CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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Cahiers de topologie et géométrie différentielle catégoriques, tome 28, nº 3 (1987), p. 173-181

<http://www.numdam.org/item?id=CTGDC_1987__28_3_173_0>

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Vol. XXVIII-3 (1987)

FINITE OBJECTS AND EXTENSIONAL RELATIONS

BY Osvaldo ACUÑA-ORTEGA

RÉSUMÉ. Dans cet article, on caractérise les cardinaux finis dans un topos arbitraire au moyen des relations extensionnelles (Corollaire 11 du Théorème 10). On y démontre aussi que l'objet N des nombres naturels est bien ordonné dans le sens suivant:

$$\models \forall_{X \in \Omega^N} (X \in 2^N \wedge \exists_{a \in N}\, a \in X \Rightarrow \exists_{b \in N}\, b \in X \wedge \forall_{n \in N} (n \in X \Rightarrow b \leqslant n))$$

(Corollaire 4 du Théorème 3).

This paper presents a characterization of the finite cardinals in any topos by means of extensional relations. In the process we prove that N the natural numbers object is well ordered in the following sense:

$$\models \forall_{X \in \Omega^N} (X \in 2^N \wedge \exists_{a \in N}\, a \in X \Rightarrow \exists_{b \in N} \forall_{n \in N} (n \in X \wedge n \leqslant b \Rightarrow n = b) \wedge b \in X).$$

In the boolean case this was proved in [5].

If E is an arbitrary topos, $Y \in |E|$, $K(Y)$ is the smallest subobject of Ω^Y that contains

$$\{.\}_Y : Y \to \Omega^Y, \quad \ulcorner\emptyset\urcorner : 1 \to \Omega^Y$$

and is closed under binary unions. $K^+(Y)$ is the smallest subobject of Ω^Y that contains $\{.\}_Y : Y \to \Omega^Y$ and is closed under binary unions.

OBSERVATIONS.

(i) $K^+(Y) = \{W \in K(Y) \mid \exists_{x \in Y}\, x \in W\}$.

(ii) $K^+(Y) \subset K(Y) \xleftarrow{\ulcorner\emptyset\urcorner} 1$ is a coproduct diagram: $K(Y) = K^+(Y)+1$.

(iii) If Y is decidable

$$(\models \forall_{x,y \in Y} (x = y \vee x \neq y))$$

then $K(Y)$ is decidable. $K(Y)$ is a boolean subring of 2^Y, and moreover $K(Y)$ is an ideal of 2^Y.

The first is obvious, the second is Proposition 3.8 in [1], and the third is the union of Corollary 3.7 and Corollary 3.9 of [1].

DEFINITION. Let E be an arbitrary topos, $Y \in |E|$, $\leqslant \subset Y\times Y$ a partial order. If $< \; = \; \leqslant \cap (\neg\Delta_Y)$, then:

(a) $(Y,\leqslant)$ is *extensional* if $r: Y \to \Omega^Y$ is monic, where

$$\models r(x) = \{ y \in Y \mid y < x \}.$$

(b) $(Y,\leqslant)$ is *strongly inductive* if:

$$\models \forall_{X_1\in\Omega^Y} (\forall_{x\in Y} (\{y \in Y \mid y < x\} \subset X_1 \Rightarrow x \in X_1) \Rightarrow X_1 = \ulcorner Y \urcorner).$$

(c) $(Y,\leqslant)$ is *strongly transitive* if it is extensional and strongly inductive.

(d) If X_1 is a variable of type Ω^Y and x is a variable of type Y, then "x minimal X_1" denotes the formula

$$\forall_{y\in Y} (y \leqslant x \wedge y \in X_1 \Rightarrow y = x) \wedge x \in X_1,$$

analogously define "x maximal X_1".

(e) $L(Y)$ and $L^{op}(Y)$ denote

$$\{X \in 2^Y \mid \forall_{X_1\in\Omega^Y} (X_1 \subset X \wedge X_1 \in K^+(Y) \Rightarrow \exists_{x\in Y}\ x \text{ minimal } X_1)\}$$

and

$$\{X \in 2^Y \mid \forall_{X_1\in\Omega^Y} (X_1 \subset X \wedge X_1 \in K^+(Y) \Rightarrow \exists_{x\in Y}\ x \text{ maximal } X_1)\}$$

respectively.

(f) "X_1 is l.or." denotes the formula:

$$\forall_{x,y\in Y}\ x,y \in X_1 \Rightarrow x = y \vee x < y \vee y < x.$$

LEMMA 1. *Let* E *be an arbitrary topos,* $(Y,\leqslant)$ *a partially ordered object,* Y *decidable. Then* $K(Y) \subset L(Y)\cap L^{op}(Y)$.

PROOF. It suffices to prove that $L(Y)$ is closed under binary unions and contains

$$\ulcorner \emptyset \urcorner : 1 \to \Omega^Y \quad \text{and} \quad \{.\}_Y : Y \to \Omega^Y.$$

(i) It is clear that $\ulcorner\emptyset\urcorner : 1 \to \Omega^Y$ factorizes through $L(Y)$.

(ii) $\{.\}_Y : Y \to \Omega^Y$ factorizes through $L(Y)$:

$$\begin{aligned}\models x \in \ulcorner Y \urcorner \wedge X_1 \subset \{x\} \wedge X_1 \in K^+(Y) &\Rightarrow X_1 = \{x\}\\ &\Rightarrow (\forall_{y\in Y}\; y \in X_1 \wedge y \leqslant x \Rightarrow y = x) \wedge x \in X_1\\ &\Rightarrow \exists_{y\in Y}\; y \text{ minimal } X_1 .\end{aligned}$$

Therefore

$$\models x \in Y \Rightarrow \forall_{X_1 \in \Omega^Y} (X_1 \subset \{x\} \wedge X_1 \in K^+(Y) \Rightarrow \exists_{y\in Y}\; y \text{ minimal } X_1).$$

(Y is decidable: $\models \{x\} \in 2^Y) \Rightarrow \{x\} \in L(Y)$).

(iii) $L(Y)$ is closed under binary unions:

$$\models a,b \in L(Y) \wedge I \subset a \cup b \wedge I \in K^+(Y)$$

$$\begin{aligned}
&\Rightarrow a,b \in L(Y) \wedge a \cap I \subset a \wedge b \cap I \subset b \wedge I \in K^+(Y) \wedge I \subset a \cup b\\
&\Rightarrow a,b \in L(Y) \wedge (a \cap I \in K^+(Y) \vee a \cap I = \ulcorner\emptyset\urcorner) \wedge a \cap I \subset a \wedge b \cap I\\
&\qquad\qquad \subset b \wedge I \in K^+(Y) \wedge I \subset a \cup b\\
&\Rightarrow ((a,b \in L(Y) \wedge a \cap I \in K^+(Y) \wedge a \cap I \subset a \wedge b \cap I \subset b)\\
&\qquad\qquad \vee (I \subset b \wedge I \in K^+(Y) \wedge b \in L(Y))) \wedge I \subset a \cup b\\
&\Rightarrow ((a,b \in L(Y) \wedge I,\, a \cap I \in K^+(Y) \wedge a \cap I \subset a \wedge b \cap I \subset b)\\
&\qquad\qquad \vee \exists_{z\in Y}\; z \text{ minimal } I) \wedge I \subset a \cup b\\
&\Rightarrow (\exists_{x\in Y}\; x \text{ minimal } a \cap I \wedge \exists_{y\in Y}\; y \text{ minimal } b \cap I \wedge I \subset a \cup b)\\
&\qquad\qquad \vee \exists_{z\in Y}\; z \text{ minimal } I\\
&\Rightarrow (\exists_{x\in Y} \exists_{y\in Y} (x \text{ minimal } a \cap I \wedge y \text{ minimal } b \cap I \wedge I \subset a \cup b)\\
&\qquad\qquad \vee \exists_{z\in Y}\; z \text{ minimal } I\\
&\Rightarrow (\exists_{x,y\in Y}\; x \text{ minimal } a \cap I \wedge y \text{ minimal } b \cap I \wedge (\{s \in a \cap I \mid s < y\} = \ulcorner\emptyset\urcorner\\
&\qquad \vee \{s \in a \cap I \mid s < y\} \in K^+(Y)) \wedge I \subset a \cup b) \vee \exists_{z\in Y}\; z \text{ minimal } I\\
&\Rightarrow (\exists_{x,y\in Y}\; y \text{ minimal } I \vee \exists_{x,y\in Y}\; x \text{ minimal } a \cap I \wedge y \text{ minimal } b \cap I\\
&\qquad \wedge (\{s \in a \cap I \mid s \leqslant y\} \in K^+(Y)) \wedge I \subset a \cup b) \vee \exists_{z\in Y}\; z \text{ minimal } I\\
&\Rightarrow (\exists_{x,y,w\in Y}\; x \text{ minimal } a \cap I \wedge y \text{ minimal } b \cap I\\
&\qquad \wedge w \text{ minimal } \{s \in a \cap I \mid s \leqslant y\} \wedge I \subset a \cup b) \vee \exists_{z\in Y}\; z \text{ minimal } I.
\end{aligned}$$

On the other hand:

$$\begin{aligned}
&\models x \text{ minimal } a \cap I \wedge y \text{ minimal } b \cap I \wedge w \text{ minimal } \{s \in a \cap I \mid s \leqslant y\}\\
&\qquad\qquad \wedge I \subset a \cap b \wedge r \leqslant w \wedge r \in I\\
&\Rightarrow x \text{ minimal } a \cap I \wedge y \text{ minimal } b \cap I \wedge w \text{ minimal } \{s \in a \cap I \mid s \leqslant y\}\\
&\qquad\qquad \wedge r \leqslant w \wedge (r \in a \cap I \vee r \in b \cap I)\\
&\quad \Rightarrow r = w \vee (x \text{ minimal } a \cap I \wedge y \text{ minimal } b \cap I\\
&\qquad\qquad \wedge w \text{ minimal } \{s \in a \cap I \mid s \leqslant y\} \wedge r \leqslant w \wedge r \in b \cap I\\
&\qquad \Rightarrow r = w \vee (r \leqslant w \wedge w \leqslant y \wedge r = y)\\
&\qquad \Rightarrow r = w \vee (r = w) \qquad\qquad \Rightarrow r = w.
\end{aligned}$$

Then we have

$\models x$ minimal $a \cap b \wedge y$ minimal $b \cap \mathrm{I} \wedge w$ minimal $\{s \in a \cap \mathrm{I} \mid s \leqslant y\}$
$\Rightarrow w$ minimal I.

Therefore:

$$\models a,b \in \mathrm{L}(\mathrm{Y}) \wedge \mathrm{I} \subset a \cup b \wedge \mathrm{I} \in \mathrm{K}^{+}(\mathrm{Y}) \Rightarrow \exists_{w \in \mathrm{Y}}\ w \text{ minimal I}.$$

Then:

$$\models a,b \in \mathrm{L}(\mathrm{Y}) \Rightarrow \forall_{\mathrm{I} \in \Omega^{\mathrm{Y}}} \ (\mathrm{I} \subset a \cup b \wedge \mathrm{I} \in \mathrm{K}^{+}(\mathrm{Y}) \Rightarrow \exists_{w \in \mathrm{Y}}\ w \text{ minimal I}).$$
$$\Rightarrow a \cup b \in \mathrm{L}(\mathrm{Y}).$$

We have proved that $\mathrm{K}(\mathrm{Y}) \subset \mathrm{L}(\mathrm{Y})$. Replacing $\leqslant$ by

$$\leqslant^{op} \ (= \{(x,y) \mid (y,x) \in \leqslant\})$$

and applying the previous argument we have

$$\mathrm{K}(\mathrm{Y}) \subset \mathrm{L}^{op}(\mathrm{Y}), \quad \text{and so} \quad \mathrm{K}(\mathrm{Y}) \subset \mathrm{L}(\mathrm{Y}) \cap \mathrm{L}^{op}(\mathrm{Y}).$$

COROLLARY 2. *If* E *is an arbitrary topos,* $\mathrm{Y} \in |\mathrm{E}|$ *decidable,* $\leqslant \subset \mathrm{Y} \times \mathrm{Y}$ *a partial order. Then:*

(a) If $(\mathrm{Y},\leqslant)$ *is linearly ordered then there exist* $f,g: \mathrm{K}^{+}(\mathrm{Y}) \to \mathrm{Y}$ *such that:*

$$\models \mathrm{X}_1 \in \mathrm{K}^{+}(\mathrm{Y}) \Rightarrow (f(\mathrm{X}_1) \in \mathrm{X}_1 \wedge g(\mathrm{X}_1) \in \mathrm{X}_1$$
$$\wedge \forall_{x \in \mathrm{Y}} \ (x \in \mathrm{X}_1 \Rightarrow f(\mathrm{X}_1) \leqslant x \wedge x \leqslant g(\mathrm{X}_1)).$$

(b) If $r: \mathrm{Y} \to \Omega^{\mathrm{Y}}$ *is such that*

$$\models r(y) = \{ x \in \mathrm{Y} \mid x < y \}$$

and r factors through $\mathrm{K}(\mathrm{Y})$ *then:*

$$\models \mathrm{Y}_1 \in (2^{\mathrm{Y}})^{+} \Rightarrow \exists_{x \in \mathrm{Y}}\ x \text{ minimal } \mathrm{Y}_1$$

where

$$\models (2^{\mathrm{Y}})^{+} = \{\mathrm{X}_1 \in 2^{\mathrm{Y}} \mid \exists_{x \in \mathrm{Y}}\ x \in \mathrm{X}_1\}.$$

PROOF. (a) It follows immediately from Lemma 1.

(b) $\models \mathrm{Y}_1 \in (2^{\mathrm{Y}})^{+} \Rightarrow \exists_{y \in \mathrm{Y}}\ \mathrm{Y}_1 \in 2^{\mathrm{Y}} \wedge y \in \mathrm{Y}_1$
$\Rightarrow \exists_{y \in \mathrm{Y}}\ \mathrm{Y}_1 \in 2^{\mathrm{Y}} \wedge \{x \mid x \leqslant y\} \in \mathrm{K}(\mathrm{Y}) \wedge y \in \mathrm{Y}_1$
$\Rightarrow \exists_{y \in \mathrm{Y}}\ y \in \mathrm{Y}_1 \cap \{x \mid x \leqslant y\} \wedge \mathrm{Y}_1 \cap \{x \mid x \leqslant y\} \in \mathrm{K}(\mathrm{Y})$
$\Rightarrow \mathrm{Y}_1 \cap \{x \mid x \leqslant y\} \in \mathrm{K}^{+}(\mathrm{Y}) \wedge y \in \mathrm{Y}_1$
$\Rightarrow \exists_{x \in \mathrm{Y}}\ x \text{ minimal } \mathrm{Y}_1 \cap \{x \mid x \leqslant y\} \wedge y \in \mathrm{Y}_1$
$\Rightarrow \exists_{x \in \mathrm{Y}}\ x \text{ minimal } \mathrm{Y}_1$.

THEOREM 3. If E *is an arbitrary topos,* $Y \in |E|$ *decidable,* $\leqslant \subset Y\times Y$ *a linear order and if* $r\colon Y \to \Omega^Y$ *factors through* $K(Y)$. *Then there exists* $f\colon (2^Y)^+ \to Y$ *such that*

$$\models Y_1 \in (2^Y)^+ \Rightarrow f(Y_1) \in Y_1 \wedge \forall_{x\in Y} (x \in Y_1 \Rightarrow f(Y_1) \leqslant x).$$

Proof. Immediate from Corollary 2.

COROLLARY 4. If E *is an arbitrary topos and* $X \in |E|$ *is a finite cardinal or the natural numbers object and* $\leqslant \subset X\times X$ *is the canonical order. Then there exists* $f\colon (2^X)^+ \to X$ *such that:*

$$\models A \in (2^X)^+ \Rightarrow f(A)\in A \wedge \forall_{n\in X} (n \in A \Rightarrow f(A) \leqslant n).$$

PROOF. Obvious.

Corollary 4 shows that the natural numbers object is well ordered in the classical sense when E is boolean, this result was proved in [5].

PROPOSITION 5. *Let* E *be an arbitrary topos,* $Y \in |E|$, K-*finite, decidable and* $\leqslant \subset Y\times Y$ *a partial order. Then:*

$$\models \forall_{X_1\in 2^Y} (\forall_{x\in Y} (\{y \in Y \mid y < x\} \subset X_1 \Rightarrow x \in X_1) \Rightarrow X_1 = \ulcorner Y \urcorner).$$

PROOF.

$$\begin{aligned}
&\models X_1\in 2^Y \wedge \forall_{x\in Y} (\{y \in Y \mid y < x\} \subset X_1 \Rightarrow x \in X_1)\\
&\Rightarrow X_1\in 2^Y \wedge \forall_{x\in Y} (\{y \in Y \mid y < x\} \subset X_1 \Rightarrow x \in X_1)\\
&\qquad\qquad \wedge (X_1 = \ulcorner Y \urcorner \vee \ulcorner Y \urcorner - X_1 \neq \ulcorner \emptyset \urcorner)\\
&\Rightarrow X_1 = \ulcorner Y \urcorner \wedge \forall_{x\in Y} (\{y \in Y \mid y < x\} \subset X_1 \Rightarrow x \in X_1)\\
&\qquad\qquad \wedge \exists_{z\in Y}\ z \text{ minimal } (\ulcorner Y \urcorner - X_1)\\
&\Rightarrow X_1 = \ulcorner Y \urcorner \vee \forall_{x\in Y} (\{y \in Y \mid y < x\} \subset X_1 \Rightarrow x \in X_1)\\
&\qquad\qquad \wedge \exists_{z\in Y} \{y \in Y \mid y < z\} \subset X_1 \wedge z \notin X_1\\
&\Rightarrow X_1 = \ulcorner Y \urcorner \vee z \in X_1 \wedge z \notin X_1 \Rightarrow X_1 = \ulcorner Y \urcorner \vee \text{false}\\
&\Rightarrow X_1 = \ulcorner Y \urcorner.
\end{aligned}$$

COROLLARY 6. *Let* E *be boolean,* $Y \in |E|$, K-*finite and* $\leqslant \subset Y\times Y$ *a partial order. Then* $(Y,\leqslant)$ *is strongly inductive.*

PROOF. Apply Proposition 5.

PROPOSITION 7. *Let* X *be decidable,* $\leqslant \subset X \times X$ *a partial order. If* $r: X \to \Omega^X$ *factors through* K(X) *and* (X,≤) *is extensional then*

$$\models x \in X \wedge y \in \ulcorner X \urcorner \wedge \{z \mid z \leqslant x\} \text{ is l.or.} \wedge \{z \mid z \leqslant y\} \text{ is l.or.} \Rightarrow x < y \vee y < x \vee y = x.$$

PROOF. Consider

$$\begin{aligned} &\models x \in \ulcorner X \urcorner \wedge p \text{ minimal } \ulcorner X \urcorner \Rightarrow x \in \ulcorner X \urcorner \wedge \{y \mid y < p\} = \ulcorner \emptyset \urcorner \\ &\Rightarrow (\{y \mid y < x\} = \ulcorner \emptyset \urcorner \vee \{y \mid y < x\} \in K^{+}(X) \wedge \{y \mid y < p\} = \ulcorner \emptyset \urcorner \\ &\Rightarrow x = p \vee (\exists_{q \in X}\ q \text{ minimal } \{y \mid y < x\} \wedge \{y \mid y < p\} = \ulcorner \emptyset \urcorner) \\ &\Rightarrow x = p \vee (\{y \mid y < q\} = \ulcorner \emptyset \urcorner \wedge \{y \mid y < p\} = \ulcorner \emptyset \urcorner \wedge q < x) \\ &\Rightarrow x = p \vee (q = p \wedge q < x) \Rightarrow x = p \vee p < x. \end{aligned}$$

Therefore

$$\begin{aligned} &\models x,y \in X \wedge p \text{ minimal } \ulcorner X \urcorner \Rightarrow (p < x \vee p = x) \wedge (p < y \vee p = y) \\ &\Rightarrow (p < x \wedge p < y) \vee y < x \vee x < y \vee x = y \\ &\Rightarrow (\exists_{p \in X}\ p < x \wedge p < y) \vee y < x \vee x < y \vee x = y \\ &\Rightarrow \{z \mid z < x \wedge z < y\} \in K^{+}(X) \vee y < x \vee x < y \vee x = y \\ &\Rightarrow (\exists_{r \in X}\ r \text{ maximal } \{z \mid z < x \wedge z < y\}) \vee y < x \vee x < y \vee x = y. \end{aligned}$$

On the other hand

$$\begin{aligned} &\models a \text{ minimal } \{z \mid r < z \wedge z \leqslant x\} \wedge b \text{ minimal } \{z \mid r < z \wedge z \leqslant y\} \\ &\qquad \wedge \{z \mid z \leqslant x\} \text{ is l.or.} \wedge \{z \mid z \leqslant y\} \text{ is l.or.} \wedge h < a \\ &\Rightarrow a \text{ minimal } \{z \mid r < z \wedge z \leqslant x\} \wedge r < b \wedge h < a \wedge r < a \\ &\Rightarrow a \text{ minimal } \{z \mid r < z \wedge z \leqslant x\} \wedge r < b \\ &\qquad \wedge (h \leqslant r \vee r < h) \wedge h \leqslant a \wedge h \neq a \\ &\Rightarrow h < b \vee (a \text{ minimal } \{z \mid r < z \wedge z \leqslant x\} \\ &\qquad \wedge r < h \wedge h \leqslant x \wedge h \leqslant a \wedge h \neq a \\ &\Rightarrow h < b \vee (h = a \wedge h \neq a) \Rightarrow h < b \vee \text{false} \Rightarrow h < b. \end{aligned}$$

By symmetry we obtain that

$$\begin{aligned} &\models a \text{ minimal } \{z \mid r < z \wedge z \leqslant x\} \wedge b \text{ minimal } \{z \mid r < z \wedge z \leqslant y\} \\ &\qquad \wedge \{z \mid z \leqslant x\} \text{ is l.or.} \wedge \{z \mid z \leqslant y\} \text{ is l.or.} \\ &\Rightarrow \{h \mid h < b\} = \{h \mid h < a\} \Rightarrow a = b. \end{aligned}$$

Then:

$$\begin{aligned} &\models x \in X \wedge y \in Y \wedge p \text{ minimal } \ulcorner X \urcorner \wedge \{z \mid z \leqslant x\} \text{ is l.or.} \\ &\qquad \wedge \{z \mid z \leqslant y\} \text{ is l.or.} \\ &\Rightarrow (\exists_{r \in X}\ r \text{ maximal } \{z \mid z < x \wedge z < y\} \\ &\wedge \exists_{a,b \in X}\ a \text{ minimal } \{z \mid r < z \wedge z \leqslant x\} \wedge b \text{ minimal } \{z \mid r < z \wedge z \leqslant y\} \\ &\wedge \{z \mid z \leqslant x\} \text{ is l.or.} \wedge \{z \mid z \leqslant y\} \text{ is l.or.} \vee x < y \vee y < x \vee x = y \end{aligned}$$

$\Rightarrow (\exists_{r,a,b\in X}\ r \text{ maximal } \{z \mid z < x \wedge z < y\} \wedge a = b$
$\wedge\ r < a \wedge a \leqslant x \wedge r < b \wedge b \leqslant y) \vee x < y \vee y < x \vee x = y$
$\Rightarrow (\exists_{r,a\in X}\ r \text{ maximal } \{z \mid z < x \wedge z < y\}$
$\wedge\ r < a \wedge a \leqslant x \wedge a \leqslant y) \vee x < y \vee y < x \vee x = y$
$\Rightarrow (\exists_{r,a\in X}\ r \text{ maximal } \{z \mid z < x \wedge z < y\} \wedge (a = x \vee a < x)$
$\wedge\ (a = y \vee a < y) \wedge r < a) \vee x < y \vee y < x \vee x = y$
$\Rightarrow (\exists_{r;a\in X}\ r \text{ maximal } \{z \mid z < x \wedge z < y\} \wedge a < x \wedge a < y$
$\wedge\ r \leqslant a \wedge r \neq a) \vee x = y \vee x < y \vee y < x \vee x = y$
$\Rightarrow (\exists_{r,a\in X}\ r = a \wedge r \neq a) \vee x < y \vee y < x \vee x = y$
$\Rightarrow \text{false} \vee x = y \vee x < y \vee y < x$
$\Rightarrow x = y \vee x < y \vee y < x\ .$

Resuming all the preceding arguments we have:

$\models x \in X \wedge y \in X \wedge \{z \mid z \leqslant x\} \text{ is l.or.} \wedge \{z \mid z \leqslant y\} \text{ is l.or.}$
$\Rightarrow x \in X \wedge y \in X \wedge \{z \mid z \leqslant x\} \text{ is l.or.} \wedge \{z \mid z \leqslant y\} \text{ is l.or.}$
$\wedge\ \{z \mid z \leqslant y\} \in K^{+}(X)$
$\Rightarrow x \in X \wedge y \in X \wedge \{z \mid z \leqslant x\} \text{ is l.or.} \wedge \{z \mid z \leqslant y\} \text{ is l.or.}$
$\wedge\ \exists_{p\in X}\ p \text{ minimal } \{z \mid z \leqslant y\}$
$\Rightarrow x \in X \wedge y \in X \wedge \{z \mid z \leqslant x\} \text{ is l.or.} \wedge \{z \mid z \leqslant y\} \text{ is l.or.}$
$\wedge\ \exists_{p\in X}\ p \text{ minimal } \ulcorner X \urcorner$
$\Rightarrow \exists_{p\in X}\ x = y \vee x < y \vee y < x \quad \Rightarrow x = y \vee x < y \vee y < x\ .$

LEMMA 8. *Let* X *be decidable,* $\leqslant\ \subset X\times X$ *a partial order. If* $r\colon X \to \Omega^X$ *factors through* K(X) *and* $(X,\leqslant)$ *is strongly transitive then*:

$$\models \forall_{z,x,y\in X}\ (x < z \wedge y < z \Rightarrow x = y \vee x < y \vee y < x).$$

PROOF. If

$$S = \{y \in X \mid \{x \in X \mid x < y\} \text{ is l.or.}\}$$

we want to prove that S = X :

$\models \{z \mid z < w\} \subset \ulcorner S \urcorner \wedge x < w \wedge y < w$
$\Rightarrow x \in X \wedge y \in X \wedge \{z \mid z < x\} \text{ is l.or.} \wedge \{z \mid z < y\} \text{ is l.or.}$
$\Rightarrow\ x = y \vee x < y \vee y < x\ .$

Therefore:

$\models \{z \mid z < w\} \subset \ulcorner S \urcorner \Rightarrow \forall_{x,y\in X}\ (x,y \in \{z \mid z < w\} \Rightarrow x = y \vee x < y \vee y < x)$
$\Rightarrow \{z \mid z < w\} \text{ is l.or.} \quad \Rightarrow w \in \ulcorner S \urcorner\ .$

Since $(X,\leqslant)$ is strongly inductive we have that S = X.

THEOREM 9. *Let* X *be decidable,* $\leq\ \subset X\times X$ *a partial order. If* $r: X \to \Omega^X$ *factors through* K(X) *and* $(X,\leq)$ *is strongly transitive, then* $(X,\leq)$ *is linearly ordered.*

PROOF.

$$\models x \in X \wedge y \in X$$
$$\Rightarrow \{z \mid z < x\} \in \ulcorner S \urcorner \wedge \{z \mid z < y\} \in \ulcorner S \urcorner \wedge x \in X \wedge y \in X$$
$$\Rightarrow x = y \vee x < y \vee y < x\ .$$

The converse is also true. Since complemented non-empty subobjects of X have a minimum, $(X,\leq)$ is extensional. Let Z be the image of

$$id \times r: X\times X \to X\times\Omega^X\ ;$$

since $1 = pr_1 \mid Z$ is a finite cardinal in E/X by Corollary 9 of [3], 1 is strongly inductive. It is easy to prove that if 1 is strongly inductive so is X.

Let E_{kfd} be the full subcategory of K-finite decidable objects of E. This category is a boolean topos, see [4].

THEOREM 10. *Let* X *be* K-*finite decidable and* $\leq\ \subset X\times X$ *a partial order. If* $r: X \to \Omega^X$ *factors through* K(X) *then the following propositions are equivalent:*

(a) $(X,\leq)$ *is linearly ordered in* E.

(b) $(X,\leq)$ *is extensional in* E.

(c) $(X,\leq)$ *is strongly transitive in* E.

PROOF. (c) ⇒ (a): Theorem 9.

(a) ⇒ (c): X is a finite cardinal and by Corollary 9 of [3] we have that $(X,\leq)$ is strongly inductive. Since $(X,\leq)$ has a successor function (Proposition 2 of [3]), then it is extensional.

(b) ⇒ (a): by Proposition 1 of [2] we know that $(X,\leq)$ is linearly ordered in E iff it is linearly ordered in E_{kfd}. On the other hand applying Proposition 5 to $(X,\leq)$ in E_{kfd} we have that $(X,\leq)$ is strongly inductive in E_{kfd} and by Theorem 9, $(X,\leq)$ is linearly ordered since $K(X) = 2^X$ and $(X,\leq)$ is extensional in E_{kfd}.

(a) ⇒ (b): already proved.

In [2] we defined X to be a finite cardinal when K is K-finite, decidable and linearly ordered.

COROLLARY 11. *Let* X *be a* K-*finite decidable object of* E. *The following propositions are equivalent:*

(a) X *is a finite cardinal.*

(b) There exists $\leqslant \subset X\times X$ *a partial order such that* $(X,\leqslant)$ *is extensional in* E *and* $r: X \to \Omega^X$ *factors through* K(X).

(c) There exists $\leqslant \subset X\times X$ *a partial order such that* $(X,\leqslant)$ *is strongly transitive and* $r: X \to \Omega^X$ *factors through* K(X).

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