CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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Cahiers de topologie et géométrie différentielle catégoriques, tome 28, nº 3 (1987), p. 235-253

<http://www.numdam.org/item?id=CTGDC_1987__28_3_235_0>

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Vol. XXVIII-3 (1987)

ON FREE TOPOLOGICAL ALGEBRAS

BY Hans-E. PORST

RÉSUMÉ. Etant donné une catégorie A d'algèbres topologiques ou uniformes, on discute sous quelles conditions:

(i) l'algèbre topologique libre $G\underline{X}$ sur un espace $\underline{X} = (X,\tau)$ par rapport à A est algébriquement l'algèbre abstraite libre sur X; et

(ii) $G\underline{X}$ contient $\underline{X}$ comme sous-espace.

INTRODUCTION.

More than forty years ago A.A. Markov [13] achieved the first result on free topological algebras in proving the existence of a free (Hausdorff-) topological group $G\underline{X}$ over an arbitrary Tychonoff space $\underline{X} = (X,\tau)$. According to his time Markov's notion of a free topological group $G\underline{X}$ was still rather uncategorical; $G\underline{X}$ was supposed to fulfill the following axioms:

(A) The algebraic structure of $G\underline{X}$ is just FX, the free (abstract) group over the underlying set X of $\underline{X}$;

(T) Topologically $\underline{X} = (X,\tau)$ is a subspace of $G\underline{X} = (FX,\sigma)$ by means of the "insertion-of-generators map" $\gamma_X: X \to FX$.

(U) $\gamma_X: \underline{X} \to G\underline{X}$ has the usual universal property.

Correspondingly, his proof was carried out by an explicit construction of a suitable topology on the free group FX.

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*) The author acknowledges the hospitality of the Department of Mathematics, Applied Mathematics and Astronomy at the University of South Africa, where he carried out most of the work presented in this article.

Only a few years later P. Samuel [17] and S. Kakutani [9] independently accomplished substantially shorter proofs by focusing on the universal property (U) (by which G$\underline{X}$ is determined uniquely for categorical reasons) and using the purely categorical idea of what later was called "Freyd's General Adjoint Functor Theorem". (See [8].) However for checking Markov's conditions (A) and (T) they had to provide some additional arguments, in particular on linear groups.

Very much in the spirit of the latter papers A.I. Mal'cev in 1957 started developing a theory of general free topological algebras [11] taking the universal property as the only defining condition of a free (Hausdorff-) topological algebra and considering Markov's additional axioms (A) and (T) secondary. He then proved the existence of these objects in general in the same way as Samuel and Kakutani, according to the categorical nature of their proofs; but he only got partial answers to the question when the axioms (A) and (T) will be satisfied.

A final step of this development was reached by O. Wyler's lifting theorem for adjunctions [22] and certain generalizations [3, 21] (see also Section 3). Here, the specific use of categorical ideas gives a description of free topological algebras, which is very easy to handle; it also makes clear why one can't expect the free (Hausdorff-) topological group over a space $\underline{X}$ to be algebraically free over X in general.

However, categorical ideas were not used so far to look for settings where the axioms (A) and (T) are fulfilled. The best (but apparently not too well known) result in this respect up to now is due to S. Świerczkowski [19], who showed that both conditions are satisfied provided $\underline{X}$ is a Tychonoff space. Swierczkowski's proof however makes use of the additional - but avoidable (see [16]) - axiom that the free topological algebra over $\underline{X}$ is algebraically generated by X ; moreover the actual contents of his construction can be expressed more explicitly (see [20] and Section 4).

One might add at this stage that one hardly can see Markov's result as a predecessor of Swierczkowski, for Markov worked with Tychonoff spaces for the simple reason that he felt this to be the most natural class of spaces in this setting, since every Hausdorff-topological group is a Tychonoff space.

It is the aim of this paper to give satisfactory answers to the questions when Markov's axioms (A) and (T) are satisfied. This is done by combining categorical methods and a specialized and at the same time strengthened version of Swierczkowski's Theorem.

1. PRELIMINARIES.

(1.1) For basic notions and facts from topology we refer to [10]. Moreover we will use the following notations: *Top* (resp. Top_i) denotes the category of all topological (resp. T_i-) spaces and continuous maps (i = 0,1,2,...), while *Tych* (resp. $Comp_2$) denotes the full subcategory of *Top* consisting of all Tychonoff (resp. compact Hausdorff) spaces. *Unif* (resp. $Unif_0$) denotes the category of all (resp. all separated) uniform spaces and uniformly continuous maps, and *Met* is the category of metric spaces and uniformly continuous maps. Of particular interest will be the category Top_{2*} of all functionally Hausdorff spaces.

A topological space is called *functionally Hausdorff* (or a T_{2*}-*space)* if every pair (equivalently every finite number, see [18]) of distinct points can be separated by a continuous real ($\underline{I}$-)valued map. Observe that the Tychonoff-reflection of a functionally Hausdorff space is a bijection, as is immediate e.g. from ([10], 3.9).

If τ is a topology on the set X the corresponding topological space will be denoted by $\underline{X}$ = (X,τ); similarly $\underline{X}$ = (X,$\mathcal{U}$) denotes a uniform space if $\mathcal{U}$ is a uniformity on X. I always denotes the closed unit interval and $\mathbb{R}_+$ the set of all positive real numbers.

(1.2) Standard facts from universal algebra can be taken from [12]; as far as categorical notions are used in this context one might consult [4], [5] and [14]. In particular we call a quasivariety A (resp. its underlying functor U: $A \to$ *Set*) *nontrivial* if for each set X the insertion of generators map γ_X: X $\to$ UFX from X into the corresponding free algebra FX over X is injective (see [12], p. 51). Throughout this paper the terms algebra, (quasi-)variety, regular functor are always meant to be finitary and nontrivial. *Free algebra* will always refer to a free algebra with respect to an arbitrary but fixed quasivariety. Differently from [12] we will denote a universal algebra of type Ω by $A^\wedge$ = (A,($f_i^{(n_i)}$)) where A is the carrier (set) of $A^\wedge$ and ($f_i^{(n_i)}$) is its family of operations $f_i^{(n_i)}$: $A^{n_i} \to$ A given by the type Ω.

For an algebra $A^\wedge$ = (A,($f_i^{(n_i)}$)) of type Ω we denote by P($A^\wedge$) its set of polynomials, i.e., the smallest set of operations on A containing all the operations $f_i^{(n_i)}$ given by the type Ω and all projections π_n: $A^n \to$ A, which is closed with respect to substitution of operations.

A subset M of A is said to generate an element $a \in$ A if there is some $f: A^n \to A \in P(A^\wedge)$ such that $a \in f[M^n]$; the *support* S_a of $a \in$ A is the intersection of all subsets of A generating the element a. The following simple observation for an algebra $A^\wedge$ which is free over a set X will be used (see [19]):

If $a = f(x_1,\ldots x_n) \in A$ for $(x_1,\ldots,x_n) \in X^n$ and $f \in P(A^\wedge)$, then $a = f(tx_1,\ldots,tx_n)$ for each map $t: X \to X$ with $t(x) = x$ for all $x \in S_a$.

(1.3) For basic categorical facts we refer to [8], and [1], [6] or [7] for the more specific notions of categorical topology. However we use the term monotopological functor instead of (regular-epi, mono)-topological functor as in [6]; explicitly: a functor T: $\mathcal{X} \to \mathcal{Y}$ is called *monotopological*, provided:

(i) $\mathcal{Y}$ is a regular category in the sense of [5] (i.e., every source $(Y, f_i: Y \to Y_i)_{i \in I}$ in $\mathcal{Y}$ admits a (unique) factorization $f_i = m_i \circ e$ with a regular epimorphism $e: Y \to Z$ and a monosource (i.e., a point-separating family in case $\mathcal{Y} = Set$) $(Z, m_i: Z \to Y_i)_{i \in I}$).

(ii) $\mathcal{X}$ has T-initial T-lifts of arbitrary monosources of the form $(Y, m_i: Y \to TX_i)_{i \in I}$.

Examples are the underlying functors of Top_1, *Tych*, $Unif_0$.

(1.4) By a topological algebra we always mean a triple

$$\underline{A}^\wedge = (A, f_i^{(n_i)}, \tau)$$

where $\underline{A} = (A,\tau)$ is a topological space and $A^\wedge = (A, f_i^{(n_i)})$ is a universal algebra such that all the operations $f_i^{(n_i)}$ are continuous with respect to the topology τ (and the product topologies); τ then might be called an algebra topology; similarly we use the notion of a uniform algebra. If $\mathcal{A}$ is a quasivariety and $\mathcal{X}$ a category of topological or uniform spaces, we denote by $\mathcal{A}(\mathcal{X})$ the category of all topological (resp. uniform) algebras $\underline{A}^\wedge$ such that $A^\wedge$ belongs to $\mathcal{A}$ and $\underline{A}$ belongs to $\mathcal{X}$; the morphisms of $\mathcal{A}(\mathcal{X})$ are the (uniformly) continuous algebra-homomorphisms. Note that in case of a (mono-)topological functor T: $\mathcal{X} \to Set$ the obvious underlying-algebra functor S: $\mathcal{A}(\mathcal{X}) \to \mathcal{A}$ will be (mono-)topological, too ([1, 22]), while the underlying-space functor V: $\mathcal{A}(\mathcal{X}) \to \mathcal{X}$ will not be regular in general, but only T-regular in the sense of [15]. The following easy observation will be of some importance:

Given a quasivariety $\mathcal{A}$ we might form the categories $\mathcal{A}(Top)$ and $\mathcal{A}(Top_2)$ with underlying space functors V, resp. V_2 which will have adjoints (see Section 3) G, resp. G_2. If now $\underline{X}$ is a Hausdorff space it might happen that $G\underline{X}$ belongs to $A(Top_2)$; then clearly $G\underline{X}$ is (up to isomorphism) the same as $G_2\underline{X}$. In general however $G\underline{X}$ and $G_2\underline{X}$ will be different; that is why we will call $G\underline{X}$ the *free topological algebra over* $\underline{X}$ and $G_2\underline{X}$ the *free Hausdorff-(topological) algebra*. Similarly for other subcategories.

2. RESULTS.

We here list some theorems which are immediate consequences of the following sections.

(2.1) **THEOREM.** *For a uniform space* (X,U) *the following are equivalent:*

(i) (X,U) *is a separated uniform space.*

(ii) The free uniform algebra and the free separated uniform algebra over (X,U) *coincide; it is algebraically the free algebra over* X, *while its uniform structure is separated and contains via "insertion of generators" the space* (X,U) *as a uniform subspace.*

PROOF. (i) implies (ii) by (5.2) and (5.3.6) while the converse is trivial.

(2.2) **PROPOSITION.** *If* $\underline{X} = (X,\tau)$ *is a functionally Hausdorff space, then the free Hausdorff (functionally Hausdorff, Tychonoff) topological algebra is algebraically the free algebra over* X.

PROOF. Clear from (5.2) and (5.3.1), resp. (5.3.4) and (5.3.5).

(2.3) **REMARK.** The converse of (2.2) obviously does not hold as the variety of algebras with no operations except projections shows. But observe (2.6).

Let V: $A(Top) \to Top$ denote the underlying-space functor with respect to any quasivariety A. Then we have:

(2.4) **THEOREM.** *For a topological space* $\underline{X} = (X,\tau)$ *the following are equivalent:*

(i) $\underline{X}$ *is functionally Hausdorff.*

(ii) $VG\underline{X}$ *is functionally Hausdorff (and the unit* γ_x *lifts to a continuous injection* $\pi_{\underline{X}}: \underline{X} \to VGX$ *).*

(iii) The free algebra FX *over* X *admits a Tychonoff algebra topology* σ *such that* γ_X *lifts to a continuous injection*

$$\pi'_X: \underline{X} \to (FX,\sigma).$$

(iv) The free topological algebra, the free Hausdorff topological algebra, and the free functionally Hausdorff topological algebra coincide; it is algebraically the free algebra over X, *while it is topologically a functionally Hausdorff space, such that the insertion of generators map is continuous.*

PROOF. The following implications are obvious: (iii) ⇒ (i) and (iv) ⇒ (ii) ⇒ (i).
Next we prove (i) ⇒ (iii): Let id: $(X,\tau) \to (X,\tau')$ be the Tychonoff reflection of $\underline{X}$ (cp. (1.1)), and let (FX,σ) be the free Tychonoff algebra over (X,τ') (cp. (2.2)). Then the insertion of generators map is continuous.
Finally (i) implies (iv) as follows: By (2.2) the free functionally Hausdorff algebra over $\underline{X}$ is of the form (FX,σ') and hence coincides with the free topological algebra (FX,σ'') algebraically (cp. (3.1)); by the universal property of (FX,σ'') the identity $(FX,\sigma'') \to (FX,\sigma')$ will be continuous, hence σ'' is a functionally Hausdorff topology as a refinement of σ'.

(2.5) **REMARK.** In view of the proof of (2.4) it might be appropriate to observe that given a functionally Hausdorff space (X,τ) with Tychonoff reflection (X,τ'), the free Hausdorff topological algebras $G_2(X,\tau)$ and $G_2(X,\tau')$ do only agree algebraically - one has $G_2(X,\tau) = (FX,\sigma)$ and $G_2(X,\tau') = (FX,\sigma')$ - but that in general σ will be finer than σ'. If however - as in the case of topological groups - the underlying Hausdorff space functor factors over *Tych*, then one has

$$G_2(X,\tau) = G_2(X,\tau') = (FX,\sigma').$$

For the sake of completeness we add the following result, basically due to Burgin [2], which is obvious in view of (2.5).

(2.6) **PROPOSITION.** *The free Hausdorff topological group over an arbitrary topological space* (X,τ) *is algebraically a free group, namely the free group over* Y, *where* Y *is the underlying set of the Tychonoff (or* $Top_{2\frac{1}{2}}$*)-reflection of* (X,τ).

(2.7) **THEOREM.** *For a topological space* $\underline{X} = (X,\tau)$ *the following are equivalent:*

(i) $\underline{X}$ *is a Tychonoff space.*

(ii) $VG\underline{X}$ *is a Tychonoff space and the unit* γ_X *lifts to an embedding* π_X: $X \to VG\underline{X}$.

(iii) The free topological algebra, the free Hausdorff topological algebra, and the free Tychonoff topological algebra over $\underline{X}$ *coincide; it is algebraically the free algebra over* X, *and topologically it is a Tychonoff space which (via insertion of generators) contains* $\underline{X}$ *as a closed subspace.*

PROOF. Obviously (iii) implies (ii), and (ii) implies (i).
By (5.2) and (5.3.4) the free Tychonoff algebra over $\underline{X}$ is algebraically the free algebra over X and contains $\underline{X}$ as a subspace. Since the free Hausdorff topological algebra over $\underline{X}$ is topologically a Tychonoff space, as can be shown using the concept of the primitive topology (see [2, 11]), it coincides with the free Tychonoff algebra on $\underline{X}$. The fact that $\underline{X}$ is actually a closed subspace of its free algebra, is shown by a simple topological argument in ([18], Proof of 0.2).

(2.8) **REMARK.** The observation mentioned above, that the free Hausdorff algebra over a Tychonoff space is Tychonoff again, is based on the fact that under certain conditions the free topological algebra functor G preserves embeddings (see [2], Thm. 1]); G will not do so in general as is shown in ([18], Ex. 3.6). It would be interesting to know more precisely when G will have this property.

(2.9) **REMARK.** Observe that for a topological space $\underline{X}$ the unit γ_X might lift to an embedding without $\underline{X}$ being Tychonoff. If for example $\underline{X}$ is a completely regular space (without T_1) and (FX,σ) denotes the free topological algebra over $\underline{X}$, then γ_X lifts to an embedding as an immediate consequence of (3.5) with $C_0 = (FI,\tau_F)$ of (4.8).

If σ is a completely regular topology (as e.g. in topological groups since they are uniformizable), one gets therefore: $\underline{X}$ is completely regular iff γ_X lifts to an embedding (see also [18] with a much more involved proof). That this will not hold in general is shown by the example of (2.3).

3. CATEGORICAL TOOLS.

The main categorical tool to achieve the results mentioned above is the following generalization of (part of) Wyler's taut lift Theorem

[22], a sketched proof of which we enclose in order to make the following corollaries comprehensible.

(3.1) THEOREM (cp. [3, 21]). *Let there be given a commutative square of functors:*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{V} & \mathcal{X} \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle T} \\ \mathcal{A} & \xrightarrow[U]{} & \mathcal{Y} \end{array}$$

such that S *and* T *are mono-topological,* U *has an adjoint* F *(with unit* γ*), and* V *preserves initiality of monosources. Then there exists an adjoint* G *of* V *(with unit* π*) and a natural transformation* χ: FT $\to$ SG *such that for each* X $\in$ ob $\mathcal{X}$ *the morphism* χ_X *is a regular epimorphism and the following diagram commutes:*

$$\begin{array}{ccc} TX & \xrightarrow{\gamma_{TX}} & UFTX \\ & {\scriptstyle T\pi_X}\searrow & \downarrow{\scriptstyle U\chi_X} \\ & & USGX = TVGX. \end{array}$$

If S *and* T *are even topological functors,* χ *will be a natural equivalence.*

PROOF. For an $\mathcal{X}$-object X consider the $\mathcal{X}$-source

$$(X{\downarrow}V) = (X,\mathcal{X}(X,VC)) = (X,\cup\{\mathcal{X}(X,VC) \mid C \in \text{ob } \mathcal{C}\}).$$

Applying T one gets the $\mathcal{Y}$-source

$$T(X{\downarrow}V) = (TX,\ Tf\colon TX \to TVC = USC)_{f\in\mathcal{X}(X,VC)}\ .$$

By adjunction there corresponds the $\mathcal{A}$-source

$$T(X{\downarrow}V)^* = (FTX,\ f^*\colon FTX \to SC)_{f\in\mathcal{X}(X,VC)}$$

with f^* being the unique $\mathcal{A}$-morphism making the diagram

$$\begin{array}{ccc} TX & \xrightarrow{\gamma_{TX}} & UFTX \\ & {\scriptstyle Tf}\searrow & \downarrow{\scriptstyle Uf^*} \\ & & USC \end{array}$$

commutative. Let

$$FTX \xrightarrow{\chi_X} A_X \xrightarrow{m_f} SC = f^*$$

be the regular factorization of $T(X{\downarrow}V)^*$ (see (1.3)) and let

$$(GX,\ m_f\colon GX \to C)_{f \in \mathcal{X}(X,VC)}$$

be the S-initial lift of the monosource $(A_X, m_f)_f$. Then - using our assumption on V - there exists a unique $\mathcal{X}$-morphism $\pi_X\colon X \to VGX$ with $T\pi_X = U\chi_X \circ \gamma_{TX}$, and which in addition is V-universal. The final assertion is a consequence of the observation that, due to the existence of indiscrete structures, the morphism γ_{TX} belongs to the source $T(X{\downarrow}V)$ and hence 1_{FTX} occurs in the source $T(X{\downarrow}V)^*$ which therefore will be a monosource.

(3.2) **APPLICATION.** The typical application of the above theorem is illustrated by the following diagram (cp. (1.4)):

$$\begin{array}{ccc} \mathcal{A}(\mathcal{X}) & \xrightarrow{\ V\ } & \mathcal{X} \\ {\scriptstyle S}\big\downarrow & & \big\downarrow{\scriptstyle T} \\ \mathcal{A} & \xrightarrow[\ U\ ]{} & \mathit{Set} \end{array}$$

where U denotes the underlying functor of a quasivariety and where $\mathcal{X}$ is an epireflective subcategory of *Top* or *Unif* (e.g., Top_2 or Unif_0).

Hence for example the free Hausdorff topological algebra $G_2\underline{X}$ over a Hausdorff space $\underline{X}$ will always exist, but it will algebraically be only a quotient of the free (abstract) algebra FX (= $FT\underline{X}$), i.e., Markov's axiom (A) will not be fulfilled in general; if however the free topological algebra $G\underline{X}$ is considered (i.e., if no separation axioms are involved), (A) will be satisfied automatically.

(3.3) **COROLLARY.** *In the situation of* (3.1) *the following are equivalent for an object* X ∈ ob $\mathcal{X}$ *:*

(i) χ_X *is an isomorphism.*

(ii) The source $T(X{\downarrow}V)^*$ *is a monosource.*

(iii) γ_{TX} *lifts to an* $\mathcal{X}$*-morphism.*

While this immediate consequence of (3.1) is crucial with respect to Markov's axiom (A) the following simple consequences will serve to discuss (T).

(3.4) **COROLLARY.** *In the situation of* (3.1) *the following are equivalent for an object* $X \in \text{ob } \mathcal{X}$ *:*

(i) π_X *is an (extremal) monomorphism.*

(ii) The source $(X \downarrow V)$ *is an (extremal) monosource.*

(iii) There exists some $C_0 \in \text{ob } \mathcal{C}$ *such that the source* $(X, \mathcal{X}(X, VC_0))$ *is an (extremal) monosource.*

(3.5) **COROLLARY.** *In the situation of* (3.1) *the following are equivalent for an object* $X \in \text{ob } \mathcal{X}$ *:*

(i) π_X *is a T-initial morphism.*

(ii) The source $(X \downarrow V)$ *is a T-initial source.*

(iii) There exists some $C_0 \in \text{ob } \mathcal{C}$ *such that the source* $(X, \mathcal{X}(X, VC_0))$ *is T-initial.*

4. THE FREE UNIFORM ALGEBRA OVER $\mathbb{I}$.

In order to show that Markov's axiom (A) is satisfied for every T_{2H}-space $\mathbb{X}$, Swierczkowski in [19] constructs explicitly a Tychonoff algebra topology on FX;

In this section we will give a description of the uniformity which is behind his construction, restricting ourselves however to the special case of the unit interval $\mathbb{I}$ (or any metric space), to which the general case can be reduced by a simple categorical argument (see Section 5). This uniformity can in fact be obtained by a single pseudometric in the sense of Bourbaki as is shown by Taylor [20], but for the following reasons we don't refer to his work: firstly our result is slightly stronger (uniform continuity of the operations instead of just continuity), secondly we feel our approach is quite natural, and finally we would like to have this explicit description at hand for further investigations.

Notational convention:

$$x_i^{(r)} = (x_{i1}, \dots, x_{ir}) \in I^r \quad \text{for} \quad i, r \in \mathbb{N} \quad \text{and} \quad P := P(FI).$$

(4.1) Construction of a uniformity on FI.

Given $\epsilon \in \mathbb{R}_+$ we denote by $\overline{D}_\epsilon$ the following subset of FI×FI:

$$\overline{D}_\epsilon := \{ (f^{(r)} x_1^{(r)}, f^{(r)} x_2^{(r)}) \mid f^{(r)} \in P,\ x_1^{(r)}, x_2^{(r)} \in I^r,\ \Sigma_{j=1}^r |x_{1j} - x_{2j}| < \epsilon \}.$$

Since any set

$$\{(x,y)\in(I^r)^2 \mid (f^{(r)}x,f^{(r)}y)\in \overline{D}_\epsilon\}$$

will belong to the natural uniformity of I^r and since we are looking for a uniformity being as fine as possible and making all the $f^{(r)}$ in P uniformly continuous, we have to look for a uniformity containing all the $\overline{D}_\epsilon$'s. Unfortunately the $\overline{D}_\epsilon$'s behave badly with respect to composition of relations; hence to obtain a (base of a) uniformity containing all these sets we define

$$D_\epsilon = \cup\{\overline{D}_{\epsilon_1}\circ\ldots\circ\overline{D}_{\epsilon_n} \mid (\epsilon_1,\ldots,\epsilon_n)\in R_+^n,\ \Sigma\epsilon_i \leq \epsilon,\ n\in \mathbb{N}\}.$$

From the obvious facts that the diagonal Δ is contained in every $\overline{D}_\epsilon$ and that the sets $\overline{D}_\epsilon$ are symmetric one concludes easily:

(1) $\Delta \subset D_\epsilon$ for each $\epsilon > 0$.
(2) $D_\epsilon = D_\epsilon^{-1}$ for each $\epsilon > 0$.

Moreover from the very definition of the D_ϵ's we conclude

(3) $D_{\epsilon/2}\circ D_{\epsilon/2} \subset D_\epsilon$ for each $\epsilon > 0$.
(4) $D_\epsilon \subset D_{\epsilon_1}\cap D_{\epsilon_2}$ with $\epsilon = \min(\epsilon_1,\epsilon_2)$ for each pair $(\epsilon_1,\epsilon_2)\in R_+^2$.

Hence the family of all D_ϵ is a base for some uniformity on FI, which will be denoted by U_1.

4.2. PROPOSITION. *All operations from* P(FI) *are uniformly continuous with respect to* U_1.

PROOF. Given $f = f^{(r)} \in$ P we only have to prove that

$$f_2^{-1}[D_\epsilon] = \{(a,b) \mid (fa,fb)\in D_\epsilon\} \subset (FI)^r\times(FI)^r$$

belongs to the product uniformity of $(FI,U_1)^r$. Choose an element

$$\{(c_1,\ldots,c_{2r}) \mid (c_i,c_{r+i})\in \overline{D}_{\epsilon_i}\} = U_\epsilon^r$$

out of this product uniformity such that $\Sigma_{i=1}^r \epsilon_i < \epsilon$. It is enough to prove $U_\epsilon^r \subset f_2^{-1}[D_\epsilon]$. In fact, given some $(c_1,\ldots,c_{2r})\in U_\epsilon^r$, for each $i \in \{1,\ldots,r\}$ we have some

$$g_i = g_i^{(n_i)} \in P,\quad x_i^{(n_i)} = x_i \quad\text{and}\quad y_i^{(n_i)} = y_i \in I^{n_i}$$

such than

$$c_i = g_i(x_i), \quad c_{r+i} = g_i(y_i), \quad \Sigma_{j=1}^{n_i} |x_{ij} - y_{ij}| < \epsilon_i .$$

With $n = \Sigma_{i=1}^{r} n_i$ the n-ary operation h with

$$h(z^{(n1)}, z^{(n2)}, \ldots, z^{(nr)}) = f(g_1(z^{(n1)}), g_2(z^{(n2)}), \ldots, g_r(z^{(nr)}))$$

belongs to P and meets the following conditions with

$$c = (c_1, \ldots, c_{2r}), \quad d = (c_{r+1}, \ldots, c_{2r}) \in (\mathrm{FI})^r :$$

$$h(x_1, \ldots, x_r) = f(c) \quad \text{and} \quad h(y_1, \ldots, y_r) = f(d)$$

where

$$\Sigma_{i,j} |x_{ij} - y_{ij}| < \Sigma_i \epsilon_i \leq \epsilon.$$

Hence we have

$$(fc, fd) \in \overline{D}_\epsilon \subset D_\epsilon$$

and therefore

$$(c_1, \ldots, c_{2r}) = (c,d) \in f_2^{-1}[D_\epsilon].$$

(4.3) ϵ-links. In order to prove some inportant additional properties of the uniform algebra (FI, U_1) just constructed we need some auxiliary notions and results which are due to [19]. Given $a,b \in \mathrm{FI}$ and $\epsilon \in \mathbb{R}_+$, a system

$$\Sigma := \Sigma(a,b) := (f_1^{(n1)}, \ldots, f_m^{(nm)};\ x_1^{(n1)}, \ldots, x_m^{(nm)};\ y_1^{(n1)}, \ldots\ y_m^{(nm)})$$

with $f_i^{(ni)} \in \mathrm{P}$ of arity n_i will be called an *ϵ-link of a and b (of length m)* provided

$$a = f_1^{(n1)}(y_1), \quad b = f_m^{(nm)}(x_m),$$

$$f_i^{(ni)}(y_i) = f_{i+1}^{(ni+1)}(x_{i+1}) \quad \text{for all} \quad i \in \{1, \ldots, m-1\},$$

$$\Sigma_{j=1}^{n_i} |x_{ij} - y_{ij}| < \epsilon_i \quad \text{for some } m\text{-tuple } (\epsilon_1, \ldots, \epsilon_m) \in \mathbb{R}_+^m \quad \text{with } \Sigma_{i=1}^{m} \epsilon_i \leq \epsilon.$$

This notion arises naturally from the definition of D_ϵ, since $(a,b) \in D_\epsilon$ iff there exists an ϵ-link of a and b. $\Sigma(a,b)$ will be shortly called a *link of a and b* if it is an ϵ-link of a and b for some ϵ.

With any link $\Sigma = \Sigma(a,b)$ there is associated a relation R_Σ on I by

$$R_\Sigma := \cup_{i=1}^{m} (M_i \cup M_i^{-1}) \quad \text{with} \quad M_i = \{(\pi_k x_i, \pi_k y_i) \mid k = 1, \ldots, n_i\}$$

where $\pi_k\colon \mathrm{I}^{n_i} \to \mathrm{I}$ is the k-th projection (i.e., elements $u,v \in \mathrm{I}$ are related by R_Σ iff u and v are the k-th coordinate of x_i and y_i respectively for some k and some i). The equivalence relation on I generated by R_Σ will be denoted by ρ_Σ. In this context the following two lemmata were proved by Swierczkowski.

(4.3.1) LEMMA ([19], L. 9). Let Σ be a link of a and b of length m, and $t: I \to I$ be a map with

(i) $x \in S_a \Rightarrow tx = x$.

(ii) $(x,y) \in R_\Sigma \Rightarrow tx = ty$.

Then

$$a = f_m^{(n_m)}(tx_{m1},\ldots,tx_{mn_m}).$$

(4.3.2) LEMMA ([19], L.10). Let Σ be an ϵ-link of a and b. Then the following implication for $x,y \in I$ holds:

$$(x,y) \in \rho_\Sigma \Rightarrow |x-y| < \epsilon.$$

(4.4) PROPOSITION. *The insertion of generators map* γ: $\underline{I} \to (FI, \mathcal{U}_I)$ *is a uniform embedding.*

PROOF. Considering γ as an injection it is enough to prove that the natural uniformity of $\underline{I}$ is the relative uniformity of $(FI,\mathcal{U}_I)$, i.e., that all the sets $D_\epsilon \cap (I\times I)$ form a base of the uniformity of $\underline{I}$. This will follow from the equality

$$D_\epsilon \cap (I\times I) = \{(a,b) \in I^2 \mid |a-b| < \epsilon\}$$

where the inclusion "$\supset$" is trivial since $id_{FI} \in P$. Assume finally that $(a,b) \in D_\epsilon \cap (I\times I)$. According to (4.3.2) we only have to prove $(a,b) \in \rho_\Sigma$ for the ϵ-link $\Sigma(a,b)$ which exists because $(a,b) \in D_\epsilon$.
Assuming the contrary we could find a map $t: I \to I$ with

$$(x,y) \in \rho_\Sigma \Rightarrow tx = ty \text{ and } x \in S_a \cup \{b\} \Rightarrow tx = x$$

(recall that ρ_Σ is an equivalence relation and that $S_a \not\subset \{a\}$). By (4.3.1) we conclude

$$a = f_m^{(n_m)}(tx_{m1},\ldots,tx_{mn_m})$$

while by the definition of $\Sigma(a,b)$ we have

$$b = f_m^{(n_m)}(x_{m1},\ldots,x_{mn_m})$$

and therefore

$$tb = tf_m^{(n_m)}(x_{m1},\ldots,x_{mn_m}) = f_m^{(n_m)}(tx_{m1},\ldots,tx_{mn_m}) ;$$

since FI is free over I and hence t extends to a homomorphism. We end up with the contradiction $b = tb = a$.

(4.5) **PROPOSITION.** $\bigcap_{\epsilon>0} D_\epsilon = \Delta$, *hence* U_I *is a separated uniformity.*

PROOF. We only have to prove that for each pair of distinct elements $a,b \in$ FI there exists some $\epsilon > 0$ such that $(a,b) \notin D_\epsilon$. To do so choose ϵ such that

$$0 < \epsilon < \min \{|x-y| \mid x,y \in S_a \cup S_b,\ x \neq y\}.$$

Then the assumption $(a,b) \in D_\epsilon$ shows for the corresponding ϵ-link Σ of a and b that no two distinct elements of $S_a \cup S_b$ are ρ_Σ-equivalent by (4.3.2). Hence there exists some $t: I \to I$ with

$$(x,y) \in \rho_\Sigma r \ \Rightarrow\ tx = ty \quad \text{and} \quad x \in S_a \cup S_b \ \Rightarrow\ tx = x.$$

From this we get by (4.3.1)

$$a = f_m^{(nm)}(tx_{m1},\ldots,tx_{mnm})$$

while the equality $b = f_m^{(nm)}(x_{m1},\ldots,x_{mnm})$ yields

$$b = f_m^{(nm)}(tx_{m1},\ldots,tx_{mnm})$$

by the final remark of (1.2). We obtain the contradiction $a = b$.

We summarize the results of this section as follows, where I̲ either denotes the unit interval with its natural uniformity or with its natural topology:

(4.6) **THEOREM.** *Let* FI *denote the free algebra over the unit interval* I *in any nontrivial finitary quasivariety. Then there exists a separated uniformity* U_I *on* FI *such that* (FI,U_I) *becomes a uniform algebra (i.e., all operations of* FI *are uniformly continuous with respect to* U_I) *and contains* I̲ *(via insertion of generators) as a uniform subspace.*

(4.7) **REMARK.** By inspecting the proof of (4.6) we see that we have only used the metric properties of I̲. Hence (4.6) holds for an arbitrary metric space instead of I̲.

(4.8) **PROPOSTION.**

(i) The free uniform algebra and the free separated uniform algebra over $\mathbb{I}$ *coincide. They are of the form* (FI, U_f) *- where* U_f *is a uniformity on* FI *not coarser than* U_I *- and contain (via insertion of generators)* $\mathbb{I}$ *as a uniform subspace.*

(ii) If $\tau(U_f)$ *denotes the uniform topology of* U_f, then $(FI, \tau(U_f))$ *is the free algebra over* $\mathbb{I}$ *in the full subcategory of all Tychonoff algebras (resp. completely regular algebras) consisting of those algebras whose operations are uniformly continuous in some uniformization.* $(FI, \tau(U_f))$ *contains* $\mathbb{I}$ *as a (closed) subspace.*

(iii) The free Tychonoff algebra over $\mathbb{I}$ *is of the form* (FI, τ_f); *its topology* τ_f *is not coarser than* $\tau(U_f)$. (FI, τ_f) *contains* $\mathbb{I}$ *as a (closed) subspace.*

(iv) The free topological algebra, the free Hausdorff algebra and the free functionally Hausdorff algebra over $\mathbb{I}$ *coincide. They are of the form* (FI, τ) *where* τ *is a topology not coarser than* τ_f, *and contain* $\mathbb{I}$ *as a closed subspace.*

(v) If $\mathbb{M} = (M, d)$ *is a metric space, then the free (separated) uniform algebra over* $\mathbb{M}$ *is of the form* (FM, U), *where* U *is not coarser than the uniformity* U_M *constructed analogously as* U_I.

PROOF. (i) Immediate from (4.6), the corollaries of (3.1) and the fact that $Unif_o$ is closed with respect to refinement of uniform structures.
(ii) follows from (i) by turning over to the uniform topology.
(iii) The fact that the free Tychonoff algebra over $\mathbb{I}$ is algebraically FI follows by means of (3.3), since id_{FI} belongs to the source $T(\mathbb{I}{\downarrow}V)^*$ because of (ii). The rest is obvious.
(iv) The final statement follows in the same way as (iii). The algebras in question coincide, since the topology of the free topological algebra refines the topology of the free functionally Hausdorff algebra.
(v) follows in the same way as (i).

(4.9) **PROBLEM.** Is U_I even the "free uniformity" U_f ?

5. FREE SEPARATED ALGEBRAS OVER CERTAIN CLASSES OF SPACES.

(5.1) **BASIC SITUATION.** We consider the situation of (3.2) and assume in addition that there is given a factorization structure *(E,M)* on *X* (in the sense of [7]) and a subcategory *Y* of *X*. By $Y^{\sim}$ we denote the

$\mathcal{E}$-reflective hull of $\mathcal{Y}$. Recall that $X \in \text{ob } \mathcal{Y}^{\frown}$ iff there exists a source in $\mathcal{M}$

$$(X, m_i: X \to Y_i)_{i \in I} \quad \text{with} \quad Y_i \in \text{ob } \mathcal{Y} \quad \text{for all } i \in I;$$

if $\mathcal{X}$ has products and is $\mathcal{E}$-cowellpowered, $X \in \text{ob } \mathcal{Y}^{\frown}$ iff X is an $\mathcal{M}$-subobject of an $\mathcal{X}$-product of $\mathcal{Y}$-objects (see [7, 8]).

These data are subject to the following conditions:

(I) χ_Y is an isomorphism for each $Y \in \text{ob } \mathcal{Y}$.

(II) $\pi_Y \in M$ for each $Y \in \text{ob } \mathcal{Y}$.

(III) Given $X \in \text{ob } \mathcal{Y}^{\frown}$ and finitely many elements in TX, then there exists some $\mathcal{X}$-morphism $m: X \to Y$ with $Y \in \text{ob } \mathcal{Y}$ such that Tm distinguishes all these elements.

Observe that (III) is a strengthening of the condition that $\mathcal{M}$ consists of monosources only, and is satisfied automatically if $\mathcal{M}$ consists of monosources only and $\mathcal{Y}$ is closed with respect to finite products: for given distinct $x_1,\dots,x_n \in TX$, there exist $\mathcal{X}$-morphisms

$$m_{ij}: X \to Y_{ij} \quad \text{with} \quad Y_{ij} \in \text{ob } \mathcal{Y} \quad \text{and} \quad Tm_{ij}(x_i) \neq Tm_{ij}(x_j) \quad \text{for each pair } i,j \text{ with } 1 \leqslant i \leqslant j \leqslant n$$

by definition of $\mathcal{Y}^{\frown}$ and our assumption on $\mathcal{M}$. The morphism $m: X \to \prod Y_{ij}$ induced by the m_{ij} then has the desired property.

(5.2) THEOREM. *Under the hypotheses of* (5.1) *the following hold:*

(i) χ_X *is an isomorphism for each* $X \in \text{ob } \mathcal{Y}^{\frown}$.

(ii) $\pi_X \in \mathcal{M}$ *for each* $X \in \text{ob } \mathcal{Y}^{\frown}$.

PROOF. (i) According to (3.3) we only have to prove that the source $T(X{\downarrow}V)^*$ is a monosource for $X \in \text{ob } \mathcal{Y}^{\frown}$. Since U is faithful it suffices to show that

$$U(T(X{\downarrow}V)^*) = (UFTX, Uf^*)_{f \in X(X,VC)}$$

is a monosource. Hence consider two different elements $a,b \in UFTX$. Since U is finitary there exists a finite set $Z \subset TX$ such that $a,b \in UFZ$. Denote by $i: Z \to TX$ the inclusion and choose $m: X \to Y$ according to (III) such that Tm distinguishes the elements of Z. The commutative diagram on the following page illustrates the situation, where the left vertical arrow $Tm \circ i$ is an injective map.

Since UF preserves injectivity of maps (see [12], p. 66), we conclude

$$\mathrm{UFT}m(a) \neq \mathrm{UFT}m(b)\ ;$$

since χ_Y is an isomorphism by assumption and

$$\chi_Y \circ \mathrm{FT}m = (\pi_Y \circ m)^{\#}$$

$$\begin{array}{ccc}
Z & \xrightarrow{\ \gamma_Z\ } & \mathrm{UFZ} \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \mathrm{UF}i} \\
\mathrm{TX} & \xrightarrow[\ \gamma_{TX}\ ]{} & \mathrm{UFTX} \\
{\scriptstyle \mathrm{T}m}\downarrow & & \downarrow{\scriptstyle \mathrm{UFT}m} \\
\mathrm{TY} & \xrightarrow[\ \gamma_{TY}\ ]{} & \mathrm{UFTY} \\
 & {\scriptstyle \mathrm{T}\pi_Y}\searrow & \wr\downarrow{\scriptstyle \mathrm{U}\chi_Y} \\
 & & \mathrm{TVGY}
\end{array}$$

we conclude that $\mathrm{U}(\mathrm{T}(X{\downarrow}V)^{\#})$ is a monosource.
(ii) Given $X \in \mathrm{ob}\ \mathcal{Y}^{\wedge}$ there exists a source $(X, m_i\colon X \to Y_i)_{i\in I} \in \mathcal{M}$. The commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{\ \pi_X\ } & \mathrm{VGX} \\
{\scriptstyle m_i}\downarrow & & \downarrow{\scriptstyle \mathrm{VG}m_i} \\
Y_i & \xrightarrow{\ \pi_{Y_i}\ } & \mathrm{VGY}_i
\end{array}$$

shows that $\pi_X \in \mathcal{M}$, since

$$(X,\ \pi_{Y_i} \circ m_i\colon X \to \mathrm{VGY}_i)_{i\in I} \in \mathcal{M}.$$

(5.3) EXAMPLES OF BASIC SITUATIONS.

(5.3.1) Consider T: *Top₂* → *Set* (i.e., $\mathcal{X}$ = *Top₂*) and let $\mathcal{Y}$ be the full subcategory of *Top₂* with $\underline{1}$ as a single object. Take the (quotient, monosource)-factorization structure (i.e., the regular factorization structure in the sense of [5]). Then $\mathcal{Y}^{\wedge}$ is the category of functionally Hausdorff spaces and condition (III) is satisfied (cp. 1.1), while (I) and (II) hold by (4.8).

(5.3.2) Replace in (5.3.1) the regular factorization structure by the (surjective, initial monosource)-factorization structure (i.e., the T-regular factorization structure in the sense of [15]). Then $Y^\frown$ = *Tych*, and (I), (II), (III) are satisfied by the same arguments as above.

(5.3.3) Replace in (5.3.1) the regular factorization structure by the (dense, closed embedding sources)-factorization structure (i.e., the (epi, extremal monosource)-factorization structure). Then $Y^\frown$ = $Comp_2$, and (I), (II), (III) hold again as above.

(5.3.4) Consider T: *Tych* → *Set* (i.e., *X* = *Tych*) and take *Y* and *(E,M)* as in (5.3.2); then $Y^\frown$ = *Tych* and (I), (II), (III) are satisfied.

(5.3.5) Consider T: Top_{2M} → *Set* (i.e., *X* = Top_{2M}) and take *Y* and *(E,M)* as in (5.3.1). Then $Y^\frown$ = Top_{2M} and (I), (II), (III) are satisfied.

(5.3.6) Consider T: $Unif_0$ → *Set* (i.e., *X* = $Unif_0$) and let *Y* be the full subcategory *Met*. Take the T-regular (i.e., the (surjective, initial monosource)-)factorization structure. Then $Y^\frown$ = $Unif_0$ by a famous result of Weil (see [10], 6.16), and conditions (I) and (II) are again satisfied by (4.8). (III) follows from the final observation of (5.1).

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