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SIMULTANEOUS REPRESENTATIONS BY METRIC SPACES

by Věra TRNKOVÁ

Dedicated to the memory of Evelyn NELSON

RÉSUMÉ. Etant donné trois monoïdes $M_1 \subset M_2 \subset M_3$, il existe un espace métrique complet P tel que toutes les applications non-constantes de P dans lui-même qui sont

non-dilatantes	forment un monoïde $\simeq M_1$
uniformément continues	forment un monoïde $\simeq M_2$
continues	forment un monoïde $\simeq M_3$.

Des résultats plus généraux et plus forts sont prouvés. On étudie aussi les foncteurs de complétion.

I. SIMULTANEOUS REPRESENTATIONS AND THE MAIN THEOREM.

1. By [2], every group can be represented as the group of all autohomeomorphisms of a topological space. This result was strengthened in the following two ways:

(i) every monoid can be represented as the monoid of all nonconstant continuous maps of a metric space into itself, by [5]. (Let us notice explicitly that all the nonconstant continuous endomaps of a space need not form a monoid; however, given a monoid M, there exists a metric space P such that all the nonconstant continuous endomaps of P do form a monoid and this monoid is isomorphic to M.)

(ii) For two arbitrary groups $G \subset H$ there exists a metric space P such that the group of all isometries of P is isomorphic to G and the group of all autohomeomorphisms is isomorphic to H, by [3].

The result mentioned in the abstract strengthens them both. This result is a consequence of the Main Theorem below.

We deal with almost full embeddings of categories. Let us recall (see [4]) that a functor F of a category $\mathcal{K}$ into a concrete category $\mathcal{H}$ is called an almost full embedding if it is faithful and

a) every morphism of $\mathcal{K}$ is mapped by F onto a nonconstant morphism of $\mathcal{H}$

and b) if $h: F(a) \to F(b)$ is a nonconstant morphism of $\mathbf{H}$, then there exists a morphism $k: a \to b$ in $\mathbf{K}$ such that $h = F(k)$.

By [5], every small category admits an almost full embedding into the category of all metric spaces and all continuous maps. If the embedded category has precisely one object, we obtain the representation of the morphism-monoid mentioned in (i) above.

2. Here, we investigate *simultaneous representations:* let D be a diagram scheme, let $\mathbf{C}$ and $\mathbf{D}$ be diagrams over D such that:

α) for every object σ of D, $\mathbf{C}(\sigma)$ and $\mathbf{D}(\sigma)$ are categories;

β) for every morphism m of D, $\mathbf{C}(m)$ and $\mathbf{D}(m)$ are functors.

We say that a natural transformation $\Phi: \mathbf{C} \to \mathbf{D}$ is a *simultaneous representation of* $\mathbf{C}$ in $\mathbf{D}$ if

(i) $\Phi_\sigma: \mathbf{C}(\sigma) \to \mathbf{D}(\sigma)$ is an almost full embedding whenever $\mathbf{D}(\sigma)$ is a concrete category in which all constants are morphisms,

(ii) $\Phi_\sigma: \mathbf{C}(\sigma) \to \mathbf{D}(\sigma)$ is a faithful and full functor (= full embedding) else.

LEMMA. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be diagrams of categories and functors over a scheme* D, *let* $\Phi: \mathbf{C} \to \mathbf{D}$ *be a simultaneous representation of* $\mathbf{C}$ *in* $\mathbf{D}$. *If, for a morphism* m *of* D, $\mathbf{D}(m)$ *is a faithful functor, then* $\mathbf{C}(m)$ *is also a faithful functor.*

PROOF. If $\Phi: \mathbf{C} \to \mathbf{D}$ is a simultaneous representation of $\mathbf{C}$ in $\mathbf{D}$, then, for every morphism $m: \sigma \to \sigma'$ of D, we have:

$$\mathbf{D}(m)\circ\Phi_\sigma = \Phi_{\sigma'}\circ\mathbf{C}(m)$$

and both Φ_σ, $\Phi_{\sigma'}$ are faithful functors. If $\mathbf{D}(m)$ is supposed to be faithful, then $\mathbf{D}(m)\circ\Phi_\sigma$ is also faithful, so that $\mathbf{C}(m)$ must be faithful. ∎

3. Let us denote by

Metr the category of all metric spaces of diameter ≤ 1 and all their nonexpanding maps (i.e., f is a morphism iff

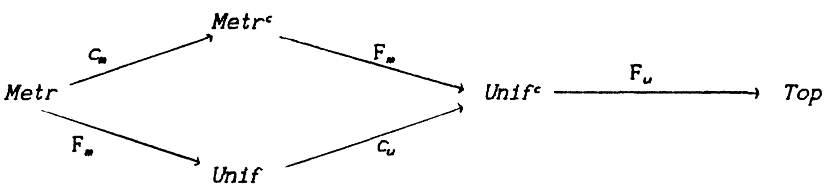
$$\operatorname{dist}((f(x),f(y)) \leq \operatorname{dist}((x,y)$$

for every x, y of the domain of f),

Metr^c its full subcategory generated by all complete spaces,

Unif the category of all uniform spaces and all uniformly continuous maps,
Unif^c its full subcategory generated by all complete spaces,
Top the category of all topological spaces and all continuous maps.

In the Main Theorem below, we investigate simultaneous representations in the following diagram $\mathcal{M}$:

$\mathcal{M}$ 

In the diagram $\mathcal{M}$, the completion functors are as follows:

c_m: *Metr* → *Metr*^{c} is the metric completion of metric spaces,

c_u: *Unif* → *Unif*^{c} is the uniform completion of uniform spaces,

the letter F always denotes the forgetful functor: $F_m(M)$ is the uniform space underlying the metric space M, and $F_u(U)$ is the topological space underlying the uniform space U.

Clearly, the square in the diagram $\mathcal{M}$ commutes, i.e., $F_m \circ c_m = c_u \circ F_m$. Moreover, the square has the following property: if M_1, M_2 are metric spaces (of diameter ≤ 1) and

$f: c_m(M_1) \to c_m(M_2)$ is a morphism in *Metr*^{c}

and $g: F_m(M_1) \to F_m(M_2)$ is a morphism in *Unif*

such that $F_m(f) = c_u(g)$, then there exists a unique morphism $h: M_1 \to M_2$ in *Metr* such that $c_m(h) = f$ and $F_m(h) = g$.

(In fact, f is a nonexpanding map of the completion $c_m(M_1)$ into the completion $c_m(M_2)$; the condition $F_m(f) = c_u(g)$ says that $f(x) = g(x)$ for every point $x \in M_1$, so that f maps M_1 into M_2; since f is nonexpanding, its domain-range-restriction $h: M_1 \to M_2$ is also nonexpanding, hence a morphism of *Metr*; then, clearly, $f = c_m(h)$ and $g = F_m(h)$.)

The last property says that "the square is a pullback on morphisms". (The square in $\mathcal{M}$ is also a "pullback on objects" but this plays no rôle in our investigations.) Let us call any commutative square with this property a *subpullback*.

4. Let us denote by *Cat* the category of all small categories and all fucntors.

OBSERVATION. Let

$$\begin{array}{ccccc} & & k_1 & & \\ & \nearrow^{G_1} & & \searrow^{\Gamma_1} & \\ k_0 & & & & k_3 \\ & \searrow_{G_2} & & \nearrow_{\Gamma_2} & \\ & & k_2 & & \end{array}$$

be a commutative square in *Cat*. Then it is a subpullback iff, forming a pullback

$$\begin{array}{ccccc} & & k_1 & & \\ & \nearrow^{H_1} & & \searrow^{\Gamma_1} & \\ h & & & & k_3 \\ & \searrow_{H_2} & & \nearrow_{\Gamma_2} & \\ & & k_2 & & \end{array}$$

the unique functor $G: k_0 \to h$ such that $H_i \circ G = G_i$ for $i = 1,2$, is a full embedding.

5. **LEMMA.** *If $\mathbf{C}$, $\mathbf{D}$ be diagrams of categories and functors over a scheme* D, *let $\Phi: \mathbf{C} \to \mathbf{D}$ be a simultaneous representation of $\mathbf{C}$ in $\mathbf{D}$. Let*

$$\begin{array}{ccccc} & & \sigma_1 & & \\ & \nearrow^{m_1} & & \searrow^{p_1} & \\ \sigma_0 & & & & \sigma_3 \\ & \searrow_{m_2} & & \nearrow_{p_2} & \\ & & \sigma_2 & & \end{array}$$

be a commutative square in D *such that its $\mathbf{D}$-image is a subpullback. Suppose that either*

a) Φ_{σ_0} is a full embedding

or b) Φ_{σ_0}, Φ_{σ_1}, Φ_{σ_2} are almost full embeddings and $\mathbf{D}(m_i)$, $i = 1,2$, preserve constant morphisms.

Then the $\mathbf{C}$-image of the square above is also a subpullback.

PROOF is quite straightforward. Let A_1, A_2 be objects of the category $\mathbf{C}(\sigma_0)$, let

$$f_i: [\mathbf{C}(m_i)](A_1) \to [\mathbf{C}(m_i)](A_2)$$

be morphisms of $\mathbf{C}(\sigma_i)$, $i = 1,2$, such that

$$[\mathcal{C}(p_1)]\,(f_1) = [\mathcal{C}(p_2)]\,(f_2).$$

The morphisms $g_i = \Phi_{\sigma_i}(f_i)$ (which are nonconstant, in the case b) fulfill

$$[\mathcal{D}(p_1)]\,(g_1) = [\mathcal{D}(p_2)]\,(g_2)$$

so that there exists a (unique!)

$$\mathfrak{l}\colon \Phi_{\sigma_0}(A_1) \to \Phi_{\sigma_0}(A_2) \quad \text{such that} \quad [\mathcal{D}(m_i)]\,(\mathfrak{l}) = g_i \ \text{ for } i = 1,2.$$

(Moreover, $\mathfrak{l}$ is nonconstant, in the case b.) We find a (unique!) morphism $h\colon A_1 \to A_2$ in $\mathcal{C}(\sigma_0)$ such that $\Phi_{\sigma_0}(h) = \mathfrak{l}$. Then, clearly,

$$[\mathcal{C}(m_i)]\,(h) = f_i \quad \text{for} \quad i = 1,2,$$

so that the $\mathcal{C}$-image of the square above is really a subpullback. ∎

6. In the terminology of simultaneous representations, the result mentioned in the abstract says the following: every diagram

$$\mathcal{C}_1\colon \qquad k_1 \xrightarrow{\quad G_1 \quad} k_2 \xrightarrow{\quad G_2 \quad} k_3,$$

where k_1, k_2, k_3 are one-object categories and G_1, G_2 are one-to-one functors, has a simultaneous representation in the diagram

$$\mathcal{D}_1\colon \quad Metr^c \xrightarrow{\quad F_m \quad} Unif^c \xrightarrow{\quad F_u \quad} Top\ .$$

As a corollary of the Main Theorem (see Remark b in 7) we obtain the following assertion: $\mathcal{C}_1$ has a simultaneous representation in $\mathcal{D}_1$, whenever k_1, k_2, k_3 are small categories and G_1, G_2 are *faithful functors*. This is an essential generalization. If we choose e.g. the categories k_1, k_2, k_3 discrete (= their only morphisms are the identities) and such that

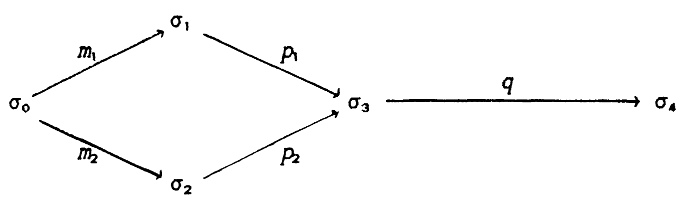
the cardinality of obj k_3 is equal to $\mathfrak{m}_3$,

G_2 sends $\mathfrak{m}_2$ objects of k_2 on each object of k_3,

G_1 sends $\mathfrak{m}_1$ objects of k_1 on each object of k_2,

where $\mathfrak{m}_1$, $\mathfrak{m}_2$, $\mathfrak{m}_3$ are given cardinals, we obtain "a rigid tree" of metric spaces: there is a set of cardinality $\mathfrak{m}_3$ of metrizable spaces without nonconstant nonidentical continuous maps; each of these spaces can be uniformized by $\mathfrak{m}_2$ metrizable uniformities such that there is no nonconstant nonidentical uniformly continuous map in the obtained set of uniform spaces; and each of these uniform spaces can be metrized by m_1 metrics such that there are no nonconstant nonidentical nonexpanding maps in this set of metric spaces.

7. Let S be the following scheme:



where $p_1 \circ m_1 = p_2 \circ m_2$. Clearly, the diagram $\mathcal{M}$ in 3 is a diagram over S.

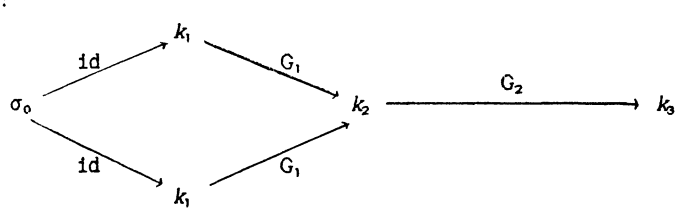
MAIN THEOREM. *Let* $\mathcal{C}$ *be a diagram in* $\mathcal{C}at$ *over* S. *Then* $\mathcal{C}$ *has a simultaneous representation in* $\mathcal{M}$ *iff all the functors* $\mathcal{C}(q)$, $\mathcal{C}(p_i)$, $\mathcal{C}(m_i)$, $i = 1,2$, *are faithful and the* $\mathcal{C}$*-image of the square in* S *is a subpullback.*

REMARKS. a) The necessity of the conditions in the Main Theorem is almost evident: since all the functors c_m, c_u, F_m, F_u in $\mathcal{M}$ are faithful, all the functors $\mathcal{C}(q)$, $\mathcal{C}(m_i)$, $\mathcal{C}(p_i)$ must be also faithful, by Lemma 2; and the $\mathcal{C}$-image of the square of S must be a subpullback, by Lemma 5. The parts II and III of the present paper are devoted to the proof that the above conditions are also sufficient. In the part IV, we present some strengthenings about representation of groups and Brandt groupoids.

b) Let us mention explicitly how the result in 6 is implied by the Main Theorem: if

$$\mathcal{C}_1: \qquad k_1 \xrightarrow{\ G_1\ } k_2 \xrightarrow{\ G_2\ } k_3,$$

is a diagram in $\mathcal{C}at$ such that G_1, G_2 are faithful, choose $\mathcal{C}$: S $\to$ $\mathcal{C}at$ as follows:



Then $\mathcal{C}$ fulfills the conditions of the Main Theorem so that there is a simultaneous representation Φ: $\mathcal{C} \to \mathcal{M}$. Then $\{\Phi_{\sigma 1}, \Phi_{\sigma 3}, \Phi_{\sigma 4}\}$ form a simultaneous representation of $\mathcal{C}_1$ in $\mathcal{D}_1$.

II. METRIC AND TOPOLOGICAL CONSTRUCTIONS.

1. Let us recall that the category *Metr* has all coproducts: if $\{(M_i, d_i) \mid i \in I\}$ is a collection of objects of *Metr*, its coproduct $\amalg_{i \in I}(M_i, d_i)$ is the space (M, d) with $M = \cup_{i \in I} M_i \times \{i\}$ and

$$d((x,i),(y,i)) = d_i(x,y) \quad \text{if} \quad x,\ y \in M_i,$$
$$d((x,i),(y,j)) = 1 \quad \text{if} \quad i \neq j,\ x \in M_i,\ y \in M_j.$$

If there is no confusion, we suppose that the sets M_i are disjoint and we omit the multiplication by the one-point set $\{i\}$ making these sets disjoint. Hence, we put simply $M = \cup_{i \in I} M_i$ and d is an extension of all the metrics d_i by the rule $d(x,y) = 1$ if x, y are in distinct M_i's. We recall that *Metr* has also quotients: let (M, d) be an object of *Metr* and q: $M \to Q$ be a surjective map; define c on Q by

$$c(x,y) = \inf \sum_{i \leq n} d(x_i, y_i)$$

where the infimum is taken over all sequences $x_0, y_0, \ldots, x_n, y_n$ of elements of M such that

$$q(x_0) = x,\ q(y_n) = y \quad \text{and} \quad q(y_i) = q(x_{i+1}) \text{ for } i = 0, \ldots, n-1;$$

then c is a pseudometric on Q and, identifying the points x, y of Q with $c(x,y) = 0$, we obtain the quotient of (M, d) in *Metr*, determined by the map q.

In the construction below, we use the above constructions in *Metr*. However, all our quotients will be so simple that the pseudometric c given by the above formula will already be a metric.

2. Let $\mathcal{C}$ be a *Cook continuum*, i.e., a compact connected *metric* space, nondegenerate (i.e., with more than one point) and such that:

if K is a subcontinuum of $\mathcal{C}$ and f: K $\to \mathcal{C}$ is a continuous map, then either f is constant or $f(x) = x$ for all $x \in$ K.

A continuum with these properties was constructed by H. Cook in [1]. A more detailed version of the construction is contained in Appendix A in [4].

Let J be the set of all integers, let

$$\mathbf{A} = \{A_i \mid i \in J\setminus\{0\}\},$$
$$\mathbf{B}_n = \{B_{n,i} \mid i \in J\setminus\{0\}\}, \quad \mathbf{C}_n = \{C_{n,i} \mid i \in J\setminus\{0\}\},$$
$$\mathbf{AB}_n = \{AB_{n,j} \mid j \in J,\ j \geq 0\}, \quad \mathbf{BC}_n = \{BC_{n,j} \mid j \in J,\ j \geq 0\},$$
$$\mathbf{CA}_n = \{CA_{n,j} \mid j \in J,\ j \geq 0\}, \quad n = 1,2,3,$$

be systems of nondegenerate subcontina of $\mathcal{C}$ such that the system

$$\mathbf{X} = \mathbf{A} \cup \cup_{n=1}^{3} (\mathbf{B}_n \cup \mathbf{C}_n \cup \mathbf{AB}_n \cup \mathbf{BC}_n \cup \mathbf{CA}_n)$$

is pairwise disjoint. Hence

(*) if $X, Y \in \mathbf{X}$, K is a subcontinuum of X and $f\colon K \to Y$ is a continuous map, then either f is constant or $X = Y$ and $f(x) = x$ for all $x \in K$.

We may suppose (by a suitable multiplication of metrics) that

$$\operatorname{diam} A_i = \operatorname{diam} B_{n,i} = \operatorname{diam} C_{n,i} = 2^{-i} \text{ for each } i \in J\setminus\{0\},$$
$$\operatorname{diam} AB_{n,j} = \operatorname{diam} BC_{n,j} = \operatorname{diam} CA_{n,j} = 2^{-(j+1)}$$
$$\text{for each } j \in J,\ j \geq 0,\ n = 1,2,3.$$

Choose, in each member of $\mathbf{X}$, two points with the distance equal to the diameter and denote them by

$\bar{a}_{i,1}$ and $\bar{a}_{i,2}$ in A_i,
$\bar{b}_{n,i,1}$ and $\bar{b}_{n,i,2}$ in $B_{n,i}$, $\bar{c}_{n,i,1}$ and $\bar{c}_{n,i,2}$ in $C_{n,i}$,
$\bar{a}\bar{b}_{n,j,1}$ and $\bar{a}\bar{b}_{n,j,2}$ in $AB_{n,j}$, $\bar{b}\bar{c}_{n,j,1}$ and $\bar{b}\bar{c}_{n,j,2}$ in $BC_{n,j}$,
$\bar{c}\bar{a}_{n,j,1}$ and $\bar{c}\bar{a}_{n,j,2}$ in $CA_{n,j}$.

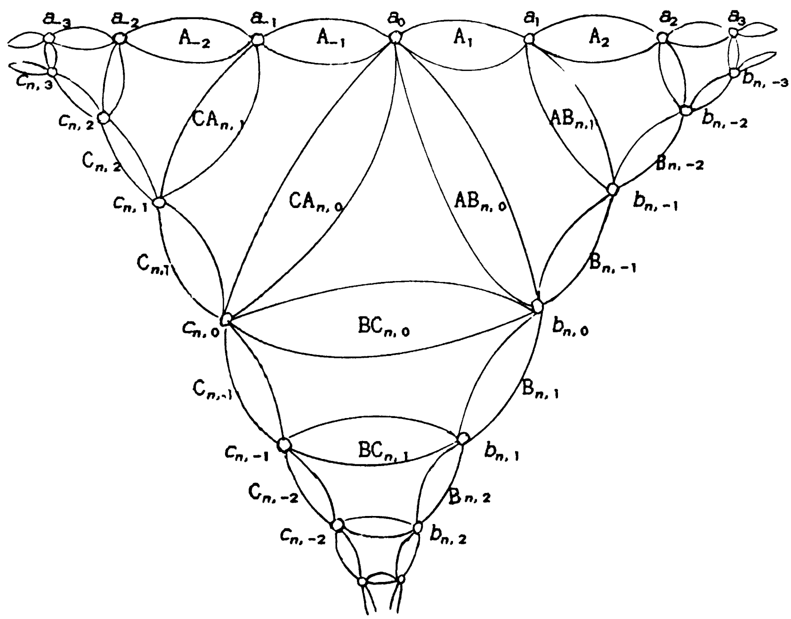
Let us denote by P the space obtained from the coproduct (in *Metr*) of all members of $\mathbf{X}$ by the following identifications (the quotient in *Metr*):

$\bar{a}_{1,1}$	with $\bar{a}_{-1,1}$;		the obtained point is denoted by a_0;
$\bar{a}_{i,2}$	with $\bar{a}_{i+1,1}$	for $i \geq 1$	the obtained point is
$\bar{a}_{i,2}$	with $\bar{a}_{i-1,1}$	for $i \leq -1$	denoted by a_i;
$\bar{b}_{n,1,1}$	with $\bar{b}_{n,-1,1}$		the obtained point is denoted by $b_{n,0}$;
$\bar{b}_{n,i,2}$	with $\bar{b}_{n,i+1,1}$	for $i \geq 1$	the obtained point is
$\bar{b}_{n,i,2}$	with $\bar{b}_{n,i-1,1}$	for $i \leq -1$	denoted by $b_{n,i}$;
$\bar{c}_{n,1,1}$	with $\bar{c}_{n,-1,1}$		the obtained point is denoted by $c_{n,0}$;
$\bar{c}_{n,i,2}$	with $\bar{c}_{n,i+1,1}$	for $i \geq 1$	the obtained point is
$\bar{c}_{n,i,2}$	with $\bar{c}_{n,i-1,1}$	for $i \leq -1$	denoted by $c_{n,i}$.

Moreover, identify, for each $j \in J$, $j \geq 0$,

$\bar{a}\bar{b}_{n,j,1}$	with a_j	and	$\bar{a}\bar{b}_{n,j,2}$	with	$b_{n,-j}$,
$\bar{b}\bar{c}_{n,j,1}$	with $c_{n,j}$	and	$\bar{b}\bar{c}_{n,j,2}$	with	$c_{n,-j}$,
$\bar{c}\bar{a}_{n,j,1}$	with $c_{n,j}$	and	$\bar{c}\bar{a}_{n,j,2}$	with	a_{-j},

The space P is indicated by the following figure



To obtain a completion cP of P, we have to add five points to P, namely

$$a_\infty = \lim_{i\to\infty} a_i, \quad a_{-\infty} = \lim_{i\to-\infty} a_i,$$
$$e_n = \lim_{j\to\infty} b_{n,j} = \lim_{j\to\infty} c_{n,-j}, \; n = 1,2,3.$$

3. In what follows, we investigate the subspace $Q = cP\setminus\{e_3\}$ of cP. Let us denote by d the metric of cP. We consider the following three metrics on Q:

$$\rho_1(x,y) = \min(1,\ d(x,y) + |\,d(x,e_3)^{-1} - d(y,e_3)^{-1}\,|),$$
$$\rho_2(x,y) = \min(1,\ 2\rho_1(x,y)),$$
$$\rho_3(x,y) = \min(1,\ 2\rho_1(x,y) + |\,d(x,e_3)^{-2} - d(y,e_3)^{-2}\,|).$$

OBSERVATIONS. a) The identity map of Q is

nonexpanding as the map $(Q,\rho_3) \to (Q,\rho_2)$ and $(Q,\rho_2) \to (Q,\rho_1)$,
uniformly continuous but not nonexpanding as $(Q,\rho_1) \to (Q,\rho_2)$,
continuous but not uniformly continuous as $(Q,\rho_2) \to (Q,\rho_3)$.

b) Q with each of the metrics ρ_1, ρ_2, ρ_3 is a complete space, the completion of $Q\setminus\{e_2\}$ in each of these metrics is Q again. *Let us denote* $Q\setminus\{e_2\}$ by Q^+ and the metric ρ_i restricted to Q^+ by ρ_i again.

4. All the metrics ρ_1, ρ_2, ρ_3 are equivalent, let us denote simply by Q the corresponding topological space.

CONVENTION. To simplify the notation, we suppose that the continua A_i, $B_{n,i}$, $C_{n,i}$, $AB_{n,j}$, $BC_{n,j}$, $CA_{n,j}$ in the family $\mathbf{X}$ are subspaces of Q, homeomorphic to the previous ones, i.e., homeomorphic to a disjoint family of subcontinua of the Cook continuum $\boldsymbol{C}$.

LEMMA. *Let* Y *be a topological space containing* Q *as a closed subspace and such that the closure* $\overline{Y\setminus Q} \subset \{a_+,a_-,e_1\} \cup (Y\setminus Q)$. *Let* X *be a continuum in* $\mathbf{X}$, *let* $f\colon X \to Y$ *be a nonconstant continuous map. Then either* $f(X) \subset Q$ *and* f *is the inclusion (i.e.,* $f(x) = x$ *for all* $x \in X$*) or* $f(X) \subset \overline{Y\setminus Q}$.

PROOF. Put

$$S = X \cup \{a_i \mid i \in J\} \cup \{b_{n,i}, c_{n,i} \mid i \in J,\ n = 1,2,3\} \cup \{a_+,a_-,e_1,e_2\} \subset Q$$

Let us suppose that $f(X)$ intersects $Q\setminus S$. Then $O = f^{-1}(Q\setminus S)$ is nonempty and open.

a) If $X\setminus O = \emptyset$, then f maps the whole X into $Q\setminus S$; since $f(X)$ is connected, it must be contained in some member K of $\mathbf{X}$, distinct from X, so that f must be constant. This is a contradiction.

b) Let us suppose that $X\setminus O \neq \emptyset$. Choose $x \in O$ and denote by C the component of O containing x. Since the closure $\bar{C}$ of C intersects the boundary of O, hence $f(\bar{C})$ intersects the boundary of $Q\setminus S$. Find the member K of $\mathbf{X}$ such that $f(x) \subset K$. Since $f(C) \subset K$, then also $f(\bar{C}) \subset K$. But $\bar{C}$ is a subcontinuum of X and K is distinct member of $\mathbf{X}$, hence f is constant on $\bar{C}$ by (*). Consequently f maps the whole $\bar{C}$ on the point $f(x) \in Q\setminus S$, which is a contradiction.

We conclude that $f(X) \cap (Q\setminus S) = \emptyset$. Since $f(X)$ is a nondegenerate connected space, necessarily either $f(X) \subset X$ (and then $f(x) = x$ for all $x \in X$) or $f(X) \subset \overline{Y\setminus Q}$.

5. **PROPOSITION.** *Let* Y *be a topological space containing* Q *as a*

closed subspace such that the $\overline{Y\backslash Q} \subset \{a_+, a_-, e_1\} \cup (Y-Q)$. *Let* $f\colon Q^+ \to Y$ *be a continuous map. Then either* $f(Q^+) \subset Q$ *and* f *is the inclusion (i.e.,* $f(x) = x$ *for all* $x \in Q^+$*) or* $f(Q^+) \subset \overline{Y\backslash Q}$ *or* f *is a constant map.*

PROOF. By II.4, Lemma, f restricted to any member X of $\mathbf{X}$ is either a constant or $f(X) \subset \overline{Y\backslash Q}$ or $f(X) = X$ and $f(x) = x$ for all $x \in X$. Let us suppose that f restricted to some X in $\mathbf{X}$ is constant, say $f(X) = \{x_0\}$. Let S be as in the proof of II.4, Lemma.

a) If $x_0 \in Q\backslash S$, then every member of $\mathbf{X}$ which intersects X has to be mapped by f on x_0. We can continue to the next members of $\mathbf{X}$. Finally, we obtain that f maps the whole Q^+ on x_0.

b) Let us suppose that

$$x_0 = a_i \text{ or } x_0 = b_{n,i} \text{ or } x_0 = c_{n,i},\ i \in J,\ n = 1,2,3.$$

b1) $x_0 \notin X$: In the definition of P (see II.2), the identifications of points in the members of $\mathbf{X}$ are chosen such that for every K in $\mathbf{X}$ not containing x_0 there exists a chain $X_0 = X, X_1, \dots, X_k = K$ of members of $\mathbf{X}$ such that none of them contains x_0 and X_j intersect X_{j+1} for $j = 0,\dots,k-1$. By II.4 Lemma, f has to map $X_0 = X, X_1, \dots, X_k = K$ onto x_0. Hence f maps any K in $\mathbf{X}$ not containing x_0 onto x_0. Consequently any member K of $\mathbf{X}$ which contains x_0 contains also a point x distinct from x_0 with $f(x) = x_0$. By II.4 Lemma, f maps K onto x_0. We conclude that f maps the whole Q^+ onto x_0.

b2) $x_0 \in X$: let $\tilde{X}$ be a member of $\mathbf{X}$, which intersects X in a point distinct from x_0; then $x_0 \notin \tilde{X}$ (see II.2). By II.4 Lemma, f maps $\tilde{X}$ onto x_0. Now, use the case b1 for $\tilde{X}$.

We conclude that if $f(Q^+)$ intersects $Q\backslash\{a_+, a_-, e_1, e_2\}$, then either f is constant or $f(x) = x$ for all $x \in Q^+$. Moreover, if $f(Q^+)$ contains e_2, then it is necessarily constant. (In fact, $f(Q^+)$ is connected and e_2 is an isolated point of

$$\overline{Y\backslash Q} \cup \{e_2\} = Y\backslash(Q\backslash\{a_+, a_-, e_1\}).)$$

In the remaining case, $f(Q^+) \subset \overline{Y\backslash Q}$.

III. THE PROOF OF THE MAIN THEOREM.

1. Let $D = (D, \leqslant)$ be a poset (= partially ordered set) with a largest element t. For every $d_0 \in D$, denote by $\mathbf{G}_D(d_0)$ the following category: objects are all pairs $(X, \{R_d \mid d \geqslant d_0\})$, where X is a set, $R_d \subset X \times X$ for every $d \in D$, $d \geqslant d_0$, and

a) the directed graph (X, R_t) is connected (i.e., for every $x, y \in X$ (not necessarily distinct) there exist $x_0 = x, x_1, \dots, x_n = y$ in X such that $(x_{i-1}, x_i) \in R_t \cup R_t^{-1}$ for $i = 1, \dots, n$), contains no loops (i.e., never $(x, x) \in R_t$) and card $X \geq 2$;

b) if $d_0 \leq d_1 \leq d_2$, then $R_{d_1} \subset R_{d_2}$.

A map $f: X \to X'$ is a morphism

$$f: (X, \{R_d \mid d \geq d_0\}) \to (X', \{R'_d \mid d \geq d_0\})$$

of $\mathbf{G}_D(d_0)$ iff, for every $d \geq d_0$, $(x,y) \in R_d$ implies $(f(x), f(y)) \in R'_d$ (i.e., f is $R_d R'_d$-compatible for every $d \geq d_0$).

If $d_0 \leq d_1$, there is a natural forgetful functor

$$\Sigma(d_0, d_1): \mathbf{G}_D(d_0) \to \mathbf{G}_D(d_1),$$

namely

$$\Sigma(d_0, d_1)(X, \{R_d \mid d \geq d_0\}) = (X, \{R_d \mid d \geq d_1\}), \quad \Sigma(d_0, d_1)(f) = f.$$

We consider the poset D as a category: if $d_0 \leq d_1$, denote the unique morphism from d_0 to d_1 by $m(d_0, d_1)$. We investigate the diagram $\mathbf{G}_D$ over D consisting of $\mathbf{G}_D(d_0)$ and

$$\mathbf{G}_D(m(d_0, d_1)) = \Sigma(d_0, d_1).$$

In [6], the following auxiliary lemma is proved.

AUXILIARY LEMMA. Let D be a poset with a last element t. Let $\mathbf{C}$: D $\to$ *Cat* be a diagram such that, for every morphism m of D, the functor $\mathbf{C}(m)$ is faithful. Then there exists a simultaneous representation Φ of $\mathbf{C}$ in $\mathbf{G}_D$. Moreover, if $d_0, d_1, d_2 \in D$, $d_0 = d_1 \wedge d_2$ in D and the square

$$\begin{array}{ccccc} & & \mathbf{C}(d_1) & & \\ & \nearrow & & \searrow & \\ \mathbf{C}(d_0) & & & & \mathbf{C}(t) \\ & \searrow & & \nearrow & \\ & & \mathbf{C}(d_2) & & \end{array}$$

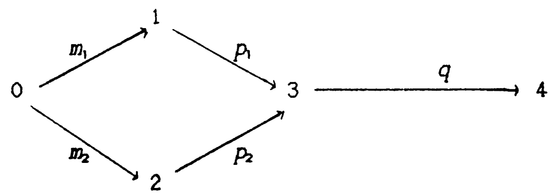
is a subpullback in *Cat*, then

$$\Phi_d : \mathbf{C}(d_i) \to \mathbf{G}_D(d_i), \quad i = 0,1,2.,$$

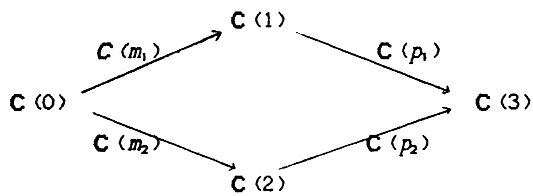
can be chosen such that, for every object of $\mathbf{C}(d_0)$, its Φ_d-image $(X, \{R_d \mid d \geq d_0\})$ fulfills $R_d = R_d \cap R_d$.

REMARK. Since constants need not be morphisms of $G_0(d)$ the functors Φ_d: $\mathbf{C}(d) \to G_0$ are full embeddings (not only almost full as it is in the case of *Metr*, *Unif*, *Top*, see the definition of the simultaneous representation in I.2).

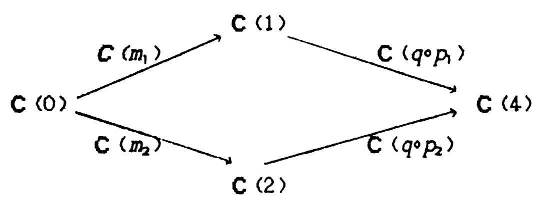
2. In the proof of the Main Theorem, we apply the auxiliary lemma to the diagram scheme S in I.7. For shortness, let us write i instead of σ_i so that S is the following scheme ($p_1 \circ m_1 = p \circ m_2$)



Let $\mathbf{C}$: S $\to$ *Cat* be a diagram such that, for every morphism m of S, the functor $\mathbf{C}(m)$ is faithful; moreover, let the square



be a subpullback in *Cat* (since the functor $\mathbf{C}(q)$ is faithful, it is equivalent to the fact that



is a subpullback in *Cat*). By the auxiliary lemma, there exists a simultaneous representation Φ: $\mathbf{C} \to G_s$, where $G_s(i)$ is as in III.1 for i = 1,2,3,4, and

$G_s(0)$ is the full subcategory of $G_s(0)$ in III.1 generated by all the (X, $\{R_0,R_1,R_2,R_3,R_4\}$) with $R_0 = R_1 \cap R_2$, and, for every morphism m of S, $\mathbf{G}_s(m)$ is as in III.1, i.e., the corresponding forgetful functor Σ.

3. A composition of a full embedding and an almost full embedding is an almost full embedding again. Hence, to prove the Main Theorem, it is sufficient to find a simultaneous representation Ψ: $\mathbf{G}_s \to \mathbf{M}$. Then $\Psi\circ\Phi$ is a simultaneous representation of $\mathbf{C}$ in $\mathbf{M}$.

4. First, we define the functor Ψ_0: $\mathbf{G}_s(0) \to \mathit{Metr}$. Let $\sigma = (X, \{R_0,R_1,R_2,R_3,R_4\})$ be an object of $\mathbf{G}_s(0)$, i.e., card X $\geq$ 2 and R_4 is a connected binary relation on X (hence in every $x \in X$ either an arrow starts or terminates) without loops,

$$R_1 \subset R_3 \subset R_4,\quad R_2 \subset R_3 \subset R_4 \quad \text{and} \quad R_0 = R_1 \cap R_2.$$

For every $r \in R_0$, let Z^r be a copy of (Q,ρ_1).
For every $r \in R_1\backslash R_0$, let Z^r be a copy of (Q^+,ρ_1).
For every $r \in R_2\backslash R_0$, let Z^r be a copy of (Q,ρ_2).
For every $r \in R_3\backslash (R_1\cup R_2)$, let Z^r be a copy of (Q^+,ρ_2).
For every $r \in R_4\backslash R_3$, let Z^r be a copy of (Q^+,ρ_3).

Let us denote by a^r_+, a^r_-, e^r_1 the points a_+, a_-, e_1 in the copy Z^r. In the category Metr, we form the coproduct $\coprod_{r\in R_4} Z^r$ and then the following identifications:

$$(**)\qquad \begin{aligned} a^{r_1}_+ &= a^{r_2}_+ && \text{iff}\quad r_1 = (x_1,y), && r_2 = (x_2,y),\\ a^{r_1}_- &= a^{r_2}_- && \text{iff}\quad r_1 = (x,y_1), && r_2 = (x,y_2),\\ a^{r_1}_- &= a^{r_2}_+ && \text{iff}\quad r_1 = (x,y), && r_2 = (y,z),\\ e^{r_1}_1 &= e^{r_2}_1 && \text{for all}\quad r_1,r_2 \in R_4. \end{aligned}$$

The obtained metric space is $\Psi_0(\sigma)$.

Let

$$f\colon \sigma \to \sigma' = (X',\{R'_0,R'_1,R'_2,R'_3,R'_4\})$$

be a morphism of $\mathbf{G}_s(0)$, i.e., $f\times f$ maps R_d into R'_d for $d = 0,1,2,3,4$. We define $\Psi_0(f) = g$ such that g maps a point z in a copy Z^r to the same point z in the copy $Z^{r'}$ with $r' = (f\times f)(r)$. Since f is $R_dR'_d$-compatible, the identifications (**) are preserved. Moreover, since f is $R_2R'_2$-compatible, the point e^r_2 has really its image $e^{r'}_2$ in $Z^{r'}$ whenever $r \in R_2$. Consequently the map $g\colon \Psi_0(\sigma) \to \Psi_0(\sigma')$ is correctly defined. Since the identity maps

$$(Q,\rho_3) \to (Q,\rho_2) \quad \text{and} \quad (Q,\rho_2) \to (Q,\rho_1)$$

are nonexpanding, g is also a nonexpanding map. We conclude that Ψ_0: $\mathbf{G}_s(0) \to$ *Metr* is a correctly defined functor. It is faithful, obviously. The fact that it is almost full will be proved in 9.

5. Now we define the functor Ψ_1: $\mathbf{G}_s(1) \to$ *Metr*^{c}. Let $\sigma = (X,\{R_1,R_3,R_4\})$ be an object of $\mathbf{G}_s(1)$, i.e., card X $\geq$ 2 and R_4 is a connected relation without loops, $R_1 \subset R_3 \subset R_4$).

For every $r \in R_1$, let Z^r be a copy of (Q,ρ_1).
For every $r \in R_3 \backslash R_1$, let Z^r be a copy of (Q,ρ_2).
For every $r \in R_4 \backslash R_3$, let Z^r be a copy of (Q,ρ_3).

In the category *Metr*, we form the coproduct $\amalg_{r \in R_4} Z^r$ and then the identifications (**) as above. The obtained metric space $\Psi_1(\sigma)$ is complete, hence it is an object of *Metr*^{c}. If

$$f\colon \sigma \to \sigma' = (X',\{R'_1,R'_3,R'_4\})$$

is a morphism of $\mathbf{G}_s(1)$, we define the map $\Psi_1(f) = g$ similarly as in 2, i.e., g maps z in a copy Z^r to the same point z in the copy $Z^{r'}$ with $r' = (f\times f)(r)$. Then g is a nonexpanding map. Clearly, Ψ_1: $\mathbf{G}_s(1) \to$ *Metr*^{c} is a correctly defined faithful functor.

6. We define the functor Ψ_2: $\mathbf{G}_s(2) \to$ *Unif* as follows: let σ = $(X, \{R_2,R_3,R_4\})$ be an object of $\mathbf{G}_s(2)$ (i.e., (X,R_4) is as in 3 and $R_2 \subset R_3 \subset R_4$).

For every $r \in R_2$, let Z^r be a copy of (Q,ρ_2).
For every $r \in R_3 \backslash R_2$, let Z^r be a copy of (Q^+,ρ_2).
For every $r \in R_4 \backslash R_3$, let Z^r be a copy of (Q^+,ρ_3).

In the category *Metr*, we form the coproduct $\amalg_{r \in R_4} Z^r$ and the identifications (**). The uniform space determined by the obtained metric space (i.e., the F_*-image of the obtained metric space) is $\Psi_2(\sigma)$.

If $f\colon \sigma \to \sigma' = (X',\{R'_2,R'_3,R'_4\})$ is a morphism of $\mathbf{G}_s(2)$, we define $\Psi_2(f) = g$ as in 2 or 3, i.e., it maps a point z in a copy Z^r to the same point z in the copy $Z^{r'}$. One can see easily that Ψ_2 is a correctly defined faithful functor.

7. The functor Ψ_3: $\mathbf{G}_s(3) \to$ *Unif*^{c} is defined as follows: let σ = $(X,\{R_3,R_4\})$ be an object of $\mathbf{G}_s(3)$.

For every $r \in R_3$, let Z^r be a copy of (Q,ρ_2).
For every $r \in R_4 \backslash R_3$, let Z^r be a copy of (Q,ρ_3).

In the category *Metr*, we form the coproduct $\coprod_{r\in R_4} Z^r$ and the identifications (**) as above. Then $\Psi_3(\sigma)$ is the uniform space (i.e., the F_*-image) of the obtained complete metric space. The Ψ_3-images of morphisms $f\colon \sigma \to \sigma'$ of $G_3(3)$ are defined as in 2 or 3, i.e., $\Psi_3(f)$ sends every z in Z^r to the same point z in $Z^{r'}$ with $r' = (f\times f)(r)$. Then Ψ_3 is a correctly defined faithful functor.

The functor $\Psi_4\colon G_3(4) \to Top$ is defined as follows: if $\sigma = (X,\{R_4\})$ is an object of $G_3(4)$, denote by Z^r a copy of (Q,ρ_3); in the category *Metr*, form the coproduct $\coprod_{r\in R} Z^r$ and the identifications (**) as above. Then $\Psi_4(\sigma)$ is the topological space of the obtained metric space (i.e., the $F_u\circ F_*$-image of the obtained metric space). If $f\colon \sigma \to \sigma'$ is a morphism of $G_3(4)$, $\Psi_4(f)$ is defined similarly as in 2 or 3.

PROPOSITION. $\Psi\colon G_3 \to M$ *is a natural transformation.*

PROOF. a) If $\sigma = (X,\{R_3,R_4\})$ is an object of $G_3(3)$, $G_3(q)$ sends it to the object $(X,\{R_4\})$. However $F_u(\Psi_3((X,\{R_3,R_4\}))$ is the same topological space as $\Psi_4((X,\{R_4\}))$ because ρ_2 and ρ_3 are equivalent metrics on Q. Consequently the square

$$\begin{array}{ccc} & \xrightarrow{\;G_3(q)\;} & \\ \Psi_3\downarrow & & \downarrow\Psi_4 \\ & \xrightarrow{\;F_u\;} & \end{array}$$

commutes.

b) The square

$$\begin{array}{ccc} G_3(1) & \xrightarrow{\;G(p_1)\;} & G_3(3) \\ \Psi_1\downarrow & & \downarrow\Psi_3 \\ Metr^c & \xrightarrow{\;F_*\;} & Unif^c \end{array}$$

commutes because the metrics ρ_1 and ρ_2 are uniformly equivalent on Q.

c) The square

$$\begin{array}{ccc} G_3(2) & \xrightarrow{\;G(p_2)\;} & G_3(3) \\ \Psi_2\downarrow & & \downarrow\Psi_3 \\ Unif & \xrightarrow{\;C_u\;} & Unif^c \end{array}$$

commutes because (Q,ρ_2) is a completion of (Q^+,ρ_2) and (Q,ρ_3) is a completion of (Q^+,ρ_3).

d) The square

$$\begin{array}{ccc} \mathcal{G}_S(0) & \xrightarrow{\;G(m_1)\;} & \mathcal{G}_S(1) \\ \Psi_0 \downarrow & & \downarrow \Psi_1 \\ \mathit{Metr} & \xrightarrow[\;C_m\;]{} & \mathit{Metr}^c \end{array}$$

commutes because (Q,ρ_i) is a completion of (Q^+,ρ_i), $i = 1,2,3$.

e) The square

$$\begin{array}{ccc} \mathcal{G}_S(0) & \xrightarrow{\;G(m_2)\;} & \mathcal{G}_S(2) \\ \Psi_0 \downarrow & & \downarrow \Psi_2 \\ \mathit{Metr} & \xrightarrow[\;F_m\;]{} & \mathit{Unif} \end{array}$$

commutes because ρ_1 and ρ_2 are uniformly equivalent metrics on Q.

8. **PROPOSITION.** *The functor* Ψ_4: $\mathcal{G}_S(4) \to$ *Top is almost full.*

PROOF. a) Let $\sigma = (X,\{R_4\})$ be an object of $\mathcal{G}_S(4)$. If h: $Q \to \Psi_4(\sigma)$ is a continuous map, then either h is constant or there exists $r \in R_4$ such that h sends Q to the copy Z^r of Q such that each $z \in Q$ is sent to the same z in Z^r. This follows immediately from II.5.

b) Let $\sigma = (X,\{R_4\})$, $\sigma' = (X',\{R'_4\})$ be objects of $\mathcal{G}_S(4)$, let h: $\Psi_4(\sigma) \to \Psi_4(\sigma')$ be a continuous map. Let us suppose that there exists $r_0 \in R_4$ such that the restriction $Z^{r_0} \to \Psi_4(\sigma')$ of h is constant, so that

$$h(a^{r_0}_+) = h(a^{r_0}_-) = h(e^{r_0}_1).$$

Since R'_4 is a connected binary relation on X', necessarily the restriction of h to any Z^r, $r \in R_4$, is constant, so that h is a constant map. If h is nonconstant, then, for every $r \in R_4$, there exists $r' \in R'_4$ such that h maps Z^r onto $Z^{r'}$ (and hence, it sends z in Z^r on z in $Z^{r'}$, by a). Since R_4 is connected (hence, in every $x \in X$, an arrow either starts or terminates) and h preserves the identifications (**), there exists a compatible map f: $\sigma \to \sigma'$ such that $h = \Psi_4(f)$.

9. **PROPOSITION.** *The functors* Ψ_3: $\mathbf{G}_8(3) \to$ *Unif*^{c}, Ψ_1: $\mathbf{G}_8(1) \to$ *Metr*^{c}, Ψ_2: $\mathbf{G}_8(2) \to$ *Unif*, Ψ_0: $\mathbf{G}_8(0) \to$ *Metr are almost full.*

PROOF. a) Let $\sigma = (X,\{R_3,R_4\})$, $\sigma' = (X',\{R'_3,R'_4\})$ be objects of $\mathbf{G}_8(3)$, let h: $\Psi_3(\sigma) \to \Psi_3(\sigma')$ be a nonconstant uniformly continuous map. Then

$$\tilde{h} = F_u(h): \Psi_4((X,\{R_4\})) \to \Psi_4((X',\{R'_4\}))$$

is a nonconstant continuous map so that there exists an $R_4R'_4$-compatible map f: $(X,\{R_4\}) \to (X',\{R'_4\})$ such that $\tilde{h} = \Psi_4(f)$, by III.8. Then both $\tilde{h}$ and h send each copy of Z^r on the copy $Z^{r'}$ with $r' = (f\times f)(r)$. Since h is uniformly continuous while the identity map $(Q,\rho_2) \to (Q,\rho_3)$ is not, necessarily r' is in R'_3 whenever $r \in R_3$, hence f is also $R_3R'_3$-compatible.

b) The proof that Ψ_1 is almost full is analogous. Given objects

$$\sigma = (X,\{R_1,R_3,R_4\}), \quad \sigma' = (X',\{R'_1,R'_3,R'_4\})$$

and a nonexpanding nonconstant map h: $\Psi_1(\sigma) \to \Psi_1(\sigma')$, we find an $R_4R'_4$-compatible map

$$f: (X,\{R_4\}) \to (X',\{R'_4\}) \text{ such that } \Psi_4(f) = \tilde{h} = F_u(F_m(h)).$$

Then f must also be $R_3R'_3$-compatible and $R_1R'_1$-compatible because the identity maps $(Q,\rho_2) \to (Q,\rho_3)$ and $(Q,\rho_1) \to (Q,\rho_2)$ are not nonexpanding.

c) The proof that Ψ_2 is almost full is analogous. It uses the facts that the identity map $(Q^+,\rho_2) \to (Q^+,\rho_3)$ is not uniformly continuous and that there is no nonconstant uniformly continuous map $(Q,\rho_2) \to (Q^+,\rho_2)$.

d) The proof that Ψ_0 is almost full is also analogous, only more facts are used, namely that:

the identity maps $(Q,\rho_2) \to (Q,\rho_3)$ and $(Q,\rho_1) \to (Q,\rho_2)$ are not nonexpanding (and analogously for Q^+)

and there are no nonconstant nonexpanding maps

$$(Q,\rho_2) \to (Q^+,\rho_2) \text{ and } (Q,\rho_1) \to (Q^+,\rho_1).$$

IV. REPRESENTATION OF GROUPS AND BRANDT GROUPOIDS.

1. In I.1 we already mentioned the result of [3]: for every two groups $G \subset H$ there exists a metric space P such that the group of all

isometries of P is isomorphic to G and the group of all autohomeomorphisms of P is isomorphic to H. The result about three monoids $M_1 \subset M_2 \subset M_3$, proved here and mentioned in the abstract, is stronger even if we choose groups M_1 and M_3, disregarding M_2. In this case we obtain that, for two arbitrary groups $M_1 \subset M_3$, there exists a metric space P such that

(***) every nonconstant nonexpanding map of P into itself is already an isometry and every nonconstant continuous map of P into itself is already an autohomeomorphism

and the isometries form a group isomorphic to M_1 and the autohomeomorphisms form a group isomorphic to M_3.

2. In [3], embeddings of a given metric space P_0 into a metric space P, representing the given groups $G \subset H$, are investigated. The author proves there that for every metric space P_0 there exists a metric space P containing P_0 and representing the given groups $G \subset H$ in the above sense. This is not true in general, if we require also the validity of (***). In fact, if the given space P_0 contains an arc, then every completely regular space P containing P_0 admits many nonconstant continuous maps into this arc so that the nonconstant continuous maps cannot represent the trivial group. However, disregarding (***), stronger embedding results can be proved by the present methods, see the proposition below.

3. Let us recall that a small category *b* is called a *Brandt groupoid* if each of its morphisms is an isomorphism. If $\mathcal{K}$ is a category, let us denote by iso $\mathcal{K}$ its subcategory formed by all objects of $\mathcal{K}$ and all isomorphisms of $\mathcal{K}$. Let us denote by F_m and F_u again the domain-range restrictions of the forgetful functors. We investigate the following diagram Iso $\mathcal{D}$:

$$\text{Iso } \mathcal{D}\text{: iso } \mathit{Metr} \xrightarrow{F_m} \mathit{iso\ Unif} \xrightarrow{F_u} \mathit{iso\ Top}$$

PROPOSITION. *Let a metric space* P_0 *with* diam $P_0 \leq 1$ *be given. Then every diagram*

$$\mathcal{B}\text{: } b_1 \xrightarrow{G_1} b_2 \xrightarrow{G_2} b_3,$$

where b_1, b_2, b_3 are Brandt groupoids and G_1, G_2 are faithful functors, has a simultaneous representation

$$\Phi = \{\Phi_1,\Phi_2,\Phi_3\}: \mathcal{B} \to \text{Iso } \mathcal{D}$$

such that for every object σ of b_1, the space P_0 is a retract of the space $\Phi_1(\sigma)$ in *Metr* (i.e., there are nonexpanding maps $e: P_0 \to \Phi_1(\sigma)$ and $r: \Phi_1(\sigma) \to P_0$ such that $r \circ e$ = identity on P_0).

REMARKS. a) The equation $r \circ e$ = identity implies that e preserves the metric of P_0 so that P_0 is a subspace of $\Phi_1(\sigma)$. Since the choice of the Brandt groupoids b_1, b_2, b_3 is rather free, we can always suppose that

$$G_1(\text{obj } b_1) = \text{obj } b_2 \quad \text{and} \quad G_2(\text{obj } b_2) = \text{obj } b_3$$

so that $F_m(P_0)$ is a retract in *Unif* of every $\Phi_2(\sigma)$, $\sigma \in \text{obj } b_2$, and $F_u(F_m(P_0))$ is a retract in *Top* of every $\Phi_3(\sigma)$, $\sigma \in \text{obj } b_3$. If each of b_1, b_2, b_3 has precisely one object, we obtain the result about representation groups. But b_1, b_2, b_3 can be chosen e.g., to be discrete and then we obtain the existence of an "isomorphism-rigid tree", analogously as in I.6.

b) The restriction that diam $P_0 \leqslant 1$ is not essential; it is presumed in the proposition for the sake of a simple formulation. Inspecting the proof below, one can see this immediately.

4. **PROOF OF THE PROPOSITION.** 1) Let P_0 be a metric space with diam $P_0 \leqslant 1$ and a diagram

$$\mathcal{B}: b_1 \xrightarrow{\quad G_1 \quad} b_2 \xrightarrow{\quad G_2 \quad} b_3,$$

of Brandt groupoids and faithful functors be given. Let h be a discrete category with obj $h = P_0$. We put $k_i = b_i \amalg h$, $i = 1,2,3$ (we suppose that b_i and h are subcategories of k_i) and extend the functors G_1, G_2 as identities on h (let us denote the extended functors by G_1 and G_2 again). By I.7 (Remark b), there exists a simultaneous representation $B = (B_1, B_2, B_3)$ of the diagram

$$\mathcal{C}_1: k_1 \xrightarrow{\quad G_1 \quad} k_2 \xrightarrow{\quad G_2 \quad} k_3,$$

into the diagram

$$\mathcal{D}_1: Metr^c \xrightarrow{\quad F_m \quad} Unif^c \xrightarrow{\quad F_u \quad} Top,$$

2) For every $z \in$ obj $h = P_0$, choose a point a_z in $B_1(z)$. Now, we define a functor Φ_1: $b_1 \to$ iso *Metr* : for every $\sigma \in$ obj b_1, $\Phi_1(\sigma)$ is obtained from the coproduct (in *Metr*)

$$B_1(\sigma) \amalg P_0 \amalg \coprod_{z \in P_0} B_1(z)$$

by the identification (in *Metr*) of each $z \in P_0$ with $a_z \in B_1(z)$. Let L denote the subspace of $\Phi_1(\sigma)$ obtained from $P_0 \amalg \coprod_{z \in P_0} B_1(z)$ and let us suppose that

$$L = P_0 \cup \bigcup_{z \in P_0} B_1(z), \quad P_0 \cap B_1(z) = \{z\} \text{ and } z = a_z,$$
$$B_1(z_1) \cap B_1(z_2) = \emptyset \quad \text{for } z_1 \neq z_2.$$

Moreover, let us suppose that

$$\Phi_1(\sigma) = B_1(\sigma) \cup L, \quad B_1(\sigma) \cap L = \emptyset.$$

(Clearly, $\Phi_1(\sigma)$ can be retracted to P_0; the nonexpanding map r sending $B_1(z)$ to a_z and $B_1(\sigma)$ to an arbitrary point of P_0 fulfills $r \circ e$ = ident, where e: $P_0 \to \Phi_1(\sigma)$ is the inclusion map.) If m: $\sigma \to \sigma'$ is a morphism of b_1, $\Phi_1(m)$ maps $B_1(\sigma)$ onto $B_1(\sigma')$ as $B_1(m)$ and it is identical on L. Thus, Φ_1 is a correctly defined faithful functor.

3) Since we may suppose that

$$G_1(\text{obj } b_1) = \text{obj } b_2 \quad \text{and} \quad G_2(\text{obj } b_2) = \text{obj } b_3,$$

the commutativity conditions

$$\Phi_2 \circ G_1 = F_\mu \circ \Phi_1 \quad \text{and} \quad \Phi_3 \circ G_2 = F_\nu \circ \Phi_2$$

already determine the functors

$$\Phi_2: b_2 \to \text{iso } \mathit{Unif} \text{ and } \Phi_3: b_3 \to \text{iso } \mathit{Top}.$$

They are faithful, obviously.

4) To finish the proof, we have to show that all the functors Φ_1, Φ_2, Φ_3 are full. Let $\sigma, \sigma' \in$ obj b_1 and let f be a homeomorphism of the metric space $\Phi_1(\sigma) = B_1(\sigma) \cup L$ onto $\Phi_1(\sigma')$. We prove that f maps $B_1(\sigma)$ onto $B_1(\sigma')$ and it is identical on L.

a) The definition of h and B implies that, for every $z \in P_0$, every continuous map of $B_1(z)$ into $B_1(\sigma')$ is constant; since f is one-to-one, necessarily $f(B_1(z)) \subset L$ for every $z \in P_0$. We conclude that

$f(L) \subset L$.

b) For every $z \in P_0$, let r_z: $L \to B_1(z)$ be the retraction sending every $x \in L \setminus B_1(z)$ to a_z. If $z_1, z_2 \in P_0$, $z_1 \neq z_2$, then the map

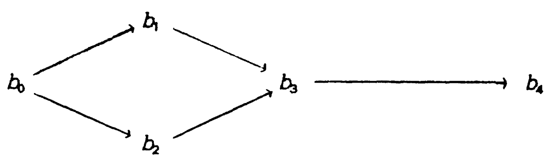
$$B_1(z_1) \xrightarrow{\text{incl}} L \xrightarrow{f} L \xrightarrow{r_{z_2}} B_1(z_2)$$

is constant (this follows from the fact that the category h is discrete). Since f is one-to-one, this constant map cannot be in $B_1(z_2) \setminus \{a_{z_2}\}$ *because* $r_{z_2}^{-1}(y) = \{y\}$ for every $y \in B_1(z_2) \setminus \{a_{z_2}\}$. Consequently $r_{z_2}(f(B_1(z_1))) = \{a_{z_2}\}$. If z_2 ranges over $P_0 \setminus \{z_1\}$, we obtain that $f(B_1(z_1)) \subset P_0 \cup B_1(z_1)$. Let r: $P_0 \cup B_1(z_1) \to B_1(z_1)$ be the retraction sending P_0 to z_1. Then $r \circ f$ maps $B_1(z_1)$ continuously into $B_1(z_1)$ so it must be either identity or a constant. If it is a constant then necessarily $r(f(B_1(z_1)) = \{z_1\}$ because $r^{-1}(y) = \{y\}$ for all $y \in B_1(z_1) \setminus \{z_1\}$; hence $f(B_1(z_1)) \subset P_0$; but this is a contradictioon because $B_1(z_1) \setminus \{z_1\}$ is open in $\Phi_1(\sigma)$, f is a homeomorphism and no subset of P_0 is open in $\Phi_1(\sigma')$. We conclude that $r \circ f$ maps $B_1(z_1)$ onto itself as the identity. Consequently f maps L onto itself as the identity. Since f is one-to-one on $\Phi_1(\sigma) = B_1(\sigma) \cup L$ and it maps L onto itself, it maps necessarily $B_1(\sigma)$ into $B_1(\sigma')$. Let us denote by g: $B_1(\sigma) \to B_1(\sigma')$ the domain-range restriction of f. If f is an isometry, necessarily $g = B_1(m)$ for a morphism m: $\sigma \to \sigma'$ in b_1, so that $f = \Phi_1(m)$. If f is a uniform homeomorphism (or homeomorphism) then it is Φ_2-image (or Φ_3-image) of m: $\sigma \to \sigma'$ in b_2 (or in b_3) - this is evident now.

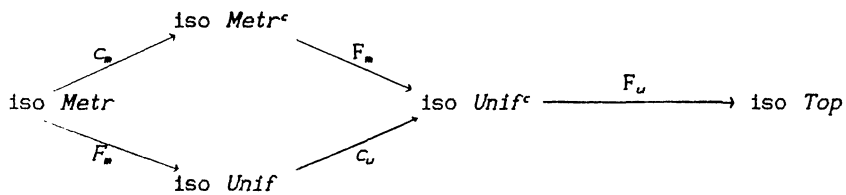
REMARK. If the given space P_0 is complete, then the constructed spaces $\Phi_1(\sigma)$ are also complete so that Φ is a simultaneous representation in the diagram

$$\text{iso } Metr^c \xrightarrow{F_m} \text{iso } Unif^c \xrightarrow{F_u} \text{iso } Top \,.$$

5) The presented proof can be easily modified to obtain the following: Let a metric space P_0 with diam $P_0 \leqslant 1$ be given. Then every diagram over S



where the square is a subpullback in *Cat*, all the functors in it are faithful and $b_0, \dots, b_4$ are Brandt groupoids, has a simultaneous representation $\Phi = \{\Phi_0, \dots, \Phi_4\}$ in iso $\mathcal{M}$:



such that P_0 is a retract in *Metr* of every space in $\Phi_0(\text{obj } b_0)$.

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