CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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Cahiers de topologie et géométrie différentielle catégoriques, tome 28, nº 3 (1987), p. 197-226

<http://www.numdam.org/item?id=CTGDC_1987__28_3_197_0>

CAHIERS DE TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE CATÉGORIQUES *Vol. XXVIII-3 (1987)*

THE SHIFT FUNCTOR AND THE COMPREHENSIVE FACTORIZATION FOR INTERNAL GROUPOIDS

by Dominique BOURN

RÉSUMÉ. On démontre que la catégorie Grd **E** des groupoïdes internes à une catégorie exacte à gauche **E** est triplable sur la catégorie Spl **E** dont les objets sont les épimorphismes scindés et les morphismes sont les transformations entre épimorphismes scindés. Trois applications sont données: un relèvement d'adjonction au niveau des groupoïdes, une caractérisation des catégories exactes au sens de Barr et la construction dans Grd **E**, lorsque **E** est exacte, de la décomposition d'un foncteur en composé de fibration discrète et de foncteur final.

Here is the first of two papers, continuation of [2] and introduction to some preliminary results necessary for a general cohomology theory for an exact category **E** (summarized in [3]) using internal n-groupoids as a non-abelian equivalent to chain complexes. Indeed when **E** is abelian, there is an equivalence between the category n-Grd **E** of internal n-groupoids and the category $C^n(\mathbf{E})$ of chain complexes of length n [4]. It turns out that, with this realization, the higher cohomology groups are classes of principal group actions exactly as it is the case at level 1.

This pair of papers could have been called as well: Internal n-groupoids vs simplicial objects. Indeed Duskin [7] and Glenn [10] have previously developed a realization of cohomology classes in an exact category **E** in terms of simplicial objects, more precisely special kind of complexes, called hypergroupoids. For such objects, there is an hypercomposition law, only possible on peculiar collections of n faces, submitted to hyperunitarity and hyperassociativity axioms.

So, on one hand we have the category Simpl **E** of simplicial objects in **E**, with good exactness properties (since **E** is exact), but with a working class of objects (the hypergroupoids) which is rather complicated. On the other hand, we have the category n-Grd **E** of internal n-groupoids with rather simple objects but not yet positively investigated right exactness properties. This is the aim of these two papers to begin the investigation of such properties.

Part 1 of this paper is devoted to a rather unexpected (for me at least) result. It is well known that the category Simpl **E** is monadic above the category SpSimpl **E** of split augmented simplicial objects [6]. When **E** is left exact, the category Grd **E**, as a subcategory of Simpl **E**, appears to be monadic above the category Spl **E** whose objects are the the split epimorphisms (with a given splitting) and morphisms the coherent squares between such split epimorphisms (or equivalently, the category Idem **E** whose objects are the idempotent morphisms in **E** and the morphisms the transformations of idempotents).

In Part 2, three applications, necessary for the construction of the cohomology groups, are given. The first one is an adjoint lifting theorem for groupoids. The second one is a characterization of exact categories: Let us denote by q (the quotient functor) the left adjoint to dis: **E** $\to$ Rel **E** whenever it exists. A category **E** is exact when the functor q exists and has the two following left exactness properties: (i) it is a fibration up to equivalence, (ii) it preserves the pullbacks in which one edge is an internal discrete fibration. Moreover the functor q can be extended to a functor π_0: Grd **E** $\to$ **E** which is again a fibration up to equivalence. The third one is the construction in Grd **E** of the associated discrete fibration of any internal functor. Precisely, any functor can be factorized, in a way unique up to isomorphism, into the composite of a discrete fibration and a final functor. It is what is called, according to Street and Walters [13], the comprehensive factorization of this internal functor.

PART 1. THE SHIFT FUNCTOR FOR GROUPOIDS

Let **E** be a left exact category.

1. Internal categories

Let us recall that an internal category in **E** is a diagram X_1 in **E**:

$$X_0 \;\begin{array}{c}\xleftarrow{\;\;d_0\;\;}\\ \xrightarrow{\;\;s_0\;\;}\\ \xleftarrow{\;\;d_1\;\;}\end{array}\; mX_1 \;\begin{array}{c}\xleftarrow{\;\;d_0\;\;}\\ \xleftarrow{\;\;d_1\;\;}\\ \xleftarrow{\;\;d_2\;\;}\end{array}\; m_2X_1$$

such that m_2X_1 is the pullback of d_0 along d_1. It must satisfy the usual axioms of unitarity and associativity, briefly those of a truncated simplicial object as far as level 3, when completed by the pullback m_3X_1 of d_0 along d_2. The internal functors are just the natural transformations between such diagrams. We shall denote by Cat $\mathbf{E}$ the category of internal categories in $\mathbf{E}$. It is left exact. There is an obvious functor $(\ )_0$: Cat $\mathbf{E} \to \mathbf{E}$ associating X_0 to X_1. It has a fully faithful right adjoint Gr and a fully faithful left adjoint dis [2]. We shall denote by $\pi_0(X_1)$ the coequalizer of d_0 and d_1: $mX_1 \rightrightarrows X_0$ whenever it exists. It is a potential left adjoint to dis.

A definition as short as this one cannot avoid the notion of simplicial objects. So let us denote by Simpl $\mathbf{E}$ the category of simplicial objects in $\mathbf{E}$ (see for instance [6]) and, as usual, by Ner the left exact embedding:

$$\text{Ner: Cat } \mathbf{E} \longrightarrow \text{Simpl } \mathbf{E}$$

which associates to X_1 the following simplicial object

$$X_0 \underset{d_1}{\overset{d_0}{\leftleftarrows}} mX_1 \underset{d_2}{\overset{d_0}{\leftleftarrows\!\!\leftarrow}} m_2X_1 \ \cdots \quad m_{n-1}X_1 \ \underset{d_n}{\overset{d_0}{\overleftarrow{\ \vdots\ }}} \ m_nX_1 \ldots$$

obtained by adding to the diagram X_1 completed by m_3X_1 its iterated simplicial kernels m_nX_1, $n \geqslant 4$, this object of $\mathbf{E}$ representing the internal object of "composable sequences of n arrows".

According to Illusie [11], let us denote by Dec X_1 the following internal category:

$$mX_1 \underset{d_1}{\overset{d_0}{\leftleftarrows}} m_2X_1 \underset{d_2}{\overset{d_0,\ d_1}{\leftleftarrows\!\!\leftarrow}} m_3X_1$$

It is then clear that the (split) coequalizer of d_0 and d_1 is d_0: $mX_1 \to X_0$ and that, consequently, $\pi_0(\text{Dec } X_1) = X_0$.

Actually this functor Dec is nothing but the value on Cat $\mathbf{E}$ of the shift functor Δec defined in Simpl $\mathbf{E}$ [8,11] which shifts X_0 and the higher degeneracy (d_i) and face (s_i) operators. In fact, the two functors Ner. Dec and Δec. Ner commute up to a natural isomorphism. The functor Δec being left exact, the same is true for Dec.

The functor Δec on Simpl $\mathbf{E}$ is endowed with a cotriple structure generated by the following adjunction (see [6] for instance):

$$\text{Simpl } \mathbf{E} \ \underset{+}{\overset{\delta\text{ec}}{\rightleftarrows}} \ \text{SpSimpl } \mathbf{E}$$

where SpSimpl $\mathbb{E}$ is the category of split augmented simplicial objects with a given splitting. The functor $\mathcal{D}ec$ is defined, for any simplicial object S, as the split augmented complex obtained by shifting the higher face operators, and the functor + by shifting the augmentation and the splitting of the simplicial object. Furthermore the functor $\mathcal{D}ec$ is precisely monadic [6], that is to say that Simpl $\mathbb{E}$ is the category of algebras of the triple θ generated on SpSimpl $\mathbb{E}$ by this adjunction.

The functor Ner being an embedding, there is again a cotriple on Cat $\mathbb{E}$ we shall denote in the following way:

$$X_1 \xleftarrow{\epsilon X_1} \mathrm{Dec}X_1 \;\begin{array}{c} \xleftarrow{\epsilon \mathrm{Dec}X_1} \\ \xrightarrow{\mu X_1} \\ \xleftarrow[\mathrm{Dec}\epsilon X_1]{} \end{array}\; \mathrm{Dec}^2X_1 \;\begin{array}{c} \longleftarrow \\ \longleftarrow \\ \longleftarrow \end{array}\; \mathrm{Dec}^3X_1 \;\ldots$$

The image of ϵX_1 by Ner being a coequalizer in Simpl $\mathbb{E}$ is again a coequalizer in Cat $\mathbb{E}$. We shall call this diagram the canonical presentation of X_1. Actually ϵX_1 is componentwise a split coequalizer as it is the case in Simpl $\mathbb{E}$ and thus any pullback of this diagram along a morphism of Cat $\mathbb{E}$ is again a coequalizer. On the other hand, the right hand part of the canonical presentation is an internal category in Cat $\mathbb{E}$ we shall denote by DECX_1.

2. Internal groupoids.

Now, following a remark of Illusie [11], an internal category X_1 will be said to be an internal groupoid when, moreover, the following square (α_1) is a pullback in $\mathbb{E}$:

$$(\alpha_1) \qquad \begin{array}{ccc} mX_1 & \xleftarrow{\;d_2\;} & m_2X_1 \\ {\scriptstyle d_1}\downarrow & & \downarrow{\scriptstyle d_1} \\ X_0 & \xleftarrow[\;d_1\;]{} & mX_1 \end{array}$$

Let us denote by Grd $\mathbb{E}$ the full subcategory of Cat $\mathbb{E}$ whose objects are the internal groupoids. A groupoid will be said to be an *equivalence relation* when $[d_0,d_1]$: $mX_1 \to X_0{\times}X_0$ is a monomorphism in $\mathbb{E}$, or equivalently when the unique internal functor $X_1 \to \mathrm{Gr}X_0$ is a monomorphism in Grd $\mathbb{E}$. At last, for each object X in $\mathbb{E}$, let us recall that disX is the equivalence relation associated to id: X → X and GrX the equivalence relation associated to the final map: X → 1.

LEMMA 1. *An internal category* X_1 *is a groupoid iff the following square* (β) *is a pullback in* Cat $\mathbf{E}$:

$$\begin{array}{ccc} \mathrm{Dec}X_1 & \xleftarrow{\ \mathrm{Dec}\epsilon X_1\ } & \mathrm{Dec}^2X_1 \\ {\scriptstyle \epsilon X_1}\downarrow & & \downarrow{\scriptstyle \epsilon \mathrm{Dec}X_1} \\ X_1 & \xleftarrow[\ \epsilon X_1\ ]{} & \mathrm{Dec}X_1 \end{array}$$

PROOF. The square (α_1) is the image of the square (β) by the functor $(\ )_0$. Then if (β) is a pullback, so is (α_1). Conversely (β) is a pullback iff its image by $(\ )_0$ (the objects level) and by m (the morphisms level) are pullbacks. Let us consider now the two following diagrams

$$\begin{array}{ccc} m_2X_1 & \xleftarrow{\ d_3\ } & m_3X_1 \\ {\scriptstyle d_2}\downarrow & (\alpha_1') & \downarrow{\scriptstyle d_2} \\ mX_1 & \xleftarrow[\ d_2\ ]{} & m_2X_1 \\ {\scriptstyle d_0}\downarrow & (1) & \downarrow{\scriptstyle d_0} \\ X_1 & \xleftarrow[\ d_1\ ]{} & mX_1 \end{array} \qquad \begin{array}{ccc} m_2X_1 & \xleftarrow{\ d_3\ } & m_3X_1 \\ {\scriptstyle d_0}\downarrow & (2) & \downarrow{\scriptstyle d_0} \\ mX_1 & \xleftarrow[\ d_2\ ]{} & m_2X_1 \\ {\scriptstyle d_1}\downarrow & (\alpha_1) & \downarrow{\scriptstyle d_1} \\ X_1 & \xleftarrow[\ d_1\ ]{} & mX_1 \end{array}$$

The vertical composites of the two diagrams are equal. The squares (1) and (2) are pullbacks since X_1 is an internal category. Now the square (α_1) is a pullback since X_1 is a groupoid and thus the square (α_1') is a pullback. But $(\beta)_0 = (\alpha_1)$ and $m(\beta) = (\alpha_1')$, and thus (β) is a pullback.

COROLLARY. *When* X_1 *is a groupoid, the following category* DEC X_1 *is an equivalence relation in* Cat $\mathbf{E}$.

$$\mathrm{Dec}X_1 \ \underset{\mathrm{Dec}\epsilon X_1}{\overset{\epsilon \mathrm{Dec}X_1}{\leftleftarrows}} \ \mathrm{Dec}^2X_1 \ \begin{array}{c} \xleftarrow{\ \mathrm{Dec}^2X_1\ } \\ \xleftarrow{\ \mathrm{Dec}\epsilon\ \mathrm{Dec}X_1\ } \\ \xleftarrow{\ \mathrm{Dec}^2\epsilon X_1\ } \end{array} \ \mathrm{Dec}^3X_1$$

PROOF. The functor Dec being left exact, the square Dec(β) is a pullback, that means exactly DEC X_1 is a groupoid. Moreover the square (β) being a pullback, DEC X_1 is the equivalence relation associated to ϵX_1.•

The invertibility. In the category Set of sets, a groupoid is usually defined as a category in which every morphism is invertible. It is clear that it implies that the square (α) is a pullback and the converse is not difficult to check. Now, by the Yoneda embedding, the two definitions coincide again in any left exact category. The first one is more economical in an internal context. Nevertheless let us sketch, here, how this property of invertibility emerges in the internal case.

DEC X_1 being the equivalence relation associated to ϵX_1, there is a twisting isomorphism σX_1: $Dec^2 X_1 \to Dec^2 X_1$, whence an isomorphism

$$\gamma = \pi_0(\sigma X_1): \mathit{m}X_1 \to \mathit{m}X_1$$

which represents the passage to the inverse.

LEMMA 2. *An internal category* X_1 *is a groupoid iff the following square* (α_0) *is a pullback:*

$$\begin{array}{ccc} \mathit{m}X_1 & \xleftarrow{d_1} & \mathit{m}_2X_1 \\ {\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle d_0} \\ X_0 & \xleftarrow[d_0]{} & \mathit{m}X_1 \end{array}$$

PROOF. The two following squares are globally equal:

$$\begin{array}{ccc} \mathit{m}X_1 & \xleftarrow{d_1} & \mathit{m}_2X_1 \\ {\scriptstyle d_0}\downarrow & (\alpha_0) & \downarrow{\scriptstyle d_0} \\ X_1 & \xleftarrow[d_0]{} & \mathit{m}X_1 \end{array} \qquad \begin{array}{ccccc} \mathit{m}X_1 & \xleftarrow{d_2} & \mathit{m}_2X_1 & \xleftarrow{(\sigma X_1)_0} & \mathit{m}_2X_1 \\ {\scriptstyle d_0}\downarrow & (1) & \downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle d_0} \\ X_1 & \xleftarrow[d_1]{} & \mathit{m}X_1 & \xleftarrow[\gamma]{} & \mathit{m}X_1 \end{array}$$

Now (1) being a pullback, the same is true for (α_0). Conversely, if (α_0) is a pullback, the dual X_1^{op} of X_1 is a groupoid and (α_1) is a pullback. •

COROLLARY. *If* X_1 *is a groupoid,* $DecX_1$ *is an equivalence relation.*

PROOF. Since (α_0) is a pullback, $DecX_1$ is the equivalence relation associated to d_0. •

REMARK. Thus, by its canonical presentation, a groupoid is a quotient of an equivalence relation on equivalence relations.

3. The category Grd E is monadic above the category Spl E.

Following the previous corollary, when X_1 is a groupoid the whole structure of dec(NerX_1) is uniquely determined by the following split epimorphism:

$$X_0 \underset{s_0}{\overset{d_0}{\rightleftarrows}} mX_1$$

since the higher components of dec(NerX_1) are obtained by iterated pullbacks.

So let us denote by Spl E the category whose objects are the split epimorphisms with a given splitting and whose morphisms are the commutative squares between such data. Let us denote by

$$d\colon \text{Grd } \mathbf{E} \longrightarrow \text{Spl } \mathbf{E}$$

the functor associating to X_1 the previous split epimorphism.

We have also a functor

$$n\colon \text{Spl } \mathbf{E} \longrightarrow \text{SpSimpl } \mathbf{E},$$

associating to a split epimorphism the nerve of its associated equivalence relation, augmented by itself. It is clearly an embedding since it has a left adjoint left inverse forgetting the higher levels of a split augmented simplicial object.

Now let us consider the following commutative up to isomorphism square (*):

$$(*)\qquad \begin{array}{ccc} \text{Grd } \mathbf{E} & \xrightarrow{\text{Ner}} & \text{Simpl } \mathbf{E} \\ {\scriptstyle d}\downarrow & \nearrow & \downarrow{\scriptstyle \delta\text{ec}} \\ \text{Spl } \mathbf{E} & \xrightarrow[n]{} & \text{SpSimpl } \mathbf{E} \end{array}$$

There is also a functor

$$r\colon \text{Spl } \mathbf{E} \longrightarrow \text{Grd } \mathbf{E}$$

which associates to a split epimorphism its associated equivalence relation and which consequently is such that $+.n$ and Ner.r commute up to isomorphism.

THEOREM 1. *The functor r is a left adjoint to the functor d.*

PROOF. The pair (d,r) commutes with the pair $(\delta\mathrm{ec},+)$ by means of the functors Ner and n, up to isomorphisms. Now Ner and n being fully faithful, the natural transformations

$$1 \Longrightarrow \delta\mathrm{ec}.+ \quad \text{and} \quad +.\mathrm{dec} \Longrightarrow 1$$

of the adjunction $(\delta\mathrm{ec},+)$ determine natural transformations:

$$1 \Longrightarrow d.r \quad \text{and} \quad r.d \Longrightarrow 1$$

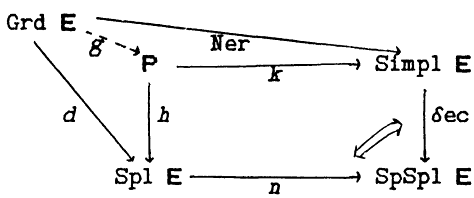
with the same equations. •

The aim, now, is to show that d is monadic. Unfortunately the Beck's criterion is not very easy to handle in this context. Another way will be used, perhaps a bit indirect, but much more enlightening, by Lemma 3, the combinatorial geometry underlying to this question.

Let us denote by T the triple generated on Spl **E** by (d,r). Now (d,r) and $(\delta\mathrm{ec},+)$ commuting up to isomorphism, there is a natural isomorphism ω: $n.\mathrm{T} \Longrightarrow \theta.n$.

THEOREM 2. *The square* (*) *is a 2-pullback and the functor d is monadic.*

PROOF. A 2-pullback (or an isocomma category) is a square like (*) with an inner isomorphism, satisfying the universal property for such squares. Let us consider the following diagram:



where **P** is the vertex of the 2-pullback. Now n being fully faithful, the same is true for k. Let g: Grd **E** $\to$ **P** be the unique factorization. There is a functor h': Spl **E** $\to$ **P**, defined for every object χ in Spl **E** by

$$h'(\chi) = (\mathrm{T}(\chi), \omega(\chi), +.n(\chi)),$$

since

$$\omega(\chi):\ n.\mathrm{T}(\chi) \longrightarrow \theta.n(\chi) = \mathrm{dec}(+.n(\chi))$$

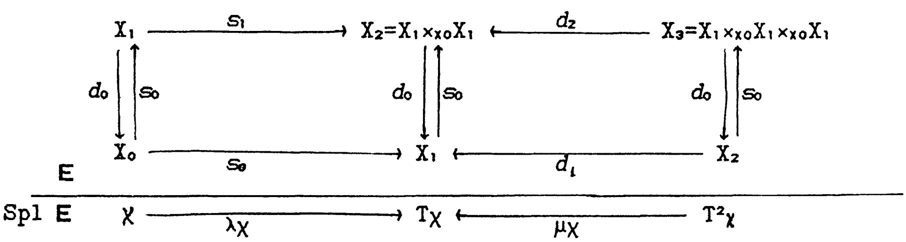
is an isomorphism. It is an adjoint to h for the same reasons as in Theorem 1, since n and k are fully faithful. The Beck precise tripleability condition being stable under 2-pullbacks (see [6]), the functor h is precisely monadic. Moreover it is clear that the triple generated by (h,h') is T and consequently is the same as the triple generated by (d,r).

Now, we have $k.g$ = Ner. The functor k being faithful and the functor Ner being fully faithful and monomorphic on objects, the same holds for g. To show that g is an isomorphism, we must now prove that it is epimorphic on objects. That means that each algebra on T determines a groupoid. The category $\mathbb{E}$ being left exact, it is sufficient, thanks to the Yoneda embedding, to prove it in Set. That is the aim of the following lemma.

LEMMA 3. *In* Set, *any algebra on* T *determines a groupoid.*

PROOF. Let $\chi = (d_0, s_0)$ be a split epimorphism. The triple T is described by the following diagram, where

$$d_2(x,y,z) = (y,z) \quad \text{and} \quad s_1(x) = (s_0 d_0(x), x):$$



An algebra for T is a morphism α: $T\chi \to \chi$ such that

1. $\alpha.\lambda\chi = 1$,   2. $\alpha.T\alpha = \alpha.\mu\chi$.

It is given, here, by a pair (δ_1, δ_2), δ_1: $X_1 \to X_0$, δ_2: $X_2 \to X_1$, determining a morphism in Spl $\mathbb{E}$, i.e. satisfying:

0.1. $d_0.\delta_2(x,y) = \delta_1(x)$,   0.2. $s_0.\delta_1(x) = \delta_2(x,x)$.

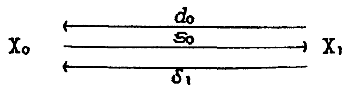
The axiom 1 becomes

1.1. $\delta_1.s_0 = 1_{X_0}$,   1.2. $\delta_2(s_0 d_0(x), x) = x$.

The axiom 2 becomes

2.1. $\delta_1 . \delta_2 (x,y) = \delta_1 (y)$, 2.2. $\delta_2(\delta_2 (x,y), \delta_2 (x,z)) = \delta_2 (y,z)$.

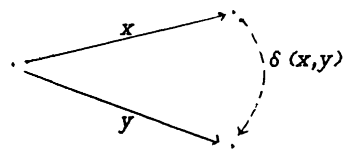
Now (d_0, s_0) being a split epimorphism and axiom 1.1 being satisfied, we have a graph:



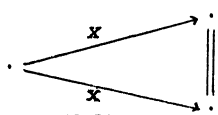
The morphism δ_2 determines an operation on pairs of arrows with the same domain. Let us denote it, for short, by δ. This operation is such that axioms 0.1 and 2.1 hold, that means:

$$d_0(\delta (x,y)) = \delta_1 (x), \quad \delta_1 (\delta (x,y))= \delta_1 (y).$$

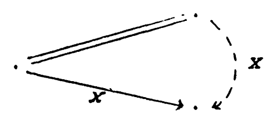
It is then possible to describe this operation by the following diagram



Let us review the three other axioms, representing the image of an object of X_0 by s_0 by the symbol '='.

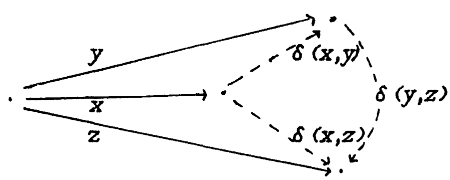


(0.2)



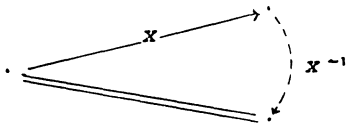
(1.2)

The stronger axiom 2.2 is represented by the following diagram:



Let us set, by now,

$$x^{-1} = \delta (x, s_0 d_0 (x)):$$



Result 1. $(x^{-1})^{-1} = x.$

Proof. $(x^{-1})^{-1} = \delta(x^{-1}, s_0 d_0 (x^{-1})) =$
$\delta(\delta(x, s_0 d_0(x), s_0 \delta_1(x)) = \delta(\delta(x, s_0 d_0(x)), \delta(x,x)) = \delta(s_0 d_0(x), x) = x.$

Corollary. $x = \delta(x^{-1}, s_0 d_0(x^{-1})).$

The composition law of two arrows. Whenever $d_0(y) = \delta_1(x)$, let us set $y.x = \delta(x^{-1}, y)$.

Result 2. $y.x = t \Leftrightarrow y = \delta(x,t).$

Proof. Let us suppose that $y = \delta(x,t)$. Then

$$y.x = \delta(x^{-1}, \delta(x,t)) = \delta(\delta(x, s_0 d_0(x)), \delta(x,t))$$
$$= \delta(s_0 d_0(x), t) = \delta(s_0 d_0(t), t) = t.$$

Conversely

$$\delta(x, y.x) = \delta(x, \delta(x^{-1}, y)) = \delta(\delta(x^{-1}, s_0 d_0(x^{-1})), \delta(x^{-1}, y)) =$$
$$\delta(s_0 d_0(x^{-1}), y) = \delta(s_0 d_0(y), y) = y.$$

The invertibility axiom: $y^{-1}.y = s_0 d_0(y)$ *and* $y.y^{-1} = s_0 \delta_1(y)$.

Proof. By Result 2, the first equality is equivalent to $y^{-1} = \delta(y, s_0 d_0(y))$, which is true. The second equality is obtained from the first one by Result 1.

The unitarity axiom: $x.s_0 d_0(x) = x$ *and* $s_0 d_0(x) = x$.

Proof. The first equality is equivalent to $\delta(s_0 d_0(x), x) = x$, which is axiom 1.2, and the second to $\delta(x,x) = s_0 \delta_1(x)$, which is axiom 0.2.

The associativity axiom: $z.(y.x) = (z.y).x.$

Proof. By Result 2, we must prove that $\delta(y.x, (z.y).x)) = z$. Now

$$\delta(y.x, (z.y).x) = \delta(\delta(x^{-1}, y), \delta(x^{-1}, z.y)) = \delta(y, z.y) =$$
$$\delta(s_0 \delta_1(y).y, z.y) = \delta(s_0 \delta_1(y), z) = \delta(s_0 d_0(z), z) = z.$$

We have therefore constructed a groupoid whose image by g is the algebra α. •

As a consequence, we obtain two important corollaries.

COROLLARY 1. *A simplicial object S is isomorphic to the nerve of an internal groupoid iff* ΔecS *is isomorphic to the nerve of the equivalence relation associated to* $d_0: S_1 \to S_0$.

PROOF. We saw it is true for a groupoid. Conversely if ΔecS is isomorphic to the nerve of the equivalence relation associated to d_0: $S_1 \to S_0$ then $\delta ecS \rightrightarrows r(d_0, s_0)$. That the square (*) is a 2-pullback implies that S is isomorphic to the nerve of a groupoid. •

COROLLARY 2: ***Intrinsic characterization of groupoids among simplicial objects.*** *A simplicial object S is isomorphic to the nerve of a groupoid iff the following square* (γ) *is a pullback in* Simpl **E**:

$$(\gamma)\qquad \begin{array}{ccc} \Delta ecS & \xleftarrow{\Delta ec\, \epsilon S} & \Delta ec^2S \\ {\scriptstyle \epsilon S}\downarrow & & \downarrow{\scriptstyle \epsilon \Delta ecS} \\ S & \xleftarrow[\epsilon S]{} & \Delta ecS \end{array}$$

PROOF. If S is the nerve of a groupoid, it is true by Lemma 1 and the fact that the functor $(\ )_0$ is exact. Conversely if (γ) is a pullback, the following simplicial object in Simpl **E** is, by Corollary 1 and Δec being exact, isomorphic to the nerve of an equivalence relation (namely that associated to ϵS):

$$S \xleftarrow{\ \epsilon S\ } \Delta ecS \underset{\Delta ec\epsilon S}{\overset{\epsilon\Delta ecS}{\leftleftarrows}} \Delta ec^2S \;\overset{\epsilon\Delta ec^2S}{\underset{\Delta ec^2\epsilon S}{\overset{\Delta ec\epsilon\Delta ecS}{\leftleftarrows\!\!\leftarrow}}}\; \Delta ec^3S \ \ldots$$

Consequently, its projection by $(\ )_0$ is isomorphic to the nerve of an equivalence relation (namely that associated to d_1):

$$S_0 \xleftarrow{\ d_1\ } S_1 \underset{d_2}{\overset{d_1}{\leftleftarrows}} S_2 \;\overset{d_1}{\underset{d_3}{\overset{d_2}{\leftleftarrows\!\!\leftarrow}}}\; S_3 \ \ldots$$

Thus by Corollary 1, the dual of S is a groupoid and consequently S is a groupoid. •

REMARK. This monadicity theorem tells us that the notion of internal groupoid is strongly algebraic, much more than the notion of internal category. Perhaps it is why this notion occurs in so many different branches of Mathematics, in Differential Geometry as well as in Homological Algebra for instance.

PART II. APPLICATIONS

1. Extensions to groupoids of an adjunction.

The monadicity theorem will be very useful to extend an adjunction

$$\mathbf{E}' \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{E}$$

to the level of groupoids. Let us suppose $\mathbf{E}'$ and $\mathbf{E}$ are left exact. The functor U being left exact, it determines a commutative square:

$$\begin{array}{ccc} \mathrm{Grd}\ \mathbf{E}' & \xrightarrow{\mathrm{Grd}\ U} & \mathrm{Grd}\ \mathbf{E} \\ {\scriptstyle (\ )_0}\downarrow & & \downarrow{\scriptstyle (\ )_0} \\ \mathbf{E}' & \xrightarrow[U]{} & \mathbf{E} \end{array}$$

Moreover this diagram commutes, up to isomorphism, with the functors Gr. Now the problem is: does the functor Grd U admit a left adjoint? We know that we have again an adjunction:

$$\mathrm{Spl}\ \mathbf{E}' \underset{\mathrm{Spl}\ U}{\overset{\mathrm{Spl}\ F}{\rightleftarrows}} \mathrm{Spl}\ \mathbf{E}$$

Furthermore the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Grd}\ \mathbf{E}' & \xrightarrow{\mathrm{Grd}\ U} & \mathrm{Grd}\ \mathbf{E} \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ \mathrm{Spl}\ \mathbf{E}' & \xrightarrow[\mathrm{Spl}\ U]{} & \mathrm{Spl}\ \mathbf{E} \end{array}$$

Now by the Adjoint Lifting Theorem (see [12]), the functor d being monadic, the functor Grd U has a left adjoint as soon as Grd $\mathbf{E}'$ has coequalizers of reflexive pairs. This is the case for instance when $\mathbf{E}$

is exact, $\mathbf{E}' = Ab(\mathbf{E})$ the category of internal abelian groups in $\mathbf{E}$, and F is the free abelian group functor. It does exist when $\mathbf{E}$ is a topos with a natural number object.

2. Aspects of internal discrete fibrations and final functors.

2.1. *Discrete fibrations.*

The two next parts dealing with the notion of discrete fibrations, let us gather here some brief recalls.

Let $f_1: X_1 \to Y_1$ be a morphism in Cat $\mathbf{E}$ and let us consider the following square (δ_i), $i = 0,1$:

$$(\delta_i) \qquad \begin{array}{ccc} mX_1 & \xrightarrow{mf_1} & mY_1 \\ {\scriptstyle d_i}\downarrow & & \downarrow{\scriptstyle d_i} \\ X_0 & \xrightarrow[f_0]{} & Y_0 \end{array}$$

If this square is a pullback, when $i = 0$, f_1 is called a *discrete cofibration*, when $i = 1$, f_1 is called a *discrete fibration*. The discrete (co)fibrations are stable under composition and pullback. Moreover if $g_1.f_1$ and g_1 are discrete (co)fibrations, f_1 is a discrete (co)fibration.

EXAMPLES AND PROPERTIES. 1. ϵX_1: Dec$X_1 \to X_1$ is a discrete cofibration.

2. $f_1: X_1 \to Y_1$ is a discrete fibration iff the following square (σ) is a pullback in Cat $\mathbf{E}$:

$$(\sigma) \qquad \begin{array}{ccc} X_1 & \xleftarrow{\epsilon X_1} & \mathrm{Dec}X_1 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle \mathrm{Dec}f_1} \\ Y_1 & \xleftarrow[\epsilon Y_1]{} & \mathrm{Dec}Y_1 \end{array}$$

Consequently the discrete fibrations are preserved by the functor Dec.

3. Let $f_1: X_1 \to Y_1$ be a discrete (co)fibration. If Y_1 is a groupoid, then X_1 is a groupoid. If Y_1 is an equivalence relation, then X_1 is an equivalence relation.

4. If X_1 and Y_1 are two groupoids, f_1 is a discrete fibration iff f_1 is a discrete cofibration.

5. An internal category X_1 is called discrete if all the maps of the diagram X_1 are invertible. A discrete category is a groupoid. Now

let f_1: $X_1 \to Y_1$ be a functor and Y_1 a discrete groupoid. Then f_1 is a discrete fibration iff X_1 is discrete.

2.2. *Initial functors.*

If Σ is a class of morphisms in a category $\mathbf{V}$, then Σ^+ (see [5, 14]) is the class of morphisms h in $\mathbf{V}$ satisfying the following property (diagonality condition): for any commutative square

$$\begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle h}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ Y & \longrightarrow & Y' \end{array}$$

when f is in Σ, there is a unique dotted arrow making the two triangles commutative. This class is stable under composition. If, furthermore, the morphisms $k.h$ and h are in Σ^+, then the morphism k is in Σ^+. When a morphism is in Σ and in Σ^+, it is clearly an isomorphism. Let us denote by Df the class of discrete fibrations and let us call final a morphism in $(\mathrm{Df})^+$.

3. A characterisation of exact categories.

A second application of the result of Part I is a characterisation of the exact categories. For that we shall need the following notion.

3.1. *The fibred reflexion.*

Let us consider the following general situation

$$\mathbf{V}' \underset{d}{\overset{q}{\leftrightarrows}} \mathbf{V}$$

where d is fully faithful and q is a left adjoint of d. The difference with what is called a basic situation in [2] is that no more left exact properties for $\mathbf{V}'$, $\mathbf{V}$, q are required. Let us recall that a morphism f: $X \to X'$ in $\mathbf{V}$ is called *q-cartesian* when the following square is a pullback:

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & X' \\ {\scriptstyle \lambda X}\downarrow & & \downarrow{\scriptstyle \lambda X'} \\ dqX & \xrightarrow[dq(f)]{} & dqX' \end{array}$$

A q-cartesian morphism is cartesian in the usual sense if q were a fibration. The q-cartesian morphisms are stable under composition. If the morphisms $q.f$ and g are q-cartesian, so is the morphism f. At last, a morphism $d(h)$: $dU \to dV$ is always q-cartesian.

A morphism f is said *q-invertible* when its image by q is invertible. If any two of the three morphisms f, g, $g.f$ are q-invertible, the third one is q-invertible.

Let us denote by q-C and q-I the two previous classes of morphisms. Then $(q\text{-C})^{+}$ is q-I. Furthermore if, in a square of morphisms, a parallel pair is in q-C and the other one in q-I, then this square is a pullback.

DEFINITION 1. A functor q: $\mathbb{V} \to \mathbb{V}'$ is called a *fibred reflexion* when it has a fully faithful right adjoint d and when furthermore the pullback of any q-invertible morphism along a q-cartesian morphism does exist, the parallel pairs in this square being in the same classes.

PROPOSITION 1. *In the general situation, q is a fibred reflexion iff it is a fibration up to equivalence.*

PROOF. Let q be a fibred reflexion and k: $U \to qX$ a morphism in $\mathbb{V}'$, then the higher edge of the following square is clearly a q-cartesian morphism whose image is the morphism k up to isomorphism:

$$\begin{array}{ccc} \bar{U} & \xrightarrow{\ \bar{k}\ } & X \\ \downarrow & & \downarrow{\scriptstyle \lambda X} \\ dU & \xrightarrow[d(k)]{} & dqX \end{array}$$

Conversely, let f: $Y \to Y'$ be a q-cartesian morphism (what means also cartesian according to the fibration q) and g: $X' \to Y'$ be a q-invertible morphism. Now let us consider the cartesian morphism $\bar{f}$ associated to X' and $(dg)^{-1}.df$: $dY \to dX'$; it determines a square:

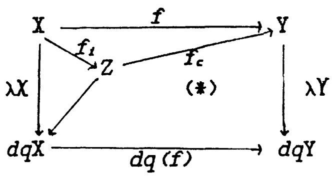
$$\begin{array}{ccc} X & \xrightarrow{\ \bar{f}\ } & X' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & Y' \end{array}$$

where g' is q-invertible, and consequently this square is a pullback.•

COROLLARY. *The fibred reflexions are stable under composition.*

PROPOSITION 2. *If q is a fibred reflexion, then any morphism has a unique decomposition $f_c.f_i$, up to isomorphism, with f_i q-invertible and f_c q-cartesian.*

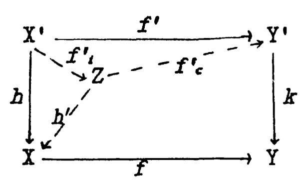
PROOF. The unicity is given by the property $(q\text{-C})^{\uparrow} = q\text{-I}$. The decomposition is given by the following diagram where the square (*) is a pullback:



•

PROPOSITION 3. *Let q be a fibred reflexion. Then the q-cartesian morphisms are stable under pullbacks, whenever they exist, and such pullbacks are preserved by q.*

PROOF. Let us consider the following pullback where f is q-cartesian:



The diagonality condition yields a morphism $h': Z \to X$ such that:

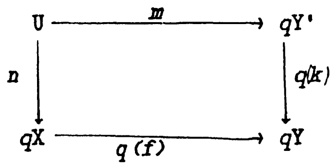
$$f.h' = k.f'_c \quad \text{and} \quad h'.f'_i = h.$$

Now the square being a pullback, there is a morphism backward

$$j\colon Z \to X' \text{ such that } h.j = h' \text{ and } f'.j = f'_c.$$

To prove that f'_i and j are mutually inverse is pure diagram chasing.

If now we consider the following commutative square:



the q-cartesian morphism $\bar{m}$ associated to Y' and m: U → qY' determines a square $k.\bar{m} = f.n'$ in **V** whose universal factorization through X' gives us, by means of its image by q, the universal factorization U → qX'. •

3.2. *The characterization.*

Let us recall that a category **E** is *exact* in the sense of Barr [1] when the following three axioms are satisfied:

EX1. Every morphism has an associated equivalence relation whose quotient does exist.

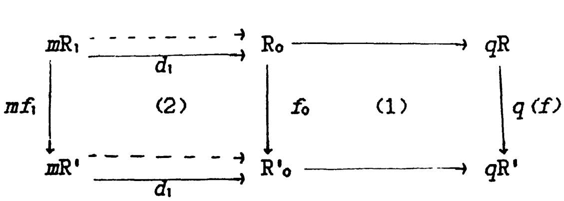
EX2. The pullback of any regular epimorphism (i.e., quotient of its associated relation) along any morphism does exist and is a regular epimorphism.

EX3. Every equivalence relation is effective (i.e., associated to some morphism).

The axioms EX1 and EX3 imply that the functor dis: **E** → Rel **E** has a left adjoint q (the quotient of the equivalence relation).

LEMMA 4. *When* **E** *is Barr-exact, the q-cartesian morphisms are the discrete fibrations.*

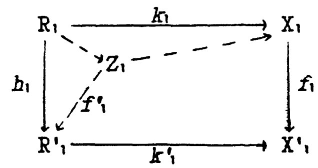
PROOF. Let f_1: R_1 → R'_1 be a q-cartesian morphism and let us consider the following diagram:



The morphism f_1 is q-cartesian iff the squares (1) and (2)+(1) are pullbacks. Then (2) is a pullback and f_1 is a discrete fibration. The converse is a consequence of [1], Example page 73. •

LEMMA 5. *The q-invertible morphisms, viewed as morphisms in* Cat **E**, *are final.*

PROOF. Let us consider the following square with h_1 q-invertible and f_1: $X_1 \to X'_1$ a discrete fibration in Cat **E**:



Let f'_1 be the pullback of f_1 along k'_1. Then f'_1 is a discrete fibration and R'_1 being in Rel **E**, the same is true for Z_1. Then f'_1 is q-cartesian. It is uniquely split by the diagonality condition in Rel **E** and it determines a splitting of the square in Cat **E** satisfying the diagonality condition in Cat **E**. •

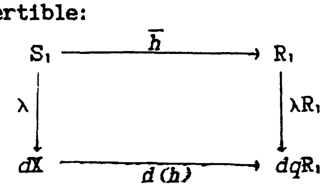
THEOREM 3. *A category* **E** *is exact in the sense of Barr iff the two following conditions are satisfied:*

A1. *Every morphism has an associated equivalence relation.*

A2. *The functor* dis: **E** $\to$ Rel **E** *has a left adjoint q which is a fibred reflexion.*

A3. *The functor q preserves the pullbacks in which one edge is a discrete fibration.*

PROOF. Let us suppose that **E** is an exact category. The axiom EX1 contains A1. Now EX1 and EX3 imply that the functor dis has a left adjoint q. Given an equivalence relation R_1 and a morphism h: $X \to qR_1$, the axiom EX2 means exactly that the following pullback does exist in Rel **E**, with λ q-invertible:



Then $\bar{h}$ is q-cartesian above h and q is a fibred reflexion (A2). Now by Lemma 4 the q-cartesian morphisms are the discrete fibrations and by Proposition 3 the functor q preserves the pullbacks in which one edge is a discrete fibration.

Conversely let us suppose A1, A2 and A3 satisfied. The axiom A1 and the existence of q imply EX1. That the functor q is a fibred reflexion implies EX2. We must now prove EX3. Let us consider the canonical presentation of an equivalence R_1:

$$R_1 \xleftarrow{\epsilon R_1} \mathrm{Dec}R_1 \underset{\mathrm{Dec}\epsilon R_1}{\overset{\epsilon\mathrm{Dec}R_1}{\leftleftarrows}} \mathrm{Dec}^2R_1$$

R_1 being a groupoid, the discrete cofibration ϵR_1 is also a discrete fibration. On the other hand, this is a pullback. Then by A3, its image by q is a pullback:

$$qR_1 \longleftarrow R_0 \underset{d_1}{\overset{d_0}{\leftleftarrows}} \pi R_1$$

and R_1 is effective. •

Remark. It may be asked whether A1 and A2 are sufficient or not.

3.3. *The functor* π_0 *for groupoids.*

Let $\mathbb{E}$ be a left exact category. We will now prove that, whenever $\mathbb{E}$ is moreover Barr-exact, the functor q can be extended to a functor π_0: Grd $\mathbb{E} \to \mathbb{E}$ left adjoint of the functor dis: $\mathbb{E} \to$ Grd $\mathbb{E}$ and that, furthermore, π_0 is a fibred reflexion.

Let us recall from [2], that if $\mathbb{E}$ is Barr-exact then the fibration $(\ )_0$: Grd $\mathbb{E} \to \mathbb{E}$ is Barr-exact, that is: each fibre is Barr-exact and each change of base functor is Barr-exact. Now let X_1 be an internal groupoid. The final object in the fibre over X_0 is Gr X_0 and the final map in the fibre is: $X_1 \to$ Gr X_0. It can be factorized in the Barr-exact fibre above X_0 into a composite of a monomorphism and an epimorphism:

$$X_1 \twoheadrightarrow \mathrm{Supp}\ X_1 \rightarrowtail \mathrm{Gr}\ X_0$$

where Supp X_1 is called the $(\ )_0$-*support of* X_1. Consequently Supp X_1 is an equivalence relation and it determines a functor

$$\mathrm{Supp}: \mathrm{Grd}\ \mathbb{E} \longrightarrow \mathrm{Rel}\ \mathbb{E}$$

left adjoint to the inclusion i: Rel $\mathbb{E} \to$ Grd $\mathbb{E}$.

PROPOSITION 4. *The functor* Supp: Grd $\mathbb{E} \to$ Rel $\mathbb{E}$ *is a fibred reflexion.*

PROOF. It is sufficient to prove that, in the following pullback in Grd $\mathbb{E}$ which always exists since $\mathbb{E}$ is left exact, γ is Supp-invertible as soon as R_1 is an equivalence relation:

$$\begin{array}{ccc} Y_1 & \xrightarrow{\;k\;} & X_1 \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle \gamma X_1} \\ R_1 & \xrightarrow[\;h\;]{} & \mathrm{Supp}\ X_1 \end{array}$$

Now γX_1 being $(\ )_0$-invertible, the same is true for γ. The morphism γX_1 being a regular epimorphism in the fibre above X_0, γ is a $(\ )_0$-invertible regular epimorphism since the fibration $(\ )_0$ is Barr-exact. Consequently R_1 being an equivalence relation, it is isomorphic to Supp Y_1 and γ is thus Supp-invertible. ♦

COROLLARY. *The functor* dis: $\mathbf{E} \to \mathrm{Grd}\ \mathbf{E}$ *has a left adjoint which is a fibred reflexion.*

PROOF. This functor dis can be decomposed:

$$\mathbf{E} \xrightarrow{\;\mathrm{dis}\;} \mathrm{Rel}\ \mathbf{E} \xrightarrow{\;i\;} \mathrm{Grd}\ \mathbf{E}$$

which both have left adjoints which are fibred reflexions. Their composite π_0 is therefore a left adjoint to dis and a fibred reflexion. ♦

The π_0-cartesian morphisms. A π_0-cartesian morphism is then a Supp-cartesian morphism f_1 such that Suppf_1 is q-cartesian. It is therefore a morphism such that the following square is a pullback and Suppf_1 a discrete fibration:

$$\begin{array}{ccc} X_1 & \xrightarrow{\;f_1\;} & Y_1 \\ {\scriptstyle \gamma X_1}\downarrow & & \downarrow{\scriptstyle \gamma Y_1} \\ \mathrm{Supp}X_1 & \xrightarrow{\;\mathrm{Supp}f_1\;} & \mathrm{Supp}Y_1 \end{array}$$

Consequently f_1 is certainly a discrete fibration. But a discrete fibration is not in general π_0-cartesian. Indeed if X_1 is a groupoid, the discrete fibration ϵX_1 is π_0-cartesian iff X_1 is an equivalence relation.

Now f_1 is π_0-cartesian iff f_1 and Suppf_1 are discrete fibrations. Indeed, if f_1 and Suppf_1 are discrete fibrations, the morphisms γX_1 and γY_1 being $(\ )_0$-invertible the previous square is necessarily a pullback.

When $\mathbf{E}$ = Set, a π_0-cartesian functor is a discrete fibration f_1: $X_1 \to Y_1$ such that any map in X_1 whose image by f_1 is an endomap in

Y_1 is itself an endomap. Equivalently the π_0-cartesian functors are the functors such that any connected component of X_1 is isomorphic to its image by f_1.

4. The comprehensive factorization in Grd E.

Let $\mathbb{E}$ be a left exact and Barr-exact category.

The aim of this section is to show that any functor f_1: $X_1 \to Y_1$ in Grd $\mathbb{E}$ can be factorized (necessarily in a way unique up to isomorphism) into a composite of a discrete fibration and a final functor.

4.1. *The regular epic discrete fibration in* Cat $\mathbb{E}$.

DEFINITION 2. A discrete fibration f_1: $X_1 \to Y_1$ in Cat $\mathbb{E}$ is said to be *regular epic* when f_0: $X_0 \to Y_0$ is a regular epimorphism in $\mathbb{E}$.

It is clear then that mf_1: $mX_1 \to mY_1$ is a regular epimorphism in $\mathbb{E}$ and consequently that f_1 is a regular epimorphism in Cat $\mathbb{E}$, preserved by the functor Ner. Now given an equivalence relation R_1 in Cat $\mathbb{E}$:

$$X_1 \underset{p_1}{\overset{p_0}{\leftleftarrows}} S_1 \overset{p_0,\ p_1,\ p_2}{\Lleftarrow} T_1$$

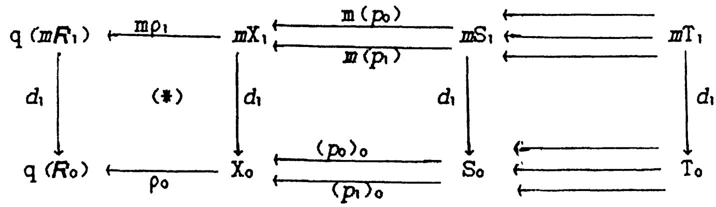
with p_1 a discrete fibration. Then any structural map of R_1 is a discrete fibration.

PROPOSITION 5. *Such an equivalence relation* R_1 *in* Cat $\mathbb{E}$ *has a quotient* ρ_1: $X_1 \to Q_1$ *which is a regular epic discrete fibration. Such quotients are stable under pullback. If furthermore* g_1: $X_1 \to K_1$ *is a discrete fibration coequalizing* p_0 *and* p_1, *the unique factorization* g_1: $Q_1 \to K_1$ *is a discrete fibration. When* X_1 *and* S_1 *are groupoids, then* Q_1 *is a groupoid.*

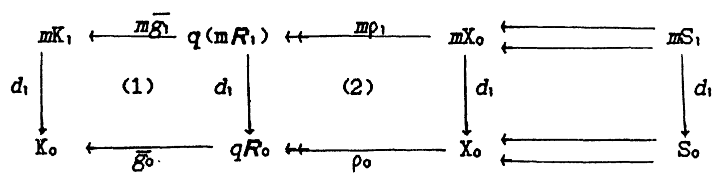
PROOF. Let us denote by R_0, mR_1 and m_2R_1 the images of R_1 in $\mathbb{E}$ by the functors $(\ )_0$, m and m_2. We obtain the following diagram in Rel $\mathbb{E}$, which is an internal category in Rel $\mathbb{E}$:

$$R_0 \underset{\delta_0}{\overset{\delta_1}{\leftleftarrows}} mR_1 \overset{\delta_0,\ \delta_1,\ \delta_2}{\Lleftarrow} m_2R_1$$

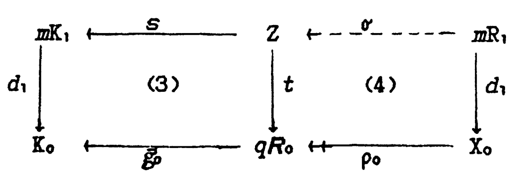
where δ_0 and δ_1 are induced by the d_0 and the d_1. Now the fact that p_1 is a discrete fibration is equivalent to the fact that δ_1 is a discrete fibration and thus a q-cartesian morphism. The image by q of the previous diagram is therefore an internal category in $\mathbb{E}$ since, δ_1 being q-cartesian, the functor q preserves the pullbacks along δ_1. It is then the componentwise quotient of R_1. The morphism ρ_1: $X_1 \to Q_1$ is determined by the left hand part of the following diagram:



Now δ_1, being a discrete fibration, is q-cartesian and therefore the square (*) is a pullback and ρ_1 is a discrete fibration. Clearly such quotients are stable under pullbacks. Given a discrete fibration g_1: $X_1 \to K_1$ coequalizing p_0 and p_1, the unique factorization $\bar{g}_1$: $Q_1 \to K_1$ determines the following diagram:



The whole square (1)+(2) is a pullback since g_1 is a discrete fibration. Now take the pullback of d_1 along $\bar{g}_0$:



There is a unique σ making (4) a pullback and consequently σ a regular epimorphism. Furthermore p_0 and p_1 being discrete fibrations, the equivalence relation associated to σ is mR_1 and consequently $q(mR_1)$ is isomorphic to Z, the square (1) is a pullback and $\bar{g}_1$ is a discrete fibration. It is clear that when X_1 and S_1 are groupoids, then Q_1 is a groupoid. •

Let X_1 be an internal category and let us denote by Fib/X_1 the category of discrete fibrations with codomain X_1 and whose morphisms are the commutative triangles.

COROLLARY. *If* $\mathbb{E}$ *is Barr-exact, then* Fib/X_1 *is Barr-exact.*

4.2. *The comprehensive factorization.*

THEOREM 4. *Given* $f_1\colon X_1 \to Y_1$ *a morphism in* Grd $\mathbb{E}$ *there is a unique, up to isomorphism, factorization* $f_1 = g_1.h_1$ *with* g_1 *a discrete fibration and* h_1 *a final functor.*

PROOF. Let us consider the following diagram:

$$\begin{array}{ccccccc}
X_1 & \xleftarrow{\epsilon X_1} & DecX_1 & \overset{\epsilon DecX_1}{\underset{Dec\epsilon X_1}{\leftleftarrows}} & Dec^2X_1 & \leftleftarrows\!\!\leftarrow & Dec^3X_1 \ \dots \\
 & & \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_1^2} & & \downarrow{\scriptstyle \tau_1^3} \\
\downarrow{\scriptstyle f_1} & & U_1 & \overset{p_0}{\underset{p_1}{\leftleftarrows}} & U_1^2 & \leftleftarrows\!\!\leftarrow & U_1^3 \ \dots \\
 & & \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \sigma_1^2} & & \downarrow{\scriptstyle \sigma_1^3} \\
Y_1 & \xleftarrow{\epsilon Y_1} & DecY_1 & \overset{\epsilon DecY_1}{\underset{Dec\epsilon Y_1}{\leftleftarrows}} & Dec^2Y_1 & \leftleftarrows\!\!\leftarrow & Dec^3Y_1 \ \dots
\end{array}$$

where $\sigma_1{}^n.\tau_1{}^n$ is the decomposition of $Dec^n(f_1)$ in a q-cartesian and a q-invertible morphism. Indeed Dec^nX_1 and Dec^nY_1 lie in Rel $\mathbb{E}$. Therefore (Lemmas 4 and 5) $\sigma_1{}^n$ is a discrete fibration and $\tau_1{}^n$ is final. Now each $p_i\colon U_1{}^n \to U_1{}^{n-1}$ is a discrete fibration as closing a square whose other edges are discrete fibrations:

$$\begin{array}{ccc}
U_1{}^{n-1} & \xleftarrow{\quad p_i \quad} & U_1{}^n \\
\downarrow{\scriptstyle \sigma_1{}^{n-1}} & & \downarrow{\scriptstyle \sigma_1{}^n} \\
Dec^{n-1}Y_1 & \xleftarrow[Dec^i\epsilon.Dec^{n-i-1}Y_1]{} & Dec^nY_1
\end{array}$$

On the other hand, the following diagram is the equivalence relation associated to $p_1\colon U_1{}^2 \to U_1$:

$$U_1 \xleftarrow{\ p_1\ } U_1{}^2 \overset{p_1}{\underset{p_2}{\leftleftarrows}} U_1{}^3 \leftleftarrows\!\!\leftarrow U_1{}^4 \ \dots$$

Indeed, it is a q-cartesian diagram above the following equivalence relation associated to $d_1\colon mX_1 \to X_0$:

$$qU_1 = X_0 \xleftarrow{\ d_1\ } qU_1{}^2 = mX_1 \overset{d_1}{\underset{d_2}{\leftleftarrows}} m_2X_1 \;\ldots$$

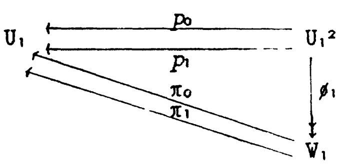
Consequently the simplicial object determined by the $U_1{}^n$ is the nerve of an internal groupoid in Grd $\mathbf{E}$.

LEMMA 6. *The coequalizer of p_0 and p_1 does exist in* Grd $\mathbf{E}$.

PROOF. Let us denote by V_1 the vertex of the pullback of $\epsilon Y_1.\sigma_1$ along itself. Now $\epsilon Y_1.\sigma_1$ coequalizing p_0 and p_1, the unique factorization v_1: $U_1{}^2 \to V_1$ is a discrete fibration since all the other maps involved in the universal property are discrete fibrations. Then the associated equivalence relation of

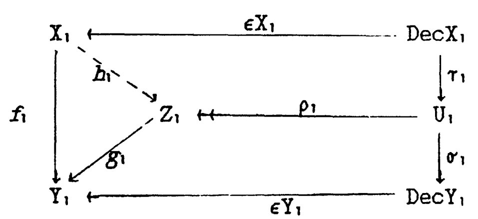
$$[p_0,p_1]\colon U_1{}^2 \longrightarrow U_1{\times}U_1 \;=\; U_1{}^2 \longrightarrow V_1 \rightarrowtail U_1{\times}U_1$$

is the same as the one associated to v_1. Therefore all its structural maps are discrete fibrations and it is consequently possible to get its quotient ϕ_1: $U_1{}^2 \twoheadrightarrow W_1$ which is a discrete fibration. Now the factorizations of p_0 and p_1 are again discrete fibrations:



The pair (π_0,π_1) is underlying to an equivalence relation (see Corollary of Proposition 4) of which it is possible to exhibit the quotient ρ_1: $U_1 \twoheadrightarrow Z_1$. It is therefore the coequalizer of p_0 and p_1. Now $\epsilon Y_1.\sigma_1$ being a discrete fibration and coequalizing π_0 and π_1, it admits a factorization g_1: $Z_1 \to Y_1$ which is a discrete fibration. •

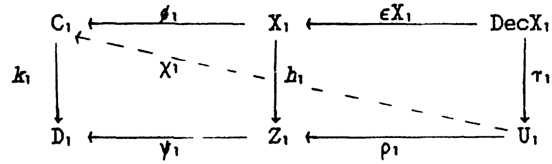
Let us now consider the following diagram:



We have just shown that g_1 is a discrete fibration. Let us denote by h_1 the factorization of the τ_1's through the coequalizers ϵX_1 and ρ_1.

LEMMA 7. *The morphism h_1 is final.*

PROOF. Let us consider the following squares where k_1 is a discrete fibration:



The morphism τ_1 being final, there is a unique splitting $\chi_1: U_1 \to C_1$ of the composite square. Let us show that $\chi_1.p_0 = \chi_1.p_1$. Now k_1 being a discrete fibration and $\tau_1{}^2$ being final, this last equality is equivalent to the following two ones:

$$k_1.\chi_1.p_0 = k_1.\chi_1.p_1 \quad \text{and} \quad \chi_1.p_0.\tau_1{}^2 = \chi_1.p_1.\tau_1{}^2$$

which are obviously satisfied. Whence a unique $\omega_1: Z_1 \to C_1$ such that $\omega_1.\rho_1 = \chi_1$. It is pure diagram chasing to prove that this ω_1 is the unique splitting of the square $k_1.\phi_1 = \psi_1.h_1$.

Thus we have the required factorization of f_1. Its unicity is a consequence of the diagonality condition. •

COROLLARY. *Given an internal groupoid* Y_1, *the inclusion*

$$i: \mathrm{Fib}/Y_1 \longrightarrow \mathrm{Grd}\ \mathbb{E}/Y_1$$

has a left adjoint.

PROOF. Given a functor $f_1: X_1 \to Y_1$, the associated discrete fibration is the g_1 of the previous factorization, the universal property being given by the diagonality condition. •

Let us end this Section by the following remark:

PROPOSITION 6. *In* Grd $\mathbb{E}$, *the final functors are stable under pullback along discrete fibrations.*

PROOF. The result is true in Rel **E** (Theorem 3). Now let us consider the following pullback in Grd **E**, with g_1 a discrete fibration and h'_1 a final functor:

$$\begin{array}{ccc} X_1 & \xrightarrow{h_1} & Y_1 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle g_1} \\ X'_1 & \xrightarrow[h'_1]{} & Y'_1 \end{array}$$

Therefore the following square which is its image by Dec is a pullback, where the two horizontal arrows are the required factorizations of $\mathrm{Dec}(h_1)$ and $\mathrm{Dec}(h'_1)$:

$$\begin{array}{ccccc} \mathrm{Dec}X_1 & \xrightarrow{\tau_1(h_1)} & U_1(h_1) & \xrightarrow{\sigma_1(h_1)} & \mathrm{Dec}Y_1 \\ {\scriptstyle \mathrm{Dec}(f_1)}\downarrow & & \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle \mathrm{Dec}(g_1)} \\ \mathrm{Dec}X'_1 & \xrightarrow[\tau_1(h'_1)]{} & U_1(h'_1) & \xrightarrow[\sigma_1(h'_1)]{} & \mathrm{Dec}Y'_1 \end{array}$$

By the diagonality condition we have an arrow $u_1\colon U_1(h_1) \to U_1(h'_1)$ making the two squares commutative. Moreover these two squares are certainly pullbacks since in Rel **E** the pullback of a final functor along a discrete fibration is final: indeed, taking the pullback of $\mathrm{Dec}(g_1)$ along $\sigma_1(h'_1)$ would produce another factorization for $\mathrm{Dec}(h_1)$. Consequently the following composite square (*) is a pullback:

$$\begin{array}{ccccc} Y_1 & \xleftarrow{\epsilon Y_1} & \mathrm{Dec}Y_1 & \xleftarrow{\sigma_1(h_1)} & U_1(h_1) \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle \mathrm{Dec}(g_1)} & & \downarrow{\scriptstyle u_1} \\ Y'_1 & \xleftarrow[\epsilon Y'_1]{} & \mathrm{Dec}Y'_1 & \xleftarrow[\sigma_1(h'_1)]{} & U_1(h'_1) \end{array}$$

Moreover there is an analogous $u_1^2\colon U_1^2(h_1) \to U_1^2(h'_1)$, making the analogous square a pullback:

$$\begin{array}{ccc} U_1^2(h_1) & \xrightarrow{\sigma_1^2(h_1)} & \mathrm{Dec}^2Y_1 \\ {\scriptstyle u_1^2}\downarrow & & \downarrow{\scriptstyle \mathrm{Dec}^2(g_1)} \\ U_1^2(h'_1) & \xrightarrow[\sigma_1^2(h'_1)]{} & \mathrm{Dec}^2Y'_1 \end{array}$$

Consequently the two following squares are pullbacks as closing squares of pullbacks:

$$\begin{array}{ccc}
U_1(h_1) & \overset{p_0(h_1)}{\underset{p_1(h_1)}{\leftleftarrows}} & U_1^2(h_1) \\
{\scriptstyle u_h}\downarrow & & \downarrow{\scriptstyle u_h^2} \\
U_1(h'_1) & \overset{p_0(h'_1)}{\underset{p_1(h'_1)}{\leftleftarrows}} & U_1^2(h'_1)
\end{array}$$

Now h'_1 being final, $\epsilon Y'_1.\sigma_1(h'_1)$ is the coequalizer of the two lower maps. Now the square (*) being a pullback, $\epsilon Y_1.\sigma_1(h_1)$ is again the coequalizer of the two upper maps and h_1 is final. •

5. A last remark.

In Grd **E** we have two factorization systems: the (π_0-invertible, π_0-cartesian) system and the (final, discrete fibration) system. We saw that $\pi_0\text{-}C \subset Df$. Consequently, if we denote by F the class of the final morphisms, we get:

$$F = (Df)^{+} \subset (\pi_0\text{-}C)^{+} = \pi_0\text{-}I.$$

From the characterization of the π_0-cartesian morphisms, we saw that these two systems are different. However let us point out that they are produced from the same situation by two general constructions which consequently appear to be different.

The initial situation is the following one:

$$\mathbf{E} \overset{(\ )_0}{\underset{\text{dis}}{\leftrightarrows}} \text{Spl } \mathbf{E}$$

where the functor $(\ )_0$ is considered as a left adjoint of dis. Indeed, in this peculiar situation, oddly the functor $(\ )_0$ is at the same time a right adjoint of dis. It is therefore left exact and thus a fibred reflexion. Now let us consider the following commutative square:

$$\begin{array}{ccc}
\mathbf{E} & \xrightarrow{\text{dis}} & \text{Grd } \mathbf{E} \\
{\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle d} \\
\mathbf{E} & \xrightarrow[\text{dis}]{} & \text{Spl } \mathbf{E}
\end{array}$$

The functor π_0 is just the extension to the category of algebras of the functor $(\ )_0$: Spl $\mathbb{E} \to \mathbb{E}$ as in the Adjoint lifting Theorem situation [12]. This is our first general construction.

The second one arises from the following considerations: Let $\mathbb{V}$ be a category with a class Σ of morphisms, endowed with a $\Sigma^{\top}$-Σ factorization system. Now let (T,λ,μ) be a triple on $\mathbb{V}$ such that $T(\Sigma) \subset \Sigma$. We define the class Σ_A in the category Alg T of algebras of T by saying that f is in Σ_A when $U(f)$ is in Σ, where U: Alg T $\to \mathbb{V}$ is the forgetful functor (with a left adjoint F). Therefore $U(\Sigma_A) \subset \Sigma$. As a consequence of the adjunction we have $F(\Sigma^{\top}) \subset (\Sigma_A)^{\top}$. Now let us briefly describe how to lift the $\Sigma^{\top}$-Σ factorization system in $\mathbb{V}$ to a $\Sigma_A{}^{\top}$-Σ_A factorization system in Alg T. Let $f\colon (X,\alpha) \to (Y,\beta)$ be a morphism of algebras. Let us denote in the following way the factorizations of $U(f)$ and $U(T(f))$ in $\mathbb{V}$:

$$X \xrightarrow[h_0]{} Z_0 \xrightarrow[k_0]{} Y, \qquad TX \xrightarrow[h_1]{} Z_1 \xrightarrow[k_1]{} TY$$

The morphisms α and β (in $\mathbb{V}$) yield a morphism $\gamma\colon Z_1 \to Z_0$. Moreover $T(k_0)$ being in Σ, we have also a morphism $\delta\colon Z_1 \to TZ_0$. Whence two morphisms in Alg T:

$$\begin{array}{ccccc} & & (T^2Z_0,\mu TZ_0) & & \\ & {\scriptstyle T\delta}\nearrow & & \searrow{\scriptstyle \mu Z_0} & \\ (TZ_1,\mu Z_1) & & \xrightarrow[T\gamma]{\qquad\qquad} & & (TZ_0,\mu Z_0) \end{array}$$

It is exactly the situation we have got from: $\mathbb{V}$ = Spl $\mathbb{E}$, T the triple generated by the pair (d,r), Alg T = Grd $\mathbb{E}$, $\Sigma = (\ )_0$-C the class of $(\ )_0$-cartesian morphisms in Spl $\mathbb{E}$ and Σ_A = Fd the class of discrete fibrations in Grd $\mathbb{E}$.

Indeed in the previous construction of the comprehensive factorization, the object U_1 is clearly of the form $(TZ_0,\mu Z_0)$ and $U_1{}^2$ of the form $(TZ_1,\mu Z_1)$. Let us say, without detail, that the pair (p_0,p_1) corresponds to the pair $(\mu Z_0.T\delta, T\gamma)$. Therefore, if it is possible, as in the comprehensive factorization situation to exhibit a coequalizer of the following upper pair in Alg T, in such a way that $U(k)$ lies in Σ:

$$\begin{array}{ccccc} (TZ_1,\mu Z_1) & \underset{T\gamma}{\overset{\mu Z_0.T\delta}{\rightrightarrows}} & (TZ_0,\mu Z_0) & \longrightarrow & (W,\delta) \\ {\scriptstyle T(k_1)}\downarrow & & \downarrow{\scriptstyle T(k_0)} & & \downarrow{\scriptstyle k} \\ (T^2Y,\mu TY) & \underset{\beta}{\overset{\mu Y}{\rightrightarrows}} & (TY,\mu Y) & \longrightarrow & (Y,\beta) \end{array}$$

then there is a Σ_A-Σ factorization system in Alg T.

Consequently, our result in Grd **E** clearly illustrate the not too obvious fact that, in general, the extension to the algebras of the factorization system associated to a fibred reflexion is not the factorization system associated to the extension to algebras of the given fibred reflexion.

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