pairs (A,i) where A is a set and $i \in S(A)$, while $h:(A,i)\to(B,j)$ if $h: A\to B$ and $i \leq Sh(j)$. The composition in $\mathrm{Fib}(S)$ is the composition in **Set**. Then S is (naturally isomorphic to) the theory of $\mathrm{Fib}(S)$ and vice versa; for each fibration $\mathbf{C}$, $\mathbf{C}$ and $\mathrm{Fib}(F_{\mathbf{C}})$ are concretely isomorphic.

3. Consider the category **Grph** of oriented graphs without isolated vertices, U: $\mathbf{Grph}\to\mathbf{Set}$ sends each graph to its set of arrows. **Grph** is a fibration. The corresponding theory assigns to each set X the algebraic lattice of all equivalences on $X\times\{0,1\}$. Hence **Grph** is strongly finitary. Notice that the full subcategory of **Grph** consisting of all finite graphs is a weakly finitary but not a finitary fibration.

4. Each functor-structured category $S(H)$ for H: $\mathbf{Set}\to\mathbf{Set}$ [6] is a topological category, hence a fibration. The theory corresponding to $S(H)$ assigns to each set X the lattice of all subsets of $H(X)$. Hence $S(H)$ is finitary iff, for each set X,

$$2^{H(X)} = \lim(2^{H(X^f)} \mid X^f \subset \mathrm{Fin}\, X).$$

But then S(H) is strongly finitary because every lattice $2^{H(X)}$ is an algebraic lattice. Hence for functor-structured categories the notion of finitary and strongly finitary fibration coincide.

5. If a meet-semilattice with a greatest element is the carrier of a Priestley space, then it must be a complete lattice [7]. Let A be such a semilattice but not a complete lattice and let S_A: **Set**→**Pos** be a constant functor, i.e.,

$$S_A X = X, \ S_A f = \mathrm{id}_A$$

for each set X and each function f. Then $\mathrm{Fib}(S_A)$ is finitary but not strongly finitary.

The mentioned property of semilattices leads us to the following result:

PROPOSITION 5.2. *Each strongly finitary fibration with all concrete finite products is a topological category.*

PROOF. It is not hard to verify that a fibration **C** has concrete finite products iff the correponding theory $F_{\mathbf{C}}$ has the factorization:

$$\mathbf{Set}^{op} \longrightarrow \mathrm{Slat} \longrightarrow \mathbf{Pos}$$

where Slat is the category of meet-semilattices with a greatest element, and functions preserving finite infima. Since **C** is strongly finitary, every $F_{\mathbf{C}}\mathbf{n}$ is the carrier of a Priestley space, hence it is a complete lattice. Thus $F_{\mathbf{C}}$ has the factorization

$$\mathbf{Set}^{op} \longrightarrow \mathrm{Clat} \longrightarrow \mathbf{Pos}$$

where Clat is the category of complete lattices and functions preserving infima. From this it easily follows that **C** is concretely complete, that is, **C** is a topological category. ∎

6. Let (**C**,U) be a weakly finitary topological category. It is easily checked that the functor R: **C**→MODΣ(**C**) defined in Theorem 3.1 preserves and reflects products. Hence each weakly finitary topological category is representable by initially closed classes of relational systems which is also closed under formation of products.

But there exist strongly finitary topological categories which are not representable by quasi-varieties of models, i.e., classes of models of strict Hom $L_{\omega\omega}(\Sigma)$-theories. Note that if $W \subset \text{Mod}\,\Sigma$ is initially closed and a quasi-variety, then W is also closed under formation of colimits of direct systems. This implies that for every $n \in N$, $F_W\mathbf{n}$ is an algebraic lattice. Thus for each topological category **C** representable by a quasi-variety of finitary relational systems, $F_{\mathbf{C}}\mathbf{n}$ must be an algebraic lattice for every $n \in N$.

Now let A be a complete lattice but not an algebraic lattice, such that the dual lattice A^{op} is an algebraic lattice. Then $(A, \lambda(A^{op}))$ is a Priestley space. Consider the topological category $\text{Fib}(S_A)$, where

$$S_A\colon \mathbf{Set}^{op} \longrightarrow \text{Clat} \longrightarrow \mathbf{Pos}$$

is the constant functor determined by A (compare 5). Each fibre of $\text{Fib}(S_A)$ is isomorphic to A. Hence $\text{Fib}(S_A)$ is strongly finitary but it is not isomorphic to any quasi-variety of finitary relational systems.

7. The last example is due to Rosický [9]. Let M be an arbitrary infinite set. The comma-category $\mathbf{Set}\downarrow M$ considered together with the obvious forgetful functor is a finitary fibration. For every $n \in N$, the fibre in $\mathbf{Set}\downarrow M$ over $\mathbf{n}$ is the set M^n with the trivial order. Let $\underline{M} = (M,T)$ be an arbitrary Boolean space with the carrier M. Then $\underline{M}$ as well as every power of $\underline{M}$ is a Priestley space. Now it is easy to conclude that the hypotheses of Theorem 4.1 are satisfied, that is $\mathbf{Set}\downarrow M$ is a strongly finitary fibration.

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