CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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Cahiers de topologie et géométrie différentielle catégoriques, tome 28, n° 4 (1987), p. 283-302

<http://www.numdam.org/item?id=CTGDC_1987__28_4_283_0>

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Vol. XXVIII-4 (1987)

A NOTE ON LIE ALGEBROIDS WHICH ARISE FROM GROUPOID ACTIONS

by Kirill MACKENZIE

RÉSUMÉ. On calcule l'algébroïde de Lie au moyen des produits semi-directs généraux de groupoïdes différentiables, dans le sens introduit par Brown, d'après Fröhlich, pour les groupoïdes discrets. En utilisant ceci comme modèle, on définit les actions des algébroïdes de Lie sur d'autres algébroïdes de Lie, et les produits semi-directs généraux d'algébroïdes de Lie. En particulier, on détermine l'algébroïde de Lie d'un revêtement (dans le sens que lui donnent Gabriel et Zisman, Higgins et Brown) de groupoïdes différentiables, et on définit un concept correspondant de revêtement d'algébroïde de Lie; un revêtement d'algébroïde de Lie sera alors équivalent à une de ses actions sur une surmersion. Procédé au cours duquel on définit les images réciproques des algébroïdes de Lie transitifs par des applications différentiables arbitraires et on donne une définition simple d'un morphisme d'algébroïdes de Lie transitifs. Comme exemple non-standard, on calcule certains algébroïdes de Lie associés à une extension de fibrés principaux.

The concept of a groupoid action on a set fibered over its base was introduced by Ehresmann [4] and is fairly well-known. If Ω is a groupoid on base B and $p\colon M \to B$ is a map, then an action of Ω on p induces a groupoid structure on the pullback set $\Omega * M$ (of p over the source projection of Ω) and the intrinsic properties of this groupoid embody the properties of the action. Subsequently Higgins [6] defined a covering of a groupoid Ω to be a groupoid morphism $\Pi \to \Omega$ which restricts to bijections on the α-fibres, and showed that the construction $\Omega \mapsto \Omega * M$ gives (the object map of) an equivalence p of categories between the category of actions of Ω and the category of coverings of Ω. This equivalence was extended to topological groupoids by Brown et al [3].

In [2], Brown observed that this construction is a special case of a very general semi-direct product of groupoids, originally due to Frohlich. Here the groupoid which receives the action must be suitably fibered over the base of the acting groupoid, and this imposes constraints on which groupoids can act on which: for example, the only groupoids which can act on transitive groupoids are effectively groups. The first purpose of this note (§1) is to calculate the Lie algebroids of the differentiable version of these general semi-direct products. Once we have done this, we formulate in §3 a corresponding concept of the action of one Lie algebroid on another, and construct the corresponding semi-products. It is worth pointing out that this does not follow automatically from the results of §1. One is accustomed to the fact that algebraic constructions for Lie groups have analogues for Lie algebras, but this depends in part on the fact that every Lie algebra is the Lie algebra of a Lie group. It was shown by Almeida and Molino [1] that not all transitive Lie algebroids are the Lie algebroids of Lie groupoids; in fact the integrability obstruction for a transitive Lie algebroid [7] suggests that, amongst transitive Lie algebroids, the non-integrable ones are, in some sense, generic. Here we are dealing with actions and semi-direct products of general differentiable groupoids and not necessarily transitive Lie algebroids, and there is, a priori, even less reason than in the transitive case to expect that constructions for differentiable groupoids will always be modelled by constructions for general Lie algebroids.

This concept of a Lie algebroid action includes as a very special case the concept of an infinitesimal action of a Lie algebra on a manifold, as well as the concept of a Lie algebra $\mathfrak{g}$ acting on a Lie algebra $\mathfrak{h}$ by a morphism $\mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$. The semi-direct product in the former case is related to the infinitesimal graph (Palais [10]) of the action, and a general result on the integrability of Lie algebroid actions in the sense given here should be of very great interest. (That this may be very difficult is suggested by the complexities of Pradines [12].)

As a novel application we calculate in §2 certain Lie algebroids associated to the geometry of an extension of principal bundles. In [8] we showed that an extension of principal bundles is characterized by a single Lie groupoid together with a group action upon it. There are in fact several groupoid actions and semi-direct products associated with this structure, and in §2 we calculate the corresponding Lie algebroids. The importance of this will be demonstrated elsewhere.

An intermediate step in the work is to calculate the Lie algebroid of the pullback of a differentiable groupoid over a surjective submersion. Abstracting this, we define general pullbacks of transitive Lie algebroids over arbitrary smooth maps, and use them to give, in §3, a very simple definition of a morphism of transitive Lie algebroids. (The definition of a morphism is usually a simple matter, but for Lie algebroids it is a surprisingly difficult question; see Pradines [11] and Almeida and Kumpera [0].)

I am most grateful to A. Weinstein for the crucial formula in 1.2, and to P.J. Higgins for arguing that the case of general semi-direct products could be handled by the same methods as that of covering groupoids, as well as for many valuable comments throughout the evolution of the work. I also thank A.C. Ehresmann for pointing out some references.

We refer throughout to [7] for background on differentiable groupoids and Lie algebroids, and for our conventions. In particular, we omit the details of most proofs, since the techniques needed can be found in [7].

1. THE LIE ALGEBROIDS OF SEMI-DIRECT PRODUCTS.

The following definition of a smooth action of one differentiable groupoid on another is a smooth version of that given in Brown [2].

DEFINITION 1.1. Let Ω be a differentiable groupoid on B, let W be a differentiable groupoid on M, and let $p: W \to 0_B$ be a groupoid morphism and a surjective submersion, from W to the base groupoid 0_B on B. Then Ω *acts smoothly on* W *via* p if there is given a smooth map:

$$(\chi, w) \mapsto \chi w \quad \text{from} \quad \Omega * W = \{(\chi, w) \in \Omega \times W \mid \alpha\chi = pw\} \text{ to } W,$$

such that

(i) $p(\chi w) = \beta(\chi)$ for all $(\chi, w) \in \Omega * W$;

(ii) the map $w \mapsto \chi w$, $p^{-1}(\alpha\chi) \to p^{-1}(\beta\chi)$ is an isomorphism of differentiable groupoids, for all $\chi \in \Omega$;

(iii) $\omega(\chi w) = (\omega\chi)w$, whenever χw and $\omega\chi$ are defined; and

(iv) $(pw)^{\sim}w = w$, for all $w \in W$. //

For any manifold X we denote by O_X the *base groupoid* on X, in which every element is an identity, and the multiplication is the identification of the diagonal in X×X with X. The morphism p: W → O_B in 1.1 is therefore determined by the induced map p_o: M → B of the bases, for $p = p_o \circ \alpha' = p_o \circ \beta'$. Notice, too, that p_o is constant on the transitivity components of W, so that if W is transitive, then B must be a singleton and Ω a group. Lastly, each $p^{-1}(x)$, $x \in$ B, is the full subgroupoid of W on $p_o^{-1}(x)$, and is easily seen to be a differentiable subgroupoid of W.

Actions of differentiable groupoids (and categories) on fibred manifolds were defined by Ehresmann in [4]; see also [5]. Notice that in 1.1, Ω acts on the surjective submersion p_o: M → B in this sense, and the source and target projections α',β': W → M are now equivariant.

Given a smooth action of Ω on W via p, as in 1.1, the semi-direct product groupoid Ω⋉W given in Brown [2] and there attributed to Frohlich has a natural differentiable groupoid structure with base M. Here Ω×W is the manifold Ω∗W with the groupoid structure

$$\bar{\beta}(\chi, w) = \chi\beta'(w), \quad \bar{\alpha}(\chi, w) = \alpha'(w)$$

and

$$(\chi_1, w_1)(\chi_2, w_2) = (\chi_1\chi_2, (\chi_2^{-1} w_1) w_2),$$

defined if $\alpha'(w_1) = \chi_2\beta'(w_2)$. The identity corresponding to $m \in$ M is $(p_o(m)^{\sim}, m^{\sim})$ and the inverses are $(\chi, w)^{-1} = (\chi^{-1}, \chi(w^{-1}))$. It is routine to verify that Ω⋉W is a differentiable groupoid on M. Our purpose now is to calculate the Lie algebroid of Ω⋉W in terms of those of Ω and W, and we build up to this through two special cases.

CASE I. Here we suppose that Ω = G is a Lie group acting smoothly on a manifold W = M. The semi-direct product G⋉O_M was in [7] denoted G×M; we still occasionally call it the *action groupoid* corresponding to the given action. In ([7], III, 3.22) we noted that the vector bundle A(G⋉O_M) is the trivial bundle M×$\mathfrak{g}$, and the anchor M×$\mathfrak{g}$ → TM is the map $(x,\mathrm{X}) \mapsto \mathrm{X}^*(x)$ where $\mathrm{X}^* \in \Gamma$TM is the vector field on M generated by X ∈ $\mathfrak{g}$, namely

$$\mathrm{X}^*(x) = \mathrm{T}(g \mapsto gx)_1(\mathrm{X}).$$

We extend this notation to any map X: M → $\mathfrak{g}$, by defining

$$\mathrm{X}^*(x) = \mathrm{T}(g \mapsto gx)_1(\mathrm{X}(x)).$$

Thus $X^*(x) = X(x)^*(x)$.

The following result is due to A. Weinstein.

PROPOSITION 1.2. *The Lie algebroid bracket on* $A(G \ltimes O_M) = M \times \mathfrak{g}$ *is*

$$[X,Y] = X^*(Y) - Y^*(X) + [X,Y]^\bullet$$

for $X,Y\colon M \to \mathfrak{g}$. Here $X^*(Y)$, $Y^*(X)$ denote Lie dervatives, and $[\ ,\ ]^\bullet$ is the pointwise bracket of maps into $\mathfrak{g}$.

PROOF. Define a morphism ϕ from $G \ltimes O_M$ into the trivial groupoid $M \times G \times M$ by $(g,x) \mapsto (gx,g,x)$. This induces a morphism

$$\phi_*\colon M \times \mathfrak{g} \to TM \oplus (M \times \mathfrak{g})$$

of the Lie algebroids (see, for example, [7], III, §3), and since ϕ is base-preserving, ϕ_* must be of the form

$$\phi_*(x,X) = X^*(x) \oplus (x,Y).$$

In fact $Y = X$, as is easy to see, and we write, briefly, $\phi_*(X) = X^* \oplus X$. Now the result follows from the formula for the bracket on $TM \oplus (M \times \mathfrak{g})$ (see, for example, [7], III, 3.21). //

CASE II. Here we consider a general differentiable groupoid Ω on B, acting on a surjective submersion $p\colon M \to B$. The role of the trivial groupoid $M \times G \times M$ in 1.2 is taken by the inverse-image groupoid $p^*\Omega$ whose elements are all triples

$$(m_2,\chi,m_1) \quad \text{with} \quad pm_2 = \beta\chi, \quad \alpha\chi = pm_1,$$

and which has the groupoid structure

$$\beta'(m_2,\chi,m_1) = m_2, \qquad \alpha'(m_2,\chi,m_1) = m_1,$$

and

$$(m_3,\omega,m_2')(m_2,\chi,m_1) = (m_3,\omega\chi,m_1),$$

defined if $m_2' = m_2$. This construction is also due to Ehresmann [4]. Note that trivial groupoids are precisely the inverse-images of groups; to exploit this analogy we will sometimes write $p^*\Omega$ as $M * \Omega * M$.

PROPOSITION 1.3. *The Lie algebroid of* $p^*\Omega$ *is the following pullback in the category of vector bundles over* M,

$$\begin{array}{ccc} A(p^*\Omega) = TM\oplus_{p^*(TB)} p^*(A\Omega) & \longrightarrow & p^*(A\Omega) \\ \downarrow & & \downarrow \\ TM & \longrightarrow\!\!\!\!\!\rightarrow & p^*(TB), \end{array}$$

where $TM \to p^*(TB)$ *and* $p^*(A\Omega) \to p^*(TB)$ *are induced by* $T(p)\colon TM \to TB$ *and the anchor* $A\Omega \to TB$, *respectively. The anchor of* $A(p^*\Omega)$ *is the left-hand side of the square. Regarding* $\Gamma(p^*(A\Omega))$ *as the tensor product* $C(M)\otimes\Gamma A\Omega$ *over* $C(B)$ *for which* $f\otimes uU = f(u\circ p)\otimes U$, *the bracket on* $A(p^*\Omega)$ *is*

$$[X\oplus(f\otimes U),\ Y\oplus(g\otimes V)] = [X,Y] \oplus \{X(g)\otimes V - Y(f)\otimes U + fg\otimes[U,V]\},$$

where $X,Y \in \Gamma TM$, $f,g \in C(M)$, $U,V \in \Gamma A\Omega$.

PROOF. Notice that $p^*\Omega$ is the pullback manifold

$$\begin{array}{ccc} p^*\Omega & \longrightarrow & \Omega \\ \downarrow & & \downarrow{\scriptstyle(\beta,\alpha)} \\ M\times M & \xrightarrow{\ p\times p\ } & B\times B. \end{array}$$

Therefore the tangent bundle of $p^*\Omega$ is

$$\{X\oplus V\oplus Y \in TM\oplus T\Omega\oplus TM \mid T(p)(X) = T(\beta)(V),\ T(\alpha)(V) = T(p)(Y)\}.$$

Imposing the condition $Y = 0$ therefore forces $V \in T^\alpha\Omega$. Now restricting to the identity submanifold, we get the required vector bundle. The formula for the bracket is proved by the same method as for the case where Ω is a group (see, for example, [7], III, 3.21). //

If $p\colon M \to B$ is a covering $M = B^\sim \to B$ of B with group π, then $TB^\sim \to p^*(TB)$ is an isomorphism of vector bundles, and consequently $A(p^*\Omega) \to p^*(A\Omega)$ is an isomorphism also. Transferring the bracket to $\Gamma(p^*(A\Omega)) = C(B^\sim)\otimes\Gamma A\Omega$, it becomes

$$[f \otimes X,\ g \otimes Y] = fX^{\sim}(g) \otimes Y - gY^{\sim}(f) \otimes X + fg \otimes [X,Y]$$

where X, Y are the π-invariant vector fields on $B^{\sim}$ which correspond to X, Y ∈ ΓTB.

PROPOSITION 1.4. *Let* $\Omega * M \to M$, $(\chi,m) \mapsto \chi m$, *be a smooth action of a differentiable groupoid* Ω *with base* B, *on a surjective submersion* $p: M \to B$. *Then* $A(\Omega \ltimes O_M)$ *is the vector bundle* $p^*(A\Omega)$ *on* M *with anchor the map*

$$(m,\ U) \mapsto U^*(m) = T(\chi \mapsto \chi m)_{pm}(U)$$

and bracket

$$[f \otimes U,\ g \otimes V] = f\, U^*(g) \otimes V - gV^*(f) \otimes U + fg \otimes [U,V].$$

PROOF. Consider the morphism

$$\Omega \ltimes O_M \to p^*\Omega,\ (\chi,m) \mapsto (\chi m, \chi, m),$$

and proceed as in 1.2. //

CASE III. This is the general case, where a differentiable groupoid Ω with base B acts on a differentiable groupoid W with base M, via a morphism $p: W \to O_B$.

First of all, observe that the manifold underlying $\Omega \ltimes W$ is $\Omega * W$, the pullback defined by $\alpha: \Omega \to B$ and $p: W \to B$. It follows, much as in 1.3, that the vector bundle underlying $A(\Omega \ltimes W)$ is $p_o^*(A\Omega) \oplus AW$. By differentiating

$$\beta^{\sim}: \Omega \ltimes W \to M,\quad (\chi,m) \mapsto \chi\beta'(w),$$

one finds that the anchor of $A(\Omega \ltimes W)$ is $a^{\sim}: p_o(A\Omega) \oplus AW \to TM$,

$$(m,X) \oplus U \mapsto X^*(m) + a'(U) \text{ for } X \in A\Omega|_{p_o m} \text{ and } U \in AW|_m.$$

Here a denotes the anchor $A\Omega \to TB$, a' denotes the anchor $AW \to TM$, and X^* refers to the action of Ω on $p_o: M \to B$. On the section level we have

$$(f \otimes X) \oplus U \mapsto f\, X^* + a'(U),$$

where

$$f \in C(M),\quad X \in \Gamma A\Omega,\quad U \in \Gamma AW.$$

To calculate a general bracket, $[(f\otimes X)\oplus U,(g\otimes Y)\oplus V]$, break it up into

$$[(f\otimes X)\oplus 0,(g\otimes Y)\oplus 0] + [(f\otimes X)\oplus 0,0\oplus V] - [(g\otimes Y)\oplus 0,0\oplus U] + [0\oplus U,0\oplus V].$$

For the first term, observe that this is $[\phi_*(f\otimes X),\phi_*(g\otimes Y)]$ where ϕ is the morphism

$$\Omega\kappa O_M \to \Omega\kappa V,\quad (\chi,m) \mapsto (\chi,m^{\sim}).$$

From 1.4 we therefore have

$$[(f\otimes X)\oplus 0,(g\otimes Y)\oplus 0] = \{fX^*(g)\otimes Y - gY^*(f)\otimes X + fg\otimes[X,Y]\}\oplus 0.$$

Similarly, for the last term, use the morphism

$$V \to \Omega\kappa V,\quad w \mapsto (p(w)^{\sim},w),$$

to obtain $[0\oplus U,0\oplus V] = 0\oplus[U,V]$.

For the second term, we have first of all that

$$[(f\otimes X)\oplus 0,0\oplus V] = f[(1\otimes X)\oplus 0,0\oplus V] - (a'(V)(f)\otimes X)\oplus 0.$$

Next, consider an element $\chi \in \Omega_x{}^y$, any $x, y \in B$, and the isomorphism of differentiable groupoids $p^{-1}(x) \to p^{-1}(y)$ which it induces. Since $p^{-1}(x)$ is the full subgroupoid of V on $p_o^{-1}(x)$, it follows that $A(p^{-1}(x))$ is the restriction of AV to $p_o^{-1}(x)$. We denote the induced isomorphism of Lie algebroids $A(p^{-1}(x)) \to A(p^{-1}(y))$ by $\rho(\chi)$. Now let $\mathrm{Exp}\,tX$ be the exponential of X in a flow neighbourhood $N \subset B$ of a given $x \in B$. From the formula for a Lie algebroid bracket as a Lie derivative (see, for example, [7], III, 4.11(iii)) we get

$$[(1\otimes X)\oplus 0,0\oplus V] = -(d/dt)\rho(\mathrm{Exp}\,tX)(V)|_0,$$

and we denote this by $\rho_*(X)(V)$.

By $\rho(\mathrm{Exp}\,tX)(V)$ in this formula we mean the local section of AV defined at $m \in p_0^{-1}(N)$ to be

$$\rho(\mathrm{Exp}\,tX(\phi_t^{-1}(p_0(m))))(V((\mathrm{Exp}\,{-tX})(p_0(m))m))$$

where $\phi_t = \beta\circ\mathrm{Exp}\,tX$ is the local flow of $a(X)$. Notice that the action of Ω on M is involved in the point at which V is evaluated. The conventions here are those of ([7], II §5).

Putting all this together, we have the following all-inclusive result.

THEOREM 1.5. *Let the differentiable groupoid* Ω *with base* B *act on the differentiable groupoid* W *on* M *via a surjective submersive morphism* $p\colon W \to \mathcal{O}_B$. Then $A(\Omega \ltimes W) = p_0^*(A\Omega) \oplus AW$ *has anchor*

$$(f \otimes X) \oplus U \mapsto f\, X^* + a'(U)$$

and bracket

$$\begin{aligned}[(f \otimes X) \oplus U, (g \otimes Y) \oplus V] = \{& fX^*(g) \otimes Y - gY^*(f) \otimes X + fg \otimes [X,Y] \\ & - a'(V)(f) \otimes X + a'(U)(g) \otimes Y\} \oplus \{f p_*(X)(V) - g p_*(Y)(U) + [U,V]\},\end{aligned}$$

where $\qquad f,g \in C(M), \ X,Y \in \Gamma A\Omega, \quad U,V \in \Gamma AW.$ //

A note on terminology. Groupoids of the form $\Omega \ltimes \mathcal{O}_M$ were originally defined in [4]. In [7] we called them *action groupoids*; in the terminology of Higgins [6] and Brown et al [3], they are *covering groupoids*, reflecting the fact that the set-theoretic version of this concept models the theory of covering spaces. We avoid this latter terminology here, because of the danger of confusion with coverings (in the topological sense) of the various spaces involved in a groupoid.

2. THE LIE ALGEBROIDS ASSOCIATED WITH A PBG-GROUPOID.

This section gives an extended application of the results of §1. We show that action groupoids and a general semi-direct product arise in connection with extensions of principal bundles, and use this fact and the results of §1 to calculate several Lie algebroids associated with such an extension. This section may be omitted without loss of continuity.

We refer the reader to [8] for the concept of a PBG-groupoid and the equivalence between extensions of principal bundles and PBG-groupoids.

Let

$$N \rightarrowtail Q(B,H,p') \twoheadrightarrow P(B,G,p)$$

be an extension of principal bundles, and let

$$M \rightarrowtail \Phi \twoheadrightarrow \Omega$$

be the corresponding extension of Lie groupoids. Let $\Pi = (Q\times Q)/N$ be the associated PBG-groupoid.

What follows is founded on the observation that Π is naturally isomorphic to a certain action groupoid $\Phi\ltimes O_P$. Regarding P as Ω_b, for some chosen $b \in$ B, and identifying $\beta_b : \Omega_b \to$ B with p, there is a natural action of Φ on p: P $\to$ B, namely $(\gamma,\chi) \mapsto \pi(\gamma)\chi$. In terms of $\Phi = (Q\times Q)/H$, the natural isomorphism $\Pi \to \Phi\ltimes O_P$ is

$$\langle v_2, v_1\rangle^N \mapsto (\langle v_2, v_1\rangle^H, \pi(v_1)).$$

Notice that the (right) action of $G = \Omega_b{}^b$ on Π now becomes $(\gamma,u)g = (\gamma,ug)$.

It now follows immediately from 1.4 that the Lie algebroid $A\Pi$ is the vector bundle $p^*(A\Phi)$ with anchor $f\otimes X \mapsto f\overrightarrow{\pi_*(X)}$ and bracket

$$[f\otimes X, g\otimes Y] = f\overrightarrow{\pi_*(X)}(g)\otimes Y - g\overrightarrow{\pi_*(X)}(f)\otimes X + fg\otimes[X,Y],$$

where $f,g \in C(P)$, $X,Y \in \Gamma A\Phi$, $\pi_*: A\Phi \to A\Omega$ is the Lie algebroid morphism induced by π, and $\overrightarrow{\pi_*(X)}$ is the G-invariant vector field on P corresponding to $\pi_*(X) \in \Gamma A\Omega$. This expression for $A\Pi$ is given in ([8], §4), but appeared somewhat mysterious there.

Next we consider the principal bundle $\Pi(\Phi,G,q)$, where $q: \Pi \to \Phi$ is the map denoted # in [8]. In terms of $\Pi = \Phi\ltimes O_P$, the map q is the natural morphism $(\gamma,u) \mapsto \gamma$. We want to find the Atiyah sequence of this bundle, or equivalently the Lie algebroid of the associated groupoid $(\Pi\times\Pi)/G$ with base Φ; we denote this groupoid by Θ. Elements of Θ are orbits $\langle(\gamma_2,u_2),(\gamma_1,u_1)\rangle$ of the diagonal action of G on Π; thus

$$\langle(\gamma_2,u_2g),(\gamma_1,u_1g)\rangle = \langle(\gamma_2,u_2),(\gamma_1,u_1)\rangle,$$

for $g \in G$. Define a morphism from Θ to the inverse-image Lie groupoid $(\alpha')^*\Omega$, where α': $\Phi \to$ B is the source map of Φ, by

$$\langle(\gamma_2,u_2),(\gamma_1,u_1)\rangle \mapsto (\gamma_2,\langle u_2,u_1\rangle,\gamma_1).$$

This is clearly well-defined, and it is easy to check that it is in fact an isomorphism. So from 1.3, it follows that $A\Theta$ is naturally isomorphic to $T\Phi\oplus_S(\alpha')^*(A\Omega)$ where $S = (\alpha')^*(TB)$ with the bracket given there.

Thirdly, notice that Θ carries a second Lie groupoid structure, this one having base Ω. Namely, regard $\Pi\times\Pi$ as the cartesian product Lie groupoid on base $P\times P$; because G acts freely on Π by Lie groupoid automorphisms, it follows that $(\Pi\times\Pi)/G$, the quotient groupoid, is a Lie groupoid on base $(P\times P)/G = \Omega$. It is straightforward to check that this second structure commutes with the first, so that Θ is a double groupoid. Using the above identification of the manifold Θ with

$$(\alpha')^* \Omega = \Phi * \Omega * \Phi,$$

this second structure has source map $(\gamma_2,\chi,\gamma_1) \mapsto \chi$, target map

$$(\gamma_2,\chi,\gamma_1) \mapsto \pi(\gamma_2)\chi\pi(\gamma_1)^{-1},$$

and composition

$$(\gamma_2',\chi',\gamma_1')(\gamma_2,\chi,\gamma_1) = (\gamma_2'\gamma_2,\chi,\gamma_1'\gamma_1),$$

and is thus naturally isomorphic, under

$$(\gamma_2,\chi,\gamma_1) \mapsto (\gamma_2,\gamma_1,\chi),$$

to the action groupoid $(\Phi\times\Phi)\ltimes\Omega_a$, where $\Phi\times\Phi$ is the cartesian square groupoid with base $B\times B$, acting on the surmersion $(\beta,\alpha)\colon \Omega \to B\times B$ by

$$((\gamma_2,\gamma_1),\chi) \mapsto \pi(\gamma_2)\chi\pi(\gamma_1)^{-1}.$$

The Lie algebroid of this second Lie groupoid structure on Θ now follows from 1.4; as a vector bundle it is $(\beta,\alpha)^*(A\Phi\times A\Phi)$.

Lastly, recall from the remark at the end of §1 of [8], that the semi-direct product groupoid $\Pi\rtimes G$, corresponding to the right action of G on Π, is naturally isomorphic to the inverse-image groupoid $p^{**}\Phi$. It is an instructive exercise to establish directly the corresponding ismorphism of Lie algebroids

$$A\Pi\oplus(P\times\mathfrak{g}) \to TP\oplus_{p*(TB)}p^*(A\Phi).$$

Using the identification of $A\Pi$ with $p^*(A\Phi)$ given above or in ([8], 4.2), and denoting the anchor $A\Pi \to TP$ by $a^{\sim}$, this isomorphism is

$$X\oplus V \mapsto (V^*+a^{\sim}(X))\oplus X,$$

where V^* is the fundamental vector field on P corresponding to $V \in \mathfrak{g}$.

On a subsequent occasion, we will abstract the relations between the two Lie algebroids $A_1\theta \to T\Phi$ and $A_2\theta \to T\Omega$, which reflect the commutativity of the two Lie groupoid structures.

3. ACTIONS AND SEMI-DIRECT PRODUCTS OF ABSTRACT LIE ALGEBROIDS.

We begin by observing that inverse-images of transitive Lie algebroids over arbitrary smooth maps, can be defined by abstracting 1.3.

THEOREM 3.1. *Let* A *be a transitive Lie algebroid on base* B, *with anchor* a, *and let* $f\colon B' \to B$ *be a smooth map. The pullback*

$$\begin{array}{ccc} TB' \oplus_{f^*(TB)} f^*A & \longrightarrow & f^*A \\ \big\downarrow & & \big\downarrow f^*(a) \\ TB' & \longrightarrow & f^*(TB) \end{array}$$

exists in the category of vector bundles over B', *and* $TB'\oplus_{f^*(TB)}f^*A$ *is a transitive Lie algebroid with base* B' *when equipped with the left-hand arrow as anchor, and the bracket*

$$[X'\oplus(u\otimes X), Y'\oplus(v\otimes Y)] = [X',Y']\oplus(X'(v)\otimes Y - Y'(u)\otimes X + uv\otimes[X,Y]),$$

for $u,v \in C(B')$, $X',Y' \in \Gamma TB'$, $X,Y \in \Gamma A\Omega$.

PROOF. This is entirely standard, except perhaps for the fact that the bracket is well-defined with respect to the tensor-product. For this, it is necessary to observe that X' and $u\otimes X$ have the same image in $f^*(TB)$, and so too do Y' and $v\otimes Y$. //

We denote $TB' \oplus_{f^*(TB)} f^*A$ by $f^{**}(A)$ and call it the *inverse-image Lie algebroid* of A over f. Given two Lie algebroids, A' on B' and A on B, and a vector bundle map $\phi\colon A' \to A$ over $\phi_0\colon B' \to B$, there is a natural map

$$\phi^-\colon A' \to \phi_0^*A, \quad X' \mapsto (x', \phi(X')), \quad \text{for } X' \in A'_{x'}.$$

If, further, A and A' are transitive and ϕ is anchor-preserving, that is, $a'\circ\phi = T(\phi_0)\circ a$, then there is a natural map

$$\phi^{\sim}: A' \to \phi_0^{**}(A), \quad X' \mapsto a'(X')\oplus\phi^{-}(X'),$$

and it is anchor-preserving. This leads to the following natural definition.

DEFINITION 3.2. Let a': A' → TB' and a: A → TB be transitive Lie algebroids. Then a *morphism of Lie algebroids* ϕ: A' → A is a vector bundle morphism over ϕ_0: B' → B, which is anchor-preserving and which is such that $\phi^{\sim}: A' \to \phi_0^{**}(A)$ is a morphism of Lie algebroids over B'//

A detailed definition of a general morphism of (not necessarily transitive) Lie algebroids was given by Almeida and Kumpera [0], based on Pradines [11], and one can readily enough see that this definition is equivalent to theirs, in the transitive case. The point of this formulation is that it is conceptually simple. Further, the basic integrability result for general morphisms of transitive Lie algebroids can now be simply obtained from the base-preserving case - which was proved in [7] by elementary methods.

To see this, consider first a morphism of Lie groupoids ϕ: $\Omega' \to \Omega$ over ϕ_0: B' → B. Factorize ϕ into

$$\begin{array}{ccccc} \Omega' & \xrightarrow{\ \phi^{-}\ } & \phi_0^{*}\Omega & \xrightarrow{\ \phi_0{}'\ } & \Omega \\ \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\ B' & = & B' & \xrightarrow{\ \phi_0\ } & B \end{array}$$

where

$$\phi^{-} \text{ is } \chi' \mapsto (\beta'\chi', \phi(\chi'), \alpha'\chi') \text{ and } \phi_0{}' \text{ is } (y', \chi, x') \mapsto \chi.$$

Applying the Lie functor A (regarded as a functor with values in the category of vector bundles) to this diagram, one sees immediately that $A(\phi_0{}')$ is the map

$$X'\oplus(\iota\otimes X) \mapsto \iota\otimes X,$$

and that $A(\phi^-)$ is a morphism of Lie algebroids over B'; by the universality property of pullbacks, $A(\phi^-)$ must be $\phi^\sim$. Thus $A(\phi)$: $A\Omega' \to A\Omega$ is a morphism of Lie algebroids in the sense of 3.2.

THEOREM 3.3. *Let* Ω, Ω' *be Lie groupoids on* B, B', *and let* ϕ: $A\Omega' \to A\Omega$ *be a morphism of Lie algebroids over* ϕ_0: $B' \to B$. *Suppose further that* Ω' *is* α*-simply connected. Then there is a unique morphism of Lie groupoids* ψ: $\Omega' \to \Omega$ *over* ϕ_0, *such that* $\psi_* = \phi$.

PROOF. $\phi^\sim$: $A\Omega' \to \phi_0{}^{**}(A\Omega)$ is a morphism of Lie algebroids over B', and by 1.3, $\phi_0{}^{**}(A\Omega) = A(\phi_0{}^*\Omega)$. So, by the integrability result for base-preserving morphisms ([7], III, 6.5, for example), there is a unique morphism ψ': $\Omega' \to \phi_0{}^*\Omega$ over B', with $(\psi')_* = \phi^\sim$. Now define $\psi = \phi_0{}' \circ \psi'$. Clearly $\psi_* = \phi$. The uniqueness follows by reversing the argument. //

3.3 of itself shows that 3.2 is a correct definition. One can use 3.2 to give a conceptually simple definition of a general Lie subalgebroid of a transitive Lie algebroid. Compare [0], where the reverse process is followed.

This factorization in terms of inverse-images has been used recently by Pradines [13] to define and characterize quotient differentiable groupoids.

We turn now to actions of Lie algebroids, and we first consider the special case which corresponds to Case II of §1. We consider Lie algebroids which are not necessarily transitive. Notice that if f in 3.1 is a surmersion, then $f^{**}(A)$ can be defined for any Lie algebroid, not necessarily transitive. (Most generally, of course, one needs only a transversality condition on a and $T(f)$.)

DEFINITION 3.4. Let a: $A \to TB$ be a Lie algebroid and let p: $M \to B$ be a surmersion. Then an *action* of A on p: $M \to B$ is a map

$$\Gamma A \to \Gamma TM, \quad X \mapsto X^*,$$

such that

(i) $[X^*,Y^*] = [X,Y]^*$ for all $X,Y \in \Gamma A$;

(ii) $(uX+vY)^* = (u\circ p)X^* + (v\circ p)Y^*$ for all $X,Y \in \Gamma A$ and $u,v \in C(B)$;

(iii) X^* is a projectable vector field on M and projects to $a(X)$ for all $X \in \Gamma A$. //

If B is a point, so that A is a Lie algebra, this reduces to the standard concept of an infinitesimal action of a Lie algebra on a manifold. If Ω is a differentiable groupoid acting smoothly on $p: M \to B$, then

$$X^*(m): T(\chi \mapsto \chi m)_{p_m}(X(pm)), \quad \text{for } X \in \Gamma A\Omega, \quad m \in M,$$

defines an action of $A\Omega$ on $p: M \to B$.

Now consider an action of A on $p: M \to B$. Notice that condition (ii) of 3.4 implies that the map

$$\Gamma(p^*A) \simeq C(M) \otimes_{C(B)} \Gamma A \to \Gamma TM, \quad f \otimes X \mapsto fX^*,$$

is well-defined and C(M)-linear and so corresponds to a vector bundle morphism $p^*A \to TM$ over M. This becomes the anchor of a Lie algebroid structure on p^*A for which the bracket is

$$[f \otimes X, g \otimes Y] = fg \otimes [X,Y] + fX^*(g) \otimes Y - gY^*(f) \otimes X \;;$$

the conditions of 3.4 ensure that p^*A is now a Lie algebroid on M. We say that the action is *transitive* if p^*A is a transitive Lie algebroid. Notice that the natural map $p^*A \to A$ is now a morphism of Lie algebroids.

DEFINITION 3.5. (i) The Lie algebroid p^*A defined above is the *covering Lie algebroid* of the action, and the natural map $p^*A \to A$ is the *covering morphism*.

(ii) A morphism of Lie algebroids ϕ: $A' \to A$ over ϕ_0: $B' \to B$ is a *covering* if ϕ is a fibrewise bijection, and ϕ_0 is a surjective submersion. //

This terminology is modelled on that of Higgins [6] and Brown et al [3] for groupoids. Alternatively, one could call p^*A an action Lie algebroid, but in the case of Lie algebroids there is little likelihood of confusion with covering spaces.

Suppose ϕ: $A' \to A$ is a covering morphism of Lie algebroids over ϕ_0: $B' \to B$. Then $\phi_0{}^*A$ is isomorphic to A' as a vector bundle, and so there is a natural isomorphism

$$C(B') \otimes_{C(B)} \Gamma A \simeq \Gamma A'.$$

Namely, given $X \in \Gamma A$ there is a unique $X^{-} \in \Gamma A'$ such that $\phi \circ X^{-} = X \circ \phi_0$, and the isomorphism is $f \otimes X \mapsto fX^{-}$. Now it is easy to check that $X \mapsto a'(X^{-})$ is an action of A on ϕ_0: $B' \to B$ and that the covering Lie algebroid is isomorphic to A'. One can easily develop in this context the refinements of the theory of coverings of groupoids as in, for example, [6].

Given an action of a Lie algebroid A on $p: M \to B$, there is a natural morphism from p^*A to the inverse-image Lie algebroid

$$p^{**}(A) = TM \oplus_{p*(TB)} p^*A,$$

namely (on the section level),

$$f \otimes X \mapsto fX^* \oplus (f \otimes X).$$

This is a morphism of Lie algebroids over M but need not be of locally constant rank - this occurs already for $A = \mathfrak{g}$ a Lie algebra. Generalizing terminology of Palais [10], one may call the image of this morphism the *infinitesimal graph* of the action.

If A is integrable, that is if $A = A\Omega$ for some differentiable groupoid Ω, then one may ask under what conditions there is an action of Ω on $p: M \to B$ inducing the given action of $A\Omega$. Since $p^{**}(A\Omega)$ is the Lie algebroid of $p^*\Omega$, one may formulate this in terms of integrating the infinitesimal graph, regarded as a subobject of $A(p^*\Omega)$. However, even when the infinitesimal graph is of locally constant rank, and thus a genuine Lie subalgebroid of $A(p^*\Omega)$, it is not clear that this is the best approach. This problem is a very large one, and we will not attack it here.

The following result comes very cheaply on account of the conceptual apparatus we have already set up.

THEOREM 3.6. *Let* ϕ: A' $\to$ A *be a covering of Lie algebroids. If both* A *and* A' *are transitive, and* A *is integrable, then* A' *is integrable.*

PROOF. Write $A = A\Omega$, with Ω a Lie groupoid. Then the map

$$A' \to \phi_0^{**}(A\Omega) = A(\phi_0^*\Omega)$$

imbeds A' as a transitive Lie subalgebroid of $A(\phi_o{}^*\Omega)$. So we can apply ([7], III, 6.1) to obtain a unique α-connected Lie subgroupoid Ω' of $\phi_o{}^*\Omega$ with $A' \to A\Omega'$ an isomorphism. //

COROLLARY 3.7. *Let A be a transitive Lie algebroid which is trivializable as a vector bundle in such a way that the bracket of constant sections is constant. Then A is integrable.*

PROOF. Let $A \simeq B\times V$ be the trivialization. Then, by identifying elements of V with constant sections of A, one obtains a Lie algebra structure on V; denote it by $\mathfrak{g}$. Now the condition that the bracket of constant sections is constant ensures that the map $A \simeq B\times\mathfrak{g} \to \mathfrak{g}$ is a morphism of Lie algebroids. It is clearly a fibrewise bijection, and since Lie algebras are integrable, the result follows. //

In [9] we gave necessary and sufficient conditions that an integrable transitive Lie algebroid, on a base which is compact with finite fundamental group, be integrable to an action groupoid of the form $G\ltimes O_B$, for G a Lie group acting (transitively) on B. 3.7 now shows that the Lie algebroid need not be assumed, at the outset, to be integrable.

It is worth noting that the viewpoint taken here does not greatly shorten the arguments of [9]. Suppose that we are in the situation of 3.6. We may assume Ω to be α-simply connected, and we may take the monodromy groupoid $\Phi = M\Omega'$ of Ω' so that we have ϕ: $A\Phi \to A\Omega$ with both Φ and Ω α-simply connected. Now (by 3.3) ϕ integrates to ψ: $\Phi \to \Omega$. Since $\psi_* = \phi$ is a fibrewise bijection, it follows that ψ, restricted to any α-fibre, is a surjective submersion with discrete fibres. Since Ω is α-simply connected, ψ is an α-diffeomorphism. However in order for ψ to induce an action of Ω on ϕ_o: $B' \to B$ (that is, for ψ to be a covering in the sense of [3]), it is necessary that the map

$$(\psi,\alpha'): \Phi \to \Omega * B, \quad \chi \mapsto (\psi(\chi),\alpha'(\chi)),$$

where

$$\Omega * B' = \{(\gamma,x') \in \Omega\times B' \mid \alpha\gamma = \phi_o(x')\},$$

be a diffeomorphism. This does not follow in general - if it did, then every transitive infinitesimal action of a Lie algebra on a simply-connected manifold would be integrable, and this is well-known not to be so ([10], p. 88). The main work of [9] is to get around this difficulty.

We come now to general actions of one Lie algebroid on another, and the resulting semi-direct products.

DEFINITION 3.8. Let A be a Lie algebroid on base B with anchor a, and let R be a Lie algebroid on base M with anchor r. Let $p: R \to B\times 0$ be a Lie algebroid morphism over a surjective submersion $p_0: M \to B$, from R to the zero (and totally intransitive) Lie algebroid $B\times 0$ on B (see below). Then an *action* of A on R via p consists of an action

$$X \mapsto X^*, \quad \Gamma A \to \Gamma TM,$$

of A on $p_0: M \to B$ in the sense of 3.4, together with a map

$$\rho: \Gamma A \to \Gamma CDO(R)$$

such that

(i) $$\rho(X)(fV) = f\rho(X)(V) + X^*(f)V$$
for all $X \in \Gamma A$, $V \in \Gamma R$, $f \in C(M)$;

(ii) $$\rho(X)([U,V]) = [\rho(X)(U),V] + [U,\rho(X)(V)]$$
for all $X \in \Gamma A$, $U,V \in \Gamma R$;

(iii) $$r(\rho(X)(V)) = [X^*,r(V)]$$
for all $X \in \Gamma A$, $V \in \Gamma R$;

(iv) $$\rho([X,Y]) = [\rho(X),\rho(Y)] \quad \text{for all } X,Y \in \Gamma A;$$

(v) $$\rho(uX+vY) = (u\circ p)\rho(X) + (v\circ p)\rho(Y)$$
for all $u,v \in C(B)$, $X,Y \in \Gamma A$. //

Here CDO(R) is the vector bundle whose sections are those first- or zeroth-order differential operators D in the vector bundle R for which there exists a vector field S on M such that

$$D(fV) = f\,D(V) + S(f)V \quad \text{for all } V \in \Gamma R, \ f \in C(M).$$

The field S is then unique and is essentially the symbol (first-order part) of D. With the map $D \mapsto S$ as anchor, and the commutator bracket, CDO(R) is a transitive Lie algebroid on M. See, for example, ([7], III, 2.5). Conditions (i), (iv) and (v) above now assert that there is a morphism of Lie algebroids over M from $p_0{}^*A$, the covering Lie algebroid corresponding to the action of A on $p_0: M \to B$, to CDO(R), namely (on the section level) $f\otimes X \mapsto f\rho(X)$.

Consider the condition that $p: R \to B\times 0$ be a Lie algebroid morphism over p_0. Since $B\times 0$ is a zero Lie algebroid, the bracket-preservation condition is vacuous. The force of this condition is thus the anchor condition which says that for any $V \in \Gamma R$, the vector field $r(V)$ on M projects to the zero vector field on B. Equivalently,

$r(V)$ is tangent to the fibres of p_0, or $r(V)(u \circ p_0) = 0$ for all $u \in C(B)$. In particular, if R is transitive, then all smooth functions on B must be constant. So if R is transitive and B is connected, then B is in fact a point and A is a Lie algebra.

THEOREM 3.9. *With* A, R *and* p *as in 3.8, let* ρ *be an action of* A *on* R *via* p. *Then the vector bundle* $p_0{}^*A \oplus R$ *on* M, *equipped with the anchor*

$$(f \otimes X) \oplus V \mapsto f X^* + r(V)$$

and the bracket

$$[(f \otimes X) \oplus U, (g \otimes Y) \oplus V] = \{fX^*(g) \otimes Y - gY^*(f) \otimes X + fg \otimes [X,Y] - r(V)(f) \otimes X + r(U)(g) \otimes Y\} \oplus \{f\rho(X)(V) - g\rho(Y)(U) + [U,V]\}$$

for $f,g \in C(M)$, $X,Y \in \Gamma A$, $U,V \in \Gamma R$, *is a Lie algebroid on* M.

PROOF. This is a routine, but instructive exercise, after the style of ([7], Chapter IV). The only novel aspect is to check that the bracket is well-defined with respect to the tensor product; it is at this point that one uses the condition

$$r(U)(u \circ p_0) = 0 \quad \text{for} \quad U \in \Gamma R, \quad u \in C(B). \qquad //$$

With this structure, we denote $p_0{}^*A \oplus R$ by $A \ltimes R$ and call it the *semi-direct product* of A and R with action ρ. Notice that $p_0{}^*A$ could now be denoted $A \ltimes (M \times 0)$, where $M \times 0$ is the zero Lie algebroid on M. There are canonical morphisms $R \to A \ltimes R$, namely $V \mapsto 0 \oplus V$, and $A \ltimes R \to A$, namely (on the bundle level) $(m,X) \oplus V \mapsto X$. It is instructive to check that the latter is a morphism, providing we broaden 3.2 to allow non-transitive Lie algebroids when the base-map is a surjective submersion. Together these give an exact sequence of Lie algebroids, in an obvious sense, and there is a morphism $A \ltimes (M \times 0) \to A \ltimes R$, namely

$$f \otimes X \mapsto (f \otimes X) \oplus 0,$$

which may be considered to split it. Lastly, the content of 1.5 becomes, of course, that $A(\Omega \ltimes W) = A\Omega \ltimes AW$; it is a routine exercise to check that conditions (i)-(v) of 3.8 hold (compare the proof of III, 4.5 in [7]).

This semi-direct product may well be of use in a general cohomology theory for Lie algebroids, in the manner developed in [2] for groupoids.

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