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A UNIFIED APPROACH TO THE LIFTING OF ADJOINTS

by A.J. POWER

RÉSUMÉ. De nombreux résultats sur les carrés d'adjoints et les triangles d'adjoints sont disséminéss dans la littérature sur la Théorie des Catégories. Leurs démonstrations sont souvent différentes. Ici, nous unifions les principaux résultats abstraits, donnant l'idée générale d'un processus qui conduit simplement d'un résultat au suivant, en commençant par un théorème fondamental de Dubuc.

0. INTRODUCTION.

Adjoint Square and Adjoint Triangle Theorems abound in the literature of Category Theory, for instance in the work of Barr [1], Dubuc [4], Johnstone [5] and Tholen with various co-authors [3, 11, 13]. However, these results are spread widely about the literature; in many cases, they have separate proofs; and on several occasions, authors have been unaware of the related results. Herein is given a unified treatment of the subject, commencing with a fundamental result of Dubuc [4] and deducing those results that follow without substantial added assumptions upon the categories involved.

We restrict attention to those results without substantial assumptions upon the categories involved because, roughly speaking, these are the results that are of a 2-categorical nature. More precisely, the results of this paper all hold in the general 2-categorical setting of Street's paper [10]. So, in particular, they apply automatically to categories enriched over bases other than **Set**. However, for simplicity of exposition, they are all expressed here in **Cat**.

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*) Research supported by Association of Universities and Colleges, Canada,

Adjoint Triangles can be regarded as special Adjoint Squares: those in which the bottom functor is an identity. So, any Adjoint Square Theorem immediately yields an Adjoint Triangle Theorem. In fact, it is often the case that an Adjoint Triangle Theorem is logically equivalent to an Adjoint Square Theorem of which it is a restriction. Dubuc's result is an instance of this, as illustrated in Section 2 herein. However, that is not always so, and Adjoint Squares do occur in nature: for instance, in diagrams relevant to algebraic functors, in the proof that a logical functor has a left adjoint iff it has a right adjoint, and in theorems on the existence of colimits in a category of algebras [6, 2, 7]. Moreover, Adjoint Squares provide the only context in which it is appropriate to study the lifting of monadicity, as in [8]. So, although we generally give only an Adjoint Triangle version of the principal results, we do include those Adjoint Square results that are not immediate corollaries of corresponding Adjoint Triangle results.

Cat, as a 2-category, has a total of four possible duals: given by reversing 1-cells, by reversing 2-cells, by both, or by neither. Reversal of 2-cells sends left adjoints to right adjoints, monadic functors to comonadic functors, and fully faithful functors to fully faithful functors. It follows that, in this sense, all the theorems herein dualize to give meaningful and interesting results. However, reversal of 1-cells sends left adjoints to right adjoints, monadic functors to Kleisli functors, and fully faithful functors to functors with a rather peculiar property. So, this dual does not generally yield meaningful and interesting results. Dubuc's Theorem is an exception. Consequently, this paper is divided into two sections: the first is on the lifting of a right adjoint; the second is on the lifting of a left adjoint. In general, the results of the second section are in spirit but not in detail the 1-cell reversal duals of those in the first section.

In both sections, we commence with Dubuc's result. We then replace the assumptions of Dubuc by various 2-categorical assumptions: for instance, a certain functor is monadic, or a certain natural transformation is an isomorphism. Under these new assumptions, we conclude that the conditions of Dubuc's Theorem are satisfied; hence we have a lifting as desired.

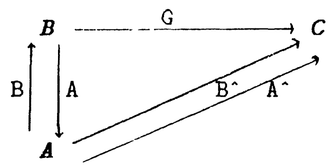
Most of these results, or mild variations of them, have appeared somewhere in the literature. So, throughout the paper, we cross reference with similar, and sometimes identical, results previously published, generally with different proofs. The final theorem of

Section 2, Theorem 2.4, is included only because it is the analogue of Theorem 1.4, and despite the fact that it is well-known and easy.

Principally, the work in this paper has been developed from part of the author's thesis [8], which in turn was based upon several theorems announced by William Butler in 1972 but without proof. I should like to acknowledge that work by Butler and express my appreciation for it. I should also like to express my thanks to Walter Tholen for his kind and generous encouragement.

1. ON THE LIFTING OF A RIGHT ADJOINT.

Consider an adjoint triangle



(1.1)

with γ,ϵ: $B \dashv A$, $\gamma^\wedge,\epsilon^\wedge$: $B^\wedge \dashv A^\wedge$, and $GB \simeq B^\wedge$.

NOTE. A is said to be of descent type if for each $X \in \mathcal{B}$,

$$BABAX \underset{\epsilon BAX}{\overset{BA\epsilon X}{\rightrightarrows}} BAX \xrightarrow{\epsilon X} X \qquad (1.2)$$

is a coequalizer, or equivalently, if the Eilenberg-Moore comparison functor is fully faithful. The latter description shows that this is essentially a 2-categorical concept, using [10]. A deeper analysis of this notion appears in [12].

THEOREM 1.1 (Dubuc [4]). *Given an adjoint triangle as above, define* θ: $AB \to A^\wedge B^\wedge$ *by*

$$\begin{array}{ccc} AB & \xrightarrow{\theta} & A^\wedge B^\wedge \simeq A^\wedge GB \\ {\scriptstyle \gamma AB}\downarrow & & \uparrow{\scriptstyle A^\wedge G\epsilon B} \\ A^\wedge B^\wedge AB & \xrightarrow{\simeq} & A^\wedge GBAB \end{array} \qquad (1.3)$$

Then, if the coequalizer

$$BABA^{\wedge} \underset{\epsilon BA^{\wedge}}{\overset{BA^{\wedge}\epsilon^{\wedge}.B\theta A^{\wedge}}{\rightrightarrows}} BA^{\wedge} \xrightarrow{\pi} V \qquad (1.4)$$

exists, if G *preserves it, if* $G\epsilon$ *is epi, and if* A *is of descent type,* $G \dashv V$.

This is the dual form of Dubuc's Theorem [4]. His formulation is slightly different, but a perusal of his proof shows that the above theorem follows directly. For instance, he has $GB = B^{\wedge}$, but his proof works equally well for $GB \simeq B^{\wedge}$.

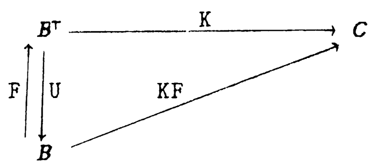
Using (1.4), it is evident how to define the counit of the adjunction: the unit is defined by using (1.2) applied to

$$BA \xrightarrow{B\gamma^{\wedge}A} BA^{\wedge}B^{\wedge}A \simeq BA^{\wedge}GBA \xrightarrow{BA^{\wedge}G\epsilon} BA^{\wedge}G \xrightarrow{\pi G} VG \qquad (1.5)$$

It is a routine calculation to check that this does induce an adjunction. (See [4] for more detail.)

THEOREM 1.2. *Given an adjoint triangle as in* (1.1), *if* A *is of descent type, and if* $\mathcal{B}$ *has and* G *preserves coequalizers of* A*-split coequalizer pairs,* G *has a right adjoint.*

This result has recently appeared in [2], § 3.7, Theorem 2 (a), with a different proof. Although an assumption is placed on $\mathcal{B}$, it is solely for simplicity of exposition. It is easy but just a trifle technical to give the general 2-categorical formulation in the spirit of [10]. An immediate application is a result by Rattray [9] that if $\mathcal{B}$ is compact and T is any monad on $\mathcal{B}$, then $\mathcal{B}^{T}$ is compact. This follows by inspection of



given any K *that preserves all colimits.*

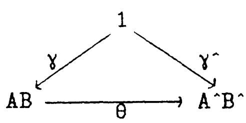
PROOF of Theorem. Everything is clear, by Theorem 1.1, once we show that the coequalizer (1.4) is A-split. Consider

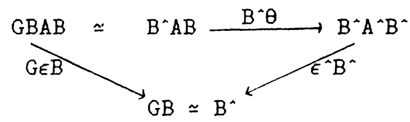
$$ABABA^{\wedge} \;\underset{\gamma ABA^{\wedge}}{\overset{ABA^{\wedge}\epsilon^{\wedge}.AB\theta A^{\wedge}}{\underset{A\epsilon BA^{\wedge}}{\rightleftarrows}}}\; ABA^{\wedge} \;\underset{\gamma A^{\wedge}}{\overset{A^{\wedge}\epsilon^{\wedge}.\theta A^{\wedge}}{\rightleftarrows}}\; A^{\wedge}$$

The only non-trivial parts are to show that

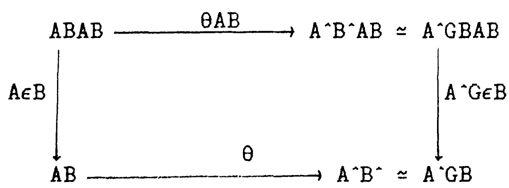
$$\hat{A}\hat{\epsilon}.\theta\hat{A}.\gamma\hat{A} = 1 \quad \text{and} \quad \hat{A}\hat{\epsilon}.\theta\hat{A}.AB\hat{A}\hat{\epsilon}.AB\theta\hat{A} = \hat{A}\hat{\epsilon}.\theta\hat{A}.A\epsilon B\hat{A}.$$

To exhibit these equations, observe how θ relates the units and counits of the two adjunctions:





and

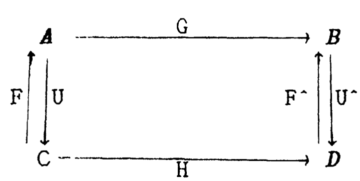


The first two of these follow easily from the definition of θ and by using naturality. The third is just a little more difficult, deduced by replacing the two horizontal composites by the composites given in the definition of θ, after which the result follows without difficulty using naturality. Now, the first of the two equations to be satisfied follows directly from the first of the above diagrams. The second equation is given by:

$$\begin{aligned}
\hat{A}\hat{\epsilon}.\theta\hat{A}.AB\hat{A}\hat{\epsilon}.AB\theta\hat{A} &= \hat{A}\hat{\epsilon}.\hat{A}\hat{B}\hat{A}\hat{\epsilon}.\hat{A}\hat{B}\theta\hat{A}.\theta AB\hat{A} \\
&= \hat{A}\hat{\epsilon}.\hat{A}\hat{\epsilon}\hat{B}\hat{A}.\hat{A}\hat{B}\theta\hat{A}.\theta AB\hat{A} \\
&= \hat{A}\hat{\epsilon}.\hat{A}G\epsilon B\hat{A}.\theta AB\hat{A} && \text{(2nd diagram)} \\
&= \hat{A}\hat{\epsilon}.\theta\hat{A}.A\epsilon B\hat{A} && \text{(3rd diagram).}
\end{aligned}$$

Thus, we do have a split coequalizer, and the result follows from Theorem 1.1.

THEOREM 1.3. *Consider*



Let γ,ϵ: $F \dashv U$, $\gamma^\wedge,\epsilon^\wedge$: $F^\wedge \dashv U^\wedge$, $GF \simeq HU$, U *monadic, and* $U^\wedge$ *of descent type. Then, if* H *has a right adjoint, so has* G.

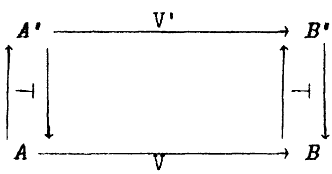
This result is a mild generalization of a result of Johnstone [5]. It follows trivially that monadic functors create limits and that logical morphisms with right adjoints have left adjoints. Observe the absence of coherence conditions.

PROOF. In order to apply Theorem 1.2, we need only show that G preserves coequalizers of U-split coequalizer pairs. In fact, any U-split coequalizer is HU-split, so $U^\wedge G$-split. Since $U^\wedge$ is of descent type, $U^\wedge$ reflects $U^\wedge$-split coequalizers [12].. Hence, G preserves coequalizers of U-split coequalizer pairs.

THEOREM 1.4. *Given an adjoint triangle as in* (1.1), *if* ϵ *is iso, then* G *has a right adjoint.*

PROOF. Consider Theorem 1.1. It is easy to show that $B\gamma A^\wedge$ is a right inverse to both $\epsilon BA^\wedge$ and $BA^\wedge\epsilon^\wedge.B\theta A^\wedge$. Indeed, this is essentially shown in the proof of Theorem 1.2. Since $\epsilon BA^\wedge$ is iso, it follows that $BA^\wedge\epsilon^\wedge.B\theta A^\wedge$ is also iso, and this coequalizer may be taken to be the identity. All coniditions of Theorem 1.1 are now obviously fulfilled. Hence, by the theorem, G has a right adjoint.

From Theorem 1.4, the hard half of Barr's Theorem [1] follows directly: given



exhibiting ***A'*** as a reflective subcategory of ***A*** and ***B'*** as a reflective subcategory of ***B***, and supposing that both functors from ***A*** to ***B'*** are the same up to isomorphism: if V has a right adjoint, so has V'.

Indeed, this half of Barr's Theorem is equivalent to Theorem 1.4: take V to be the identity in Barr's Theorem.

2. ON THE LIFTING OF A LEFT ADJOINT.

Consider

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\quad V \quad} & \mathcal{B} \\ F \uparrow \downarrow U & & F^\wedge \uparrow \downarrow U^\wedge \\ \mathcal{C} & \xrightarrow[\quad W \quad]{} & \mathcal{D} \end{array}$$

with

$$\gamma,\epsilon: F \dashv U, \quad \gamma^\wedge,\epsilon^\wedge: F^\wedge \dashv U^\wedge, \quad \phi: WU \simeq U^\wedge V,$$

and let ψ be the mate of ϕ under the adjunctions, i.e.,

$$\psi = \epsilon^\wedge VF.F^\wedge \phi F.F^\wedge W\gamma \ .$$

Now suppose $\rho,\sigma: H \dashv W$, and define χ by:

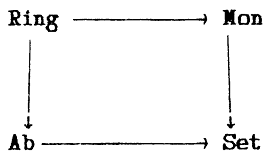
$$\begin{array}{ccccc} & & HU^\wedge VFH & & \\ & \overset{HU^\wedge \psi H}{\nearrow} & & \overset{H\phi^{-1}FH}{\searrow} & \\ HU^\wedge F^\wedge WH & & & & HWUFH \\ \uparrow HU^\wedge F^\wedge \rho & & & & \downarrow \sigma UFH \\ HU^\wedge F^\wedge & & \xrightarrow[\qquad \chi \qquad]{} & & UFH \end{array}$$

The main theorem of this section is as follows:

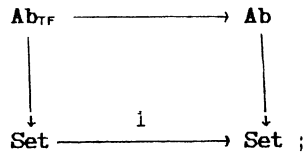
THEOREM 2.3. *Under the above conditions, if* U *is monadic, if* $U^\wedge$ *is of descent type, and if* χ *is iso,* V *has a left adjoint* G . *Moreover,* $UG \simeq HU^\wedge$.

Alas, it does not seem possible to replace χ by an arbitrary isomorphism, so applications require this coherence condition to be verified. Nevertheless, checking this coherence condition is usually straightforward. I cannot find a result very similar to this in the literature; but observe that it does bear some similarity to one of the results in [2].

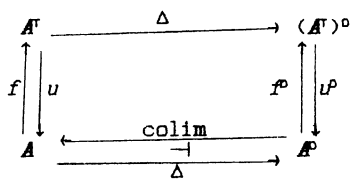
Some examples of Theorem 2.3 are:



and



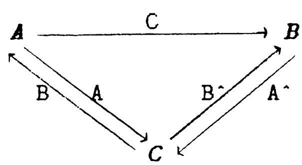
or given any cocomplete category $\mathcal{A}$ and a cocontinuous monad T on $\mathcal{A}$, the diagram



showing the $\mathcal{A}^T$ is also cocomplete.

In order to prove the theorem, we start with Dubuc's Theorem:

THEOREM 2.1. *Given*



with

$$A^{\wedge}C \simeq A, \quad \alpha,\beta\colon B \dashv A, \quad \alpha^{\wedge},\beta^{\wedge}\colon B^{\wedge} \dashv A^{\wedge}.$$

Define $\theta\colon A^{\wedge}B^{\wedge} \to AB$ *by*

$$\begin{array}{ccc} A^{\wedge}B^{\wedge} & \xrightarrow{\ \theta\ } & AB \simeq A^{\wedge}CB \\ {\scriptstyle A^{\wedge}B^{\wedge}\alpha}\downarrow & & \uparrow{\scriptstyle A^{\wedge}\beta^{\wedge}CB} \\ A^{\wedge}B^{\wedge}AB & \simeq & A^{\wedge}B^{\wedge}A^{\wedge}CB \end{array}$$

Then, if

$$BA^{\wedge}B^{\wedge}A^{\wedge} \underset{BA^{\wedge}\beta^{\wedge}}{\overset{\beta BA^{\wedge}.B\theta A^{\wedge}}{\rightrightarrows}} BA^{\wedge} \xrightarrow{\ \pi\ } D$$

is a pointwise coequalizer, and if $A^{\wedge}$ *is of descent type, then* $D \dashv C$.

Next, we translate this into an adjoint square form:

LEMMA 2.2. Given the situation stated at the start of the section, if

$$FHU^{\wedge}F^{\wedge}U^{\wedge} \underset{FHU^{\wedge}\epsilon^{\wedge}}{\overset{\epsilon FHU^{\wedge}.F\chi U^{\wedge}}{\rightrightarrows}} FHU^{\wedge} \xrightarrow{\quad\tau\quad} G$$

is a pointwise coequalizer, and if $U^{\wedge}$ is of descent type, then $G \dashv V$.

PROOF. Put $A^{\wedge} = U^{\wedge}$, $A = WU$. Then, since $B = FH$ and $\beta = \epsilon.F\sigma U$, all that need be done is show that, under these circumstances, $\chi = \sigma UFH.H\theta$. In fact, virtually by the definitions of θ and ψ, $\theta = \phi^{-1}FH.U^{\wedge}\psi H.U^{\wedge}F^{\wedge}\rho$; and from the definition of χ, it immediately follows that $\chi = \sigma UFH.H\theta$. The result follows by an application of Theorem 2.1.

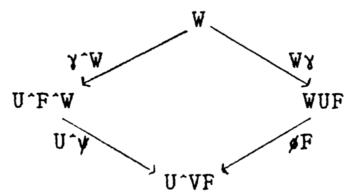
PROOF of Theorem 2.3. In order to apply the lemma, it suffices to show that the coequalizer of the lemma is U-split. Consider

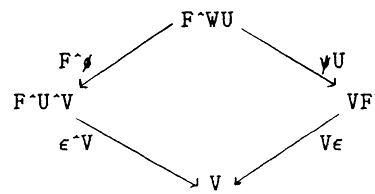
$$UFHU^{\wedge}F^{\wedge}U^{\wedge} \;\begin{array}{c} \xrightarrow{U\epsilon FHU^{\wedge}.UF\chi U^{\wedge}} \\ \xrightarrow{UFHU^{\wedge}\epsilon^{\wedge}} \\ \xleftarrow[\gamma HU^{\wedge}F^{\wedge}U^{\wedge}.\chi^{-1}U^{\wedge}]{} \end{array}\; UFHU^{\wedge} \;\begin{array}{c} \xrightarrow{HU^{\wedge}\epsilon^{\wedge}.\chi^{-1}U^{\wedge}} \\ \xleftarrow[\gamma HU^{\wedge}]{} \end{array}\; HU^{\wedge}$$

Everything is straightforward except $HU^{\wedge}\epsilon^{\wedge}.\chi^{-1}U^{\wedge}.\gamma HU^{\wedge} = 1$ and

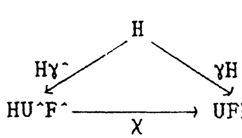
$$HU^{\wedge}\epsilon^{\wedge}.\chi^{-1}U^{\wedge}.U\epsilon FHU^{\wedge}.UF\chi U^{\wedge} = HU^{\wedge}\epsilon^{\wedge}.\chi^{-1}U^{\wedge}.UFHU^{\wedge}\epsilon^{\wedge}.$$

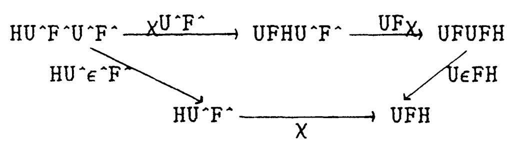
To exhibit these, first observe that, since ϕ and ψ are mates under the adjunctions (i.e., by definition of ψ), ϕ and ψ relate units and counits as follows:



,


From these diagrams, from the definition of χ, and from naturality, it follows that χ relates units and counits as follows:



and


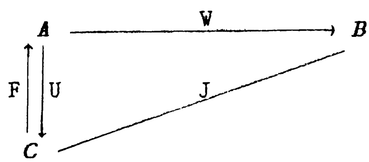
This means precisely that (H,χ) is a monad opfunctor. The first of these diagrams is easy; the second requires an enormous diagram (see [8]), but it is not inherently difficult. The first equation follows directly from the first of these two diagrams: the second equation follows by:

$$H\hat{U}\hat{\epsilon}.\chi^{-1}\hat{U}.U\epsilon FH\hat{U}.UF\chi\hat{U} = H\hat{U}\hat{\epsilon}.H\hat{U}\hat{\epsilon}\hat{F}\hat{U}.\chi^{-1}\hat{U}\hat{F}\hat{U} \qquad \text{(2nd diagram)}$$
$$= H\hat{U}\hat{\epsilon}.H\hat{U}\hat{F}\hat{U}\hat{\epsilon}.\chi^{-1}\hat{U}\hat{F}\hat{U} = H\hat{U}\hat{\epsilon}.\chi^{-1}\hat{U}.UFH\hat{U}\hat{\epsilon}.$$

So we have a split coequalizer; thus, since U is monadic, we may apply the lemma, and observe $UG \simeq H\hat{U}$.

Finally, for completeness, we mention the well-known left adjoint analogue of Theorem 1.4, namely:

THEOREM 2.4. *Given*



with $JW \simeq U$, $F \dashv U$, *and* J *fully faithful:* W *has a left adjoint.*

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Department of Pure Mathematics
University of Sydney
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