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DISCONNECTEDNESSES COGENERATED BY HAUSDORFF SPACES

by Francesca CAGLIARI

RÉSUMÉ. Nous prouvons que, si $\mathcal{P}$ est une sous-catégorie non triviale disconnexe de ***Top*** telle que $\mathcal{P}$ = U($\mathcal{P}$') où les espaces de $\mathcal{P}$' sont Hausdorff, alors $\mathcal{P}$ n'est pas l'enveloppe réflexive avec quotient d'un seul espace.

0. INTRODUCTION.

The non-simplicity of some subcategories of the category ***Top*** of topological spaces has already been studied in [8], [9] and [10]. In this paper we prove that U($\mathcal{P}$) cannot be the quotient reflective hull of a single space, when $\mathcal{P}$ is contained in the class of Hausdorff spaces. This last condition cannot be removed, since we find a class $\mathcal{P}$ (not contained in Hausdorff spaces) such that U($\mathcal{P}$)is the quotient reflective hull of $\mathcal{P}$.

1. PRELIMINARIES.

In this paper we denote by ***Top*** the category of topological spaces and maps, by ***Haus*** the category of topological Hausdorff spaces and maps, and by $\mathcal{P}$ a full and replete subcategory of ***Top***.

We recall the definitions of $\mathcal{P}$-component and of $\mathcal{P}$-quasicomponent studied by Preuss in [13] as well as the definitions of $\mathcal{P}$-epiclosed subspace and of $K^{\mathcal{P}}$-closed subspace studied in [2] and in [3].

Let X be a space, $x \in$ X and Y a subspace of X.

1.1. DEFINITION. We call *$\mathcal{P}$-component of x* in X the largest subspace Z of X containing x such that for each P $\in \mathcal{P}$ and for each f: Z $\to$ P, f is constant.

1.2. DEFINITION. We call *$\mathcal{P}$-quasicomponent* of x in X the largest subspace Z of X containing x such that for each P $\in$ $\mathcal{P}$ and for each f: X $\to$ P, the restriction $f_{|Z}$ of f to Z is constant.

1.3. DEFINITION. We call *$\mathcal{P}$-epiclosure* of Y in X (and we indicate it by $E_X{}^{\mathcal{P}}(Y)$) the largest subspace Z of X containing Y such that for each P $\in$ $\mathcal{P}$ and for each pair f,g: Z $\to$ P with $f_{|Y} = g_{|Y}$, $f = g$.

1.4. DEFINITION. We call *$K^{\mathcal{P}}$-closure* of Y in X (and we indicate with $K_X{}^{\mathcal{P}}(Y)$) the largest subspace Z of X containing Y such that for each P $\in$ $\mathcal{P}$ and for each pair f,g: $X \to P$ *with* $f_{|Y} = g_{|Y}$, $f_{|Z} = g_{|Z}$.

The following properties hold (cf. [3]):

1.5. $E_X{}^{\mathcal{P}}(x)$ is the $\mathcal{P}$-component of x in X.
1.6. $K_X{}^{\mathcal{P}}(x)$ is the $\mathcal{P}$-quasicomponent of x in X.
1.7. $K_X{}^{\mathcal{P}}(x) = E_X{}^{\mathcal{P}}(x)$ iff $K_{K^{\mathcal{P}}(x)}{}^{\mathcal{P}}(x) = K_X{}^{\mathcal{P}}(x)$.

1.8. DEFINITION. A space X is called *totally $\mathcal{P}$-disconnected* if its $\mathcal{P}$-components are singletons and *totally $\mathcal{P}$-separated* if its $\mathcal{P}$-quasicomponents are singletons.

We denote by $U\underline{\mathcal{P}}$ the class of all totally $\mathcal{P}$-disconnected spaces and by $Q\underline{\mathcal{P}}$ the class of all totally $\mathcal{P}$-separated spaces.

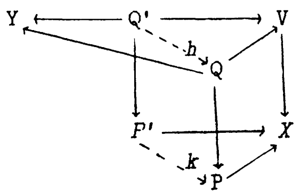
1.9. DEFINITION (cf. [5]). Let V, X, Y be topological spaces with V $\subset$ X and s: V $\to$ X be the inclusion map. The *partial product* P = $\underline{P}$(X,V,Y) of X and Y over s is a diagram

$$\begin{array}{ccccc} Y & \xleftarrow{p_2} & Q & \xrightarrow{p_1} & V \\ & & {\scriptstyle s'}\downarrow & & \downarrow{\scriptstyle s} \\ & & P & \xrightarrow{p'_1} & X \end{array}$$

such that (Q,p_1,p_2) is the product of Y and V, the previous square is a pullback and, given a diagram

$$\begin{array}{ccccc} Y & \xleftarrow{q} & Q' & \xrightarrow{f} & V \\ & & {\scriptstyle s''}\downarrow & & \downarrow{\scriptstyle s} \\ & & P' & \xrightarrow{f''} & X \end{array}$$

with the square a pullback, there is a unique pair (h,k) so that the following diagram commutes:



In this paper we consider only the partial products in which V is a singleton and, of course, s is a point embedding. If $s(V) = x$, we indicate $\underline{P}(X,V,Y)$ by $\underline{P}(X,x,Y)$ and p_1 by p_x. It may be easily verified by routine diagram technics that:

1.10. PROPOSITION. (a) *p_x is a topological quotient and each of its sections is an embedding.*

(b) *If* Z *is a subspace of* X, $\underline{P}(Z,x,Y)$ *is embeddable in* P(X,x,Y) *in a natural way.*

1.11. PROPOSITION (cf. [4]). *Let* $\boldsymbol{P} \subset \boldsymbol{T_1}$ *(the full subcategory of* ***Top*** *of* T_1-spaces) be quotient reflective in Top. $\boldsymbol{P} = \mathrm{U}\boldsymbol{P}$ *iff* $\boldsymbol{P}$ *is closed under the formation of partial products over point embeddings.*

We refer the reader to [6] for notations and definitions not explicitly given here.

2. DEGREE OF DISCONNECTION.

Having in mind 1.7 for each ordinal number λ we can define the (λ)-$\boldsymbol{P}$-component of x in X as follows:

$$K^{P}_{0}(x) = K^{P}(x),$$
$$K^{P}_{\lambda+1}(x) = K^{P}(K^{P}_{\lambda}(x)),$$
$$K^{P}_{\lambda}(x) = \cup\{K^{P}_{\beta}(x) \mid \beta < \lambda\} \quad \text{if } \lambda \text{ is limit ordinal.}$$

Denote by

$$\alpha_x = \min\{\lambda \mid K^{P}_{\lambda+1}(x) = K^{P}_{\lambda}(x)\}$$

and call α_x the *degree of* $\boldsymbol{P}$*-disconnection of* x in X. The *degree of* $\boldsymbol{P}$*-disconnection of the space* X will be:

$$\alpha_X = \sup\{\alpha_x \mid x \in X\}.$$

2.1. LEMMA. If $X = \Pi\{X_i \mid i \in I\}$ is the topological product of the family $\{X_i \mid i \in I\}$ and $x = \langle x_i \rangle_{i \in I}$ is in X, then

$$K^P(x) = \Pi\{K^P{}_i(x) \mid i \in I\}.$$

PROOF. If $\mathcal{P}$ is not contained in $\mathcal{T}_1$, $K^P(x)$ may be either $\{x\}$ or the indiscrete component of x (cf. [3]) and the lemma is trivially satisfied in this case. If $\mathcal{P} \subset \mathcal{T}_1$, $K^P(x)$ is closed (cf. [3]) and so

$$\Pi\{K^P{}_i(x) \mid i \in I\} = \cap p_i^{-1}(K^P{}_i(x))$$

is closed too; moreover it is K^P-closed, by 2.8 of [3]. Consider now a map $f: X \to P$ with $P \in \mathcal{P}$, we will prove that f is constant on $\Pi\{K^P{}_i(x) \mid i \in I\}$. Consider the subspace Y of X where

$$Y = \{\langle y_i \rangle \mid y_i \in K^P{}_i(x) \text{ and } x_i = y_i \text{ for all } i \in I \text{ but a finite number}\}.$$

Y is a dense subset of $\Pi\{K^P{}_i(x) \mid i \in I\}$. and f must be constant on Y. Suppose $f(Y) = \{p\}$: since $\{p\}$ is closed, $f^{-1}(p)$ is a closed subset of X containing Y; that implies

$$\Pi\{K^P{}_i(x) \mid i \in I\} \subset f^{-1}(p)$$

and f must be constant on $\Pi\{K^P{}_i(x) \mid i \in I\}$. •

2.2. PROPOSITION. (a) *If* $X = \Pi\{X_i \mid i \in I\}$ *and* $x = \langle x_i \rangle_{i \in I}$, *then* $\alpha_x = \sup\{\alpha_{x_i} \mid i \in I\}$.

(b) *If* $j: Y \to X$ *is a a monomorphism, for every* y *in* Y, $\alpha_Y \leq \alpha_{j(y)}$.

PROOF. (a) follows from Lemma 2.1.

(b) It follows from $K^P(y) \subset j^{-1}(K^P(j(y)))$ (cf. [3]).

2.3. COROLLARY. *If* Y *is the product of* α_x *copies of* X, Y *has a point* x *such that* $\alpha_x = \alpha_X = \alpha_Y$.

2.4. PROPOSITION. *Let* $\mathcal{P} \subset$ *Haus*, X *be a space and* x *be a non isolated point of* X, *such that* $\alpha_x = \alpha_X$. *If* $P \in \mathcal{P}$, *the partial product* $\underline{P}(X,x,Y)$ *has* α_X+1 *as degree of disconnection.*

PROOF. $p_x: \underline{P}(X,x,P) \setminus p_x^{-1}(x) \to \underline{P}(X,x,P)$ is mono and so, for any point z of $\underline{P}(X,x,P) \setminus p_x^{-1}(x)$, α_z does not exceed the degree of disconnection of $\underline{P}(X,x,P)$, by (2.2) (b). Now let $f: \underline{P}(X,x,P) \to P$ be a map. We'll prove that $f|p_x^{-1}(x)$ is constant. If $z,z' \in p_x^{-1}(x)$, then by Proposition 1.10, $(\underline{P}(X,x,P))\setminus p_x^{-1}(x))\cup\{z\}$ and $(\underline{P}(X,x,P))\setminus p_x^{-1}(x))\cup\{z'\}$ are homeomorphic to X. $\underline{P}(X,x,P)\setminus p_x^{-1}(x)$ is dense in both, so $f(z) = f(z')$ as P is Hausdorff. Therefore $K^P(z) = K^P(z')$ and both contain $p_x^{-1}(x)$, as z,z' vary in $p_x^{-1}(x)$. By 1.10 (b), the α_x-quasicomponent of z is $p_x^{-1}(x)$, which is homeomorphic to P. Consequently the α_x+1-quasicomponent of any point of $p_x^{-1}(x)$ is the point itself and this completes the proof. •

2.5. **COROLLARY.** *If* $\mathcal{P} \subset$ ***Haus*** *and* U$\mathcal{P} \neq$ ***Sing***, U$\mathcal{P}$ *is never the quotient reflective hull of a space.*

PROOF. Suppose U$\mathcal{P}$ = Q({X}). If X is a discrete space with more than one point, Q({X}) is the class of totally separated spaces, while U$\mathcal{P}$ is the class of totally disconnected spaces, and these two classes are different. So when X is discrete, U$\mathcal{P}$ must be ***Sing***. Therefore X may be supposed to have a non isolated point x and this point x is such that $\alpha_x = \alpha_X$, since Q({X}) = Q({X^{α_X}}). Now by Proposition 2.2, for any space Y in Q({X}), $\alpha_Y \leq \alpha_X$ holds. But $\underline{P}(X,x,Y)$ is in U$\mathcal{P}$ and its degree of disconnection is α_X+1 by Proposition 2.4, therefore U$\mathcal{P} \neq$ Q({X}). •

REMARK. The condition that $\mathcal{P} \subset$ ***Haus*** cannot be avoided in fact when X is a countable space with the cofinite topology, and $\mathcal{P}$ = {X}, then Q($\mathcal{P}$) = U($\mathcal{P}$).

In fact, we'll prove that for any space Y, and any $y \in$ Y, $K^P(y) = E^P(y)$ and so Q($\mathcal{P}$) = U($\mathcal{P}$) by 3.4 of [3].

Suppose there is a map f: $K^P(y) \to$ X which is not constant and consider the reflection r: Y $\to r$Y of Y in Q($\mathcal{P}$). From Proposition 3.2 of [3], $K^P(y) = r^{-1}(r(y))$. Now define g: Y $\to$ X as follows:

$g(a) = r(a)$ if a is not in $K^P(y)$,

$g(a) = f(a)$ if a is in $K^P(y)$.

Now if $b \neq y$, then $g^{-1}(b) = f^{-1}(b) \cup r^{-1}(b)$, while $g^{-1}(y) = f^{-1}(y)$; in any case, the inverse image of a point is closed, therefore g is a continuous map in contrast with the definition of K^P-closure.

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