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A SURVEY OF TOTALITY FOR ENRICHED AND ORDINARY CATEGORIES

by G.M. KELLY

RÉSUMÉ. Une V-catégorie A est dite *totale* si son plongement de Yoneda $Y : A \to [A^{op}, V]$ admet un adjoint à gauche; on peut exprimer cela sans mention explicite de $[A^{op}, V]$, qui en général n'existe pas en tant que V-catégorie. La totalité n'a guère été étudiée sauf dans le cas $V = Set$ des catégories ordinaires localement petites. Nous démontrons qu'une A totale jouit de propriétés très fortes de complétude et de cocomplétude, et nous donnons une variété de conditions suffisantes pour la totalité. Même dans le cas classique $V = Set$, notre traitement est plus complet que ceux donnés dans la littérature, et nos démonstrations souvent plus simples. Finalement, nous comparons (ce qui n'avait pas été fait, même dans le cas $V = Ab$) la totalité de la V-catégorie A et celle de sa catégorie ordinaire sous-jacente A_0 .

1. INTRODUCTION.

The notion of *totality* for a category was introduced by Street and Walters [19] in an abstract setting, wide enough to cover ordinary categories, enriched categories, and internal categories. An ordinary category A is said to be *total* if it is locally small — so that we have a Yoneda embedding $Y : A \to [A^{op}, Set]$ where *Set* is the category of small sets — and if this embedding Y admits a left adjoint Z. Totality for these ordinary categories has been further investigated by Tholen [22], Wood [24], and Street [18]; it turns out to imply very strong completeness and cocompleteness properties of A — not in fact dual to one another — and yet to be a property enjoyed by all the commonly-occurring categories of mathematical structures, and in many cases by their opposite categories as well. It is accordingly an important notion.

In the same way, a V-category A is said to be *total* if its Yoneda embedding $Y : A \to [A^{op}, V]$ has a left adjoint; this can be so re-phrased as to avoid mention of the functor-category $[A^{op}, V]$ which (since A is rarely small) does not in general exist as a V-category — just as the $[A^{op}, Set]$ of the last paragraph, not being locally small in general, is not a *Set*-category.

For instance, a locally-small *additive* category A — that is, an Ab-category A where Ab is the closed category of small abelian groups — is total if $Y : A \to [A^{op}, Ab]$ has a left adjoint, where $[A^{op}, Ab]$ is the (additive) category of *additive* functors $A^{op} \to Ab$. This totality of the additive A is not, on the face of it, at all the same thing as totality of its underlying ordinary category A_0 — although in fact it is equivalent to it, as we shall see. For more general closed categories V, however, totality of a V-category A is not equivalent to that of A_0 ; for the good V that occur in practice, the former is stricly stronger.

Our present concern is with totality in the V-enriched case, which has not been further investigated since [19], except that a forthcoming article [4] by Day and Street is set in this context. Our aims are, first, to examine the completeness and cocompleteness properties implied by totality; secondly, to give various sufficient conditions for totality; and thirdly, to pursue the relations beween the totality of a V-category A and that of its underlying ordinary category A_0. In the first two of these aims we are in large part generalizing from ordinary categories to enriched ones the results of the authors above. In these circumstances it has seemed to us proper to aim at a certain completeness, giving proofs (often new and simpler) even of those of their results peculiar to the case $V = Set$, so that the present article can serve as a compendium of the principal results on totality.

By this we mean totality *as such*. There is a more special aspect of totality which does not concern us here. The total A is said to be *lex-total* if the left adjoint Z of Y preserves finite limits. Lex-totality is as rare as totality is common: Street [17] (see also [18]) has shown that essentially the only lex-total ordinary categories are the Grothendieck toposes. Similarly a lex-total additive category, at least if it has a strong generator, is a Grothendieck abelian category. (Note that lex-totality of the additive A, unlike mere totality, is quite different from lex-totality of A_0.) The recent work [4] of Day and Street referred to above is concerned with the intermediate case where Z is required to preserve certain monomorphisms; this is still a comparatively rare property since it implies, as they show, that every small full subcategory of A that is strongly generating is in fact dense — which is false even for such well-behaved ordinary categories as Ab.

2. PRELIMINARIES ON SMALLNESS AND LIMITS.

The notion of totality for A being expressible in terms of the existence in A of certain large colimits, questions of size play a central role below. Accordingly we begin by making precise our conventions and language regarding *smallness* for ordinary categories and for V-categories. In doing so we also recall, for the convenience of readers less familiar with them, the basic ideas on indexed limits in enriched categories.

We take the view that the morphisms of any (ordinary) category A constitute a set. Choosing some universe once for all, we call its elements *small* sets, and write *Set* for the category of these. (In most contexts it is harmless to call a set "small" if it is equipotent to one in our universe; that is, if its cardinal is less than the inaccessible ∞ associated to the universe.) Then the category A is *locally small* if each hom-set $A(A, B)$ is small; while A is *small* if it is locally small and its set of objects is small — equivalently, if its set of morphisms is small. The limit of $T : K \to A$ is a *small* limit if the category K is small; and A is *complete* if it admits all *small* limits — similarly for *cocomplete*. We use *large* to mean *non-small*, or sometimes *not-necessarily-small*.

Our general reference for enriched category theory is the author's book [12]. We consider, as there, a (symmetric monoidal) closed category V, whose tensor product, identity object, and internal-hom are $\otimes$, I, and $[\ ,\ ]$, and whose underlying category V_0 is locally small, complete, and cocomplete. We *further suppose* that V_0 admits intersections of arbitrary families of subobjects, even large ones if need be; this was not demanded in [12], but it is trivially satisfied if V_0 is wellpowered, and is true in all the examples of interest — even the non-wellpowered one given by Spanier's quasi-topological spaces. As in [12], we write $V : V_0 \to Set$ for the functor $V_0(I, -)$; and we write $F : Set \to V_0$ for its left adjoint, sending the set W to the coproduct W.I of W copies of I. We are now concerned with V-categories, V-functors, and V-natural transformations; of course, a *Set*-category is the same thing as a locally-small ordinary category.

We recall from Section 1.3 of [12] that a V-category A gives rise to an "underlying" locally-small ordinary category A_0 with the same objects and with

$$A_0(A, B) = VA(A, B) ;$$

that a V-functor $T : A \to B$ gives rise to an "underlying" functor $T_0 : A_0 \to B_0$ where $(T_0)_{AB} = VT_{AB}$; and that the components of a V-natural $\alpha : T \to S$ constitute a natural transformation $\alpha_0 : T_0 \to S_0$. The value $T_0 f$ of T_0 at a morphism f of A_0 is often written Tf; especially so when T is the representable $A(C, -) : A \to V$, so that we have

$$A(C, f) : A(C, A) \to A(C, B)$$

in V_0. If A and B are V-categories and we speak of a functor $T : A \to B$ we mean of course a V-functor; if we meant an ordinary functor

$T : A_0 \to B_0$, we should say just this. Similarly, when a V-functor is said to have a left adjoint, it is a V-left-adjoint that is meant; and so on.

A V-category A is *small* when its set of objects is small; when $V = Set$, this agrees with the definition above. When A and B are V-categories with A small, we have as in Section 2.2 of [12] the V-category $[A, B]$ of V-functors $A \to B$, with V-valued-hom

$$[A, B](T, S) = \int_{A \in A} B(TA, SA) \, ; \tag{2.1}$$

then $[A, B]_0(T, S)$ is just the set of V-natural transformations $T \to S$. When A is large, the end on the right side of (2.1) does not in general exist for all T and S ; whereupon we say that $[A, B]$ *does not exist as a V-category.* Yet this end may well exist for a particular pair $T, S : A \to B$ (as it always does when $B = V$ and T is representable); then, even though there is no V-category $[A, B]$, we say that $[A, B](T, S)$ *exists in V.*

The limits appropriate to V-categories are the *indexed* ones of [12] Chapter 3. Given V-functors

$$J : K \to V \qquad \text{and} \qquad T : K \to A \, ,$$

we recall that A *admits the* J *-indexed limit* $\{J, T\}$ *of* T if, first,

$$[K, V](J, A(A, T-))$$

exists in V for each $A \in A$ (which it surely does when K is small), and if, further, the functor

$$A \mapsto [K, V](J, A(A, T-)) : A^{op} \to V$$

admits a representation

$$A(A, \{J, T\}) \simeq [K, V](J, A(A, T-)). \tag{2.2}$$

We call the V-category A *complete* if it admits all *small* (indexed) limits — that is, all those for which K is small. When $V = Set$ this definition of completeness does, by Section 3.4 of [12], coincide with that given above for a (locally-small) ordinary category A ; the point is that the classical limit of $T : K \to A$ is then just the indexed limit $\{\Delta 1, T\}$ where $\Delta 1 : K \to Set$ is the functor constant at 1, while the indexed limit $\{J, T\}$ is then the classical limit of $Td : N \to A$ where $d : N \to K$ is the discrete op-fibration corresponding to $J : K \to Set$. Indexed colimits, and cocompleteness, are defined dually: it is convenient to take as the indexing-type for a colimit a functor $J : K^{op} \to V$, and then the J-*indexed colimit* $J * T$ *of* $T : K \to A$ is defined by

$$A(J * T, A) \simeq [K^{op}, V](J, A(T-, A)). \tag{2.3}$$

The V-category V itself is complete and cocomplete, and a functor-category $[A, B]$ is complete or cocomplete if B is so, with limits or colimits formed pointwise.

We recall finally from [12] Chapter 3 that the concept of indexed limit in the V-category A contains various particular limit-notions as special cases. The *end* of $T : M^{op} \otimes M \to A$ is the limit of T indexed by $\mathrm{Hom}_M : M^{op} \otimes M \to V$. The *cotensor product* $X \pitchfork B$ of $X \in V$ and $B \in A$, defined by

$$A(A, X \pitchfork B) \simeq [X, A(A, B)], \tag{2.4}$$

is the limit of $B : I \to A$ (where I is the unit V-category with one object $*$ and $I(*,*) = I$) indexed by $X : I \to V$. Given an ordinary functor $P : L \to A_0$, a cone $(\alpha_L : B \to PL)$ is said to exhibit B as the *(conical) limit in A of* P if each

$$A(A, \alpha_L) : A(A, B) \to A(A, PL)$$

is a limit-cone in V_0; this implies that B is the limit of P in A_0, and is implied by it if A is tensored (that is, admits *tensor products* $X \otimes A$ for $X \in V$ and $A \in A$), or if V is *conservative* (that is, isomorphism-reflecting). So long as L is locally small, we have a free V-category $K = F_* L$ on L; and then the conical limit of P in A is the same thing as the indexed limit $\{J, T\}$, where $J : K \to V$ corresponds to $\Delta I : L \to V_0$ and $T : K \to A$ to $P : L \to A_0$. Completeness of A is equivalent to the existence of small conical limits and of cotensor products; by the above, it implies completeness of A_0; by Proposition 3.76 of [12], it coincides with completeness of A_0 if V is conservative and each object of V_0 has but a small set of strongly-epimorphic quotients (as, for instance, when $V = Ab$).

3. PRELIMINARIES ON FUNCTOR-CATEGORIES WITH LARGE DOMAINS

When $V = Set$, the functors $A \to B$ and the natural transformations between them constitute a functor-category $[A, B]$ even if A is large; it fails to be a Set-category because it is not locally small, yet being able to refer to it is a considerable convenience. It can of course be seen as a Set'-category, where Set' is the category of sets belonging to some universe with ob A as an element. Similarly when $V = Ab$; the additive functors $A \to B$ constitute an additive category $[A, B]$ which, while not an Ab-category, is an Ab'-category where Ab' consists of the abelian groups in a suitable Set'. Similarly, too, for most other concrete closed categories V that occur in practice; see Section 2.6 of [12]. In fact it was shown in Sections 3.11 and 3.12 of [12] that we can, for any V, construct $[A, B]$ as a V'-category where V' is a suitable extension of V into a higher universe. While it is true that the results on totality below can be formulated with no reference to $[A, B]$ itself, always being cast in terms of such $[A, B](T, S)$ as exist in V, this is at the cost of some circumlocution: it is so convenient to have an $[A, B]$ to refer to that we recall the relevant results from [12].

If Set' is the category of sets in some universe containing (not necessarily strictly) our chosen one, by a *Set'-extension of V* we mean a (symmetric monoidal) closed category V' with the following properties: (i) the underlying ordinary category V'_0 of V' is Set'-locally-small, Set'-complete and Set'-cocomplete, and admits arbitrary intersections of monomorphisms; (ii) V'_0 contains V_0 (at least to within equivalence) as a full subcategory and the inclusion $V_0 \to V'_0$ preserves all limits (small or large) that exist in V_0; (iii) the symmetric monoidal closed structure of V is the restriction of that of V'. Then every V-category can be seen as a V'-category, and similarly for functors and natural transformations. For V-categories A and B, to say that $[A, B](T, S)$ exists in V is – because $V_0 \to V'_0$ is fully-faithful and limit-preserving – precisely to say that $[A, B](T, S)$ exists in V' and that its value there lies in V. When ob $A \in Set'$, we have the existence in V' of $[A, B](T, S)$ for all T and S, and $[A, B]$ exists as a V'-category.

If $\{A_\alpha\}$ is any set of V-categories, it is possible *in many ways* to find an extension V' of V with respect to which each A_α is "small". One such extension that is always available – although in many concrete cases it may not be the most natural one – is described in Section 3.11 of [12]; we so choose the new universe that ob V_0 and each ob A_α lie in Set', and take $V'_0 = [V_0^{op}, Set']$ with the convolution symmetric monoidal closed structure. The modification of this extension given in Section 3.12 of [12] is often more natural, coinciding when V_0 is locally presentable with the naive extension of [12] Section 2.6; but for our purposes it is irrelevant which extension we choose.

We emphasize that, having introduced V' so as to give a meaning to $[A, B]$, we make no other use of V'-categories. In particular, when we speak of limits in a V-category A, we mean those indexed by some $J : K \to V$ where K is a V-category. Recall from Section 3.11 of [12] that, given such a J and a V-functor $T : K \to A$, we have both the V-limit $\{J, T\}$ and the V'-limit $\{J', T\}$, where J' is the composite of J with the inclusion $V \to V'$; but that there is no confusion since these coincide, either existing if the other does. It follows that, if the V-category B admits all limits (resp. colimits) indexed by such a J, then $[A, B]$ – even when it exists only as a V'-category – admits J'-indexed limits (resp. colimits): if now the V-category C is a full reflective subcategory of $[A, B]$, it further follows that C admits all J-indexed limits (resp. colimits).

4. PRELIMINARIES ON MONOMORPHISMS AND GENERATORS.

The notion of a monomorphism in a V-category A, given by Dubuc on page 1 of [6], was not used in the author's book [12]; so we recall it here. A map $f : B \to C$ in A_0 is said to be a *monomorphism in A* if each

$$A(A, f) : A(A, B) \to A(A, C)$$

is a monomorphism in the ordinary category V_0. By the last paragraph of Section 2 above, it clearly comes to the same thing to require that

$$\begin{tikzcd} B \arrow[r, "1"] \arrow[d, "1"'] & B \arrow[d, "f"] \\ B \arrow[r, "f"'] & C \end{tikzcd}$$

be a pullback in A. By that same paragraph, then, a monomorphism in A is one in A_0, and the converse is true if A does admit pullbacks (and thus if A is complete) or if A is tensored (and thus if A is cocomplete); so it is true in particular when A is V or V^{op} (as well as being trivially true for all A if V is faithful, and *a fortiori* if V is conservative). Of course f is an *epimorphism in A* if it is a monomorphism in A^{op}, which is to say that each $A(f, A)$ is a monomorphism in V_0.

If a family $(f_\lambda : B_\lambda \to C)$ of monomorphisms in A is such that the diagram they constitute admits the (conical) limit

$$\begin{tikzcd} & B_\lambda \arrow[dr, "f_\lambda"] & \\ B \arrow[ur, "g_\lambda"] \arrow[rr, "f"'] & & C \end{tikzcd} \qquad (4.1)$$

in A, we call the monomorphism f the *intersection in A* of the f_λ. Again by the last paragraph of Section 2 above, this implies that f is the intersection in A_0 of the f_λ, and is implied by it if A is tensored or if V is conservative.

A *small* set G of objects of the V-category A is said to be a *generator* for A if, for each $A, B \in A$, the canonical map

$$\varphi = \varphi_{AB} : A(A, B) \to \Pi_{G \in G}[A(G, A), A(G, B)] \qquad (4.2)$$

is a monomorphism in V_0; this clearly reduces to the usual notion of generator (as given on page 123 of Mac Lane [14]) when $V = Set$, in which case it may be expressed by saying that the functors

$$A(\mathrm{G}, -) : A \to Set \quad \text{for} \quad \mathrm{G} \in G$$

are *jointly faithful.* (When the generator G has but one element G, we also say that G *is a generator for* A.) If A is cocomplete, the codomain of (4.2) is isomorphic to $A(\Sigma_{\mathrm{G}} A(\mathrm{G}, \mathrm{A}) \otimes \mathrm{G}, \mathrm{B})$; and, modulo this isomorphism, φ is $A(\sigma_{\mathrm{A}}, \mathrm{B})$ where

$$\sigma_{\mathrm{A}} : \Sigma_{\mathrm{G}} A(\mathrm{G}, \mathrm{A}) \otimes \mathrm{G} \to \mathrm{A} \tag{4.3}$$

is the evident map. So G is a generator of the cocomplete A precisely when each σ_{A} is an epimorphism in A — which agrees with the definition of generator given on page 82 of Dubuc [6].

If A is complete or cocomplete, a small set of objects of A that is a generator for A_0 is also one for A ; in more detail:

Proposition 4.1. *A generator G for A_0 is also one for A if (i)* V *is faithful, or (ii) A is cotensored, or (iii) A is cocomplete.*

Proof. The image under V of the codomain of (4.2) is isomorphic to $\Pi_{\mathrm{G}} V_0(A(\mathrm{G}, \mathrm{A}), A(\mathrm{G}, \mathrm{B}))$. Suppressing this isomorphism, we have commutativity in

$$\begin{tikzcd}
A_0(\mathrm{A}, \mathrm{B}) \arrow[r, "\mathrm{V}\varphi"] \arrow[dr, "\psi"'] & \Pi_{\mathrm{G}} V_0(A(\mathrm{G}, \mathrm{A}), A(\mathrm{G}, \mathrm{B})) \arrow[d, "t"] \\
& \Pi_{\mathrm{G}} Set(A_0(\mathrm{G}, \mathrm{A}), A_0(\mathrm{G}, \mathrm{B}))
\end{tikzcd} \tag{4.4}$$

where ψ is the A_0-analogue of φ and t is a product of instances of

$$\mathrm{V}_{\mathrm{YZ}} : V_0(\mathrm{Y}, \mathrm{Z}) \to Set\,(\mathrm{VY}, \mathrm{VZ}) .$$

Since ψ is monomorphic, so is $\mathrm{V}\varphi$; whence (i) follows. As for (ii), we have for each $\mathrm{X} \in V$ commutativity in

$$\begin{tikzcd}
V_0(\mathrm{X}, A(\mathrm{A}, \mathrm{B})) \arrow[r, "V_0(\mathrm{X}, \varphi)"] \arrow[d, "\cong"'] & V_0(\mathrm{X}, \Pi_{\mathrm{G}}[A(\mathrm{G}, \mathrm{A}), A(\mathrm{G}, \mathrm{B})]) \arrow[d, "\cong"] \\
A_0(\mathrm{A}, \mathrm{X} \pitchfork \mathrm{B}) \arrow[r, "\mathrm{V}\varphi_{\mathrm{A}, \mathrm{X} \pitchfork \mathrm{B}}"'] & \Pi_{\mathrm{G}} V_0(A(\mathrm{G}, \mathrm{A}), A(\mathrm{G}, \mathrm{X} \pitchfork \mathrm{B})),
\end{tikzcd} \tag{4.5}$$

where the isomorphisms are the canonical ones. Since $\mathrm{V}\varphi$ is monomorphic, each $V_0(X, \varphi)$ is monomorphic in *Set*, so that φ is monomorphic. For (iii), recall from the last paragraph of Section 2 above that A_0 too is cocomplete. If F is the left adjoint of V, so that $FW = W.I$ for $W \in Set$, we have $W.G \simeq FW \otimes G$. In particular

$$A_0(G, A).G = VA(G, A).G \simeq FVA(G, A) \otimes G .$$

It is easy to see that, if $\varepsilon: FV \to 1$ is the counit of the adjunction, we have commutativity in

$$\begin{tikzcd}
\Sigma_G FV A(G, A) \otimes G \arrow[r, "\simeq"] \arrow[d, "\Sigma_G \varepsilon \otimes 1"'] & \Sigma_G A_0(G, A).G \arrow[d, "\tau_A"] \\
\Sigma_G A(G, A) \otimes G \arrow[r, "\sigma_A"] & A ,
\end{tikzcd}$$

where τ_A is the A_0-analogue of the σ_A of (4.3). So σ_A is epimorphic in A_0 because τ_A is so, and hence is epimorphic in the cocomplete A. ◊

A generator G for A is not in general one for A_0, even when A is complete and cocomplete; the unit object I is always a generator for the V-category V, but is one for V_0 only when V is faithful. However :

Proposition 4.2. *If the tensored A has a generator G and the ordinary category V_0 has a generator H, the objects $H \otimes G$ for $H \in H$ and $G \in G$ form a generator for A_0.*

Proof. Applying the representable V to the monomorphism (4.2) gives a monomorphism

$$V\varphi: A_0(A, B) \to \Pi_G V_0(A(G, A), A(G, B)) ;$$

thus the functors $A(G, -)_0 : A_0 \to V_0$ are jointly faithful. Since the functors $V_0(H, -) : V_0 \to Set$ are jointly faithful, so are the functors $V_0(H, A(G, -)_0)$, and so too are their isomorphs $A_0(H \otimes G, -)$. ◊

Remark 4.3. The closed categories V of practical interest all seem to have generators for V_0 ; this is certainly the case with all the examples given in Section 1.1 of [12]. For such a V, the *existence* of a generator for the cocomplete A is equivalent, by Propositions 4.1 and 4.2, to that of a generator for A_0. To this extent, the following Special Adjoint Functor Theorem for V-categories, which slightly generalizes Theorem III.2.2 of Dubuc [6], is no real advance on that for ordinary categories, as given in Mac Lane [14], from which it follows trivially if A_0, and not only A, has a cogenerator. We give it, nevertheless, so as to avoid the explicit extra hypothesis that V_0 has a generator.

Proposition 4.4. *Let the* V*-category* A *be complete, admit arbitrary intersections of monomorphisms, and have a cogenerator. Then any functor* $P : A \to V$ *that preserves small (indexed) limits and arbitrary intersections of monomorphisms is representable.*

Proof. Since P preserves cotensor products, it suffices by Theorem 4.85 of [12] to show that $P_0 : A_0 \to V_0$ admits a left adjoint. (This would be immediate from the classical Special Adjoint Functor Theorem if A_0 had a cogenerator.) Write G for the cogenerator of A, so that the σ_A of (4.3) now becomes a monomorphism

$$\sigma_A : A \longrightarrow \Pi_G A(A, G) \pitchfork G .$$

Given $X \in V$ and $f: X \to PA$, and writing g_G for the composite

$$A(A, B) \xrightarrow[P_{AG}]{} [PA, PG] \xrightarrow[[f,\ 1]]{} [X, PG],$$

we have a commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dd, "\rho"'] & & PA \arrow[d, "P\sigma_A"] \\
& & P\, \Pi_G A(A, G) \pitchfork G \arrow[d, "\beta"] \\
\Pi_G[[X, PG], PG] \arrow[rr, "\Pi_G[g_G,\ 1]"] & & \Pi_G[A(A, B), PG]
\end{tikzcd} ,$$

where ρ is the evident map and β is the isomorphism expressing the preservation by P of the product and the cotensor products. Since $P : A \to V$ preserves monomorphisms, $P\sigma_A$ is monomorphic. That P_0 has a left adjoint now follows by Day's form ([3] Theorem 2.1) of the Adjoint Functor Theorem. ◊

We make no essential use of the notion of a *strong generator* G for A, which occurs below only in counter-examples. We therefore content ourselves with the simple definition of this given in Section 3.6 of [12], which agrees in good cases with the stronger but more complicated general definition. So the small set G of objects of A is a *strong generator* when the functors $A(G, -)_0 : A_0 \to V_0$ for $G \in G$ are *jointly conservative*, meaning that $f : A \to B$ is invertible in A_0 whenever each $A(G, f)$ is invertible. If A is complete, a strong generator G is certainly a generator: the jointly-conservative $A(G, -)_0 : A_0 \to V_0$ are jointly faithful since A_0 has equalizers, so that the $V\varphi$ of (4.4) is monomorphic, whence so too is the $V_0(X, \varphi)$ of (4.5) and hence φ itself. If A is cocomplete and G is a strong generator, each σ_A of (4.3) is a strong epimorphism in A_0, by Proposition 4.3 of Im & Kelly [10], if A_0 admits all cointersections of strong epimorphisms or if A_0 is finitely complete; so that then σ_A is *a fortiori* an epimorphism in A_0, and hence in the cocomplete A; and once again G is a generator. (The definition of strong generator used by Day and Street in [4] is that σ_A be a *strong epimorphism in* A; we have not spoken here of strong epimorphisms in the V-category A, but for cocomplete A they are in fact the same thing as strong epimorphisms in A_0.)

Regarding G now as the name of the small *full* subcategory of A determined by the objects of G, we recall from Section 5.1 of [12] that G is *dense* in A if the functor

$$N : A \to [G^{op}, V] \qquad \text{given by} \qquad (NA)G = A(G, A)$$

is fully faithful. A dense G is certainly a strong generator, since the fully-faithful N_0 sends $f : A \to B$ to the V-natural transformation with components $A(G, f)$; and it is also a generator, the φ_{AB} of (4.2) being the composite of the isomorphism

$$N_{AB} : A(A, B) \to \int_G [A(G, A), A(G, B)]$$

with the inclusion of this end into $\Pi_G [A(G, A), A(G, B)]$.

5. TOTALITY AND ITS CONSEQUENCES.

Definition 5.1. The V-category A is said to be *total* when it admits, for each $U : A^{op} \to V$, the U-indexed colimit $U * 1_A$ of $1_A : A \to A$.

This definition, which makes no reference to large functor-categories, is that used by Day and Street in [4]. However :

Theorem 5.2. *Let V' be any extension of V (in the sense of Section 3 above) such that $[A^{op}, V]$ exists as a V'-category, and let*

$$Y : A \to [A^{op}, V]$$

be the Yoneda embedding. Then A is total if and only if Y admits a left adjoint Z. The value of Z is given on objects by $ZU = U * 1_A$.

Proof. If we are to have an adjunction

$$A(ZU, A) \simeq [A^{op}, V](U, YA), \tag{5.1}$$

the right side of (5.1) must exist in V, since the left side belongs to V. When this is the case, we have the desired adjunction, by Section 1.11 of [12], precisely when

$$[A^{op}, V](U, Y-) : A \to V$$

is representable for each U, say as $A(ZU, -)$. Since $YA = A(1_A -, A)$, to ask all this is exactly, by Section 2 above and (2.3) in particular, to ask for the existence of $ZU = U * 1_A$. ◊

Since V is complete and cocomplete, *so is any total* A, by the last paragraph of Section 3. However much more is true: a total A admits, besides small limits and colimits, certain large ones. Our results for limits and for colimits are not dual: we begin with the latter. A total A admits by definition the large colimits $U * 1_A$, but we have a generalization of this :

Theorem 5.3. *The following are equivalent:*

(i) A is total;

(ii) A admits the colimit $J * T$, where $J : K^{op} \to V$ and $T : K \to A$, whenever V admits the colimit $J * A(A, T-)$ for each $A \in A$.

Proof. $A(A, T-)$ is the composite

$$K \xrightarrow{\quad T \quad} A \xrightarrow{\quad Y \quad} [A^{op}, V] \xrightarrow{\quad E_A \quad} V$$

where E_A is evaluation at A. We use the observations and the language of the last paragraph of Section 3 above. If each $J * E_A YT$ exists, this is also $J' * E_A YT$, and the V'-category $[A^{op}, V]$ admits the colimit $J' * YT$. When A is total, the left adjoint Z of Y sends this to the colimit $J' * ZYT$ in A, or equally $J * ZYT$; which is $J * T$ since (Y being fully faithful) we have $ZY \cong 1$. For the converse, take

$$J = U : A^{op} \to V \quad \text{and} \quad T = 1 : A \to A ;$$

now

$$J * A(A, T-) = U * A(A, -)$$

does exist in V, being UA by the Yoneda isomorphism. So $U * 1_A$ exists, and A is total. ◊

Remark 5.4. Since $J * A(A, T-)$ certainly exists when K is small, Theorem 5.3 contains the assertion that *a total A is cocomplete.*

The following corollary of Theorem 5.3 in the case $V = Set$ expands a remark of Walters [23] ; see also Wood [24].

Theorem 5.5. *When $V = Set$ the follwing are equivalent:*

(i) A is total;

(ii) A admits the colimit of $T : K \to A$ whenever, for each $A \in A$, the set $\pi(A/T)$ of connected components of the comma category

A/T is small.

(iii) A admits the colimit of $T: K \to A$ whenever T is a discrete fibration with small fibres.

Proof. Recall from Section 2 above that colim $T = \Delta 1 * T$; so that

$$\Delta 1 * A(A, T-) = \mathrm{colim}\ A(A, T-),$$

which by (3.35) of [12] is $\pi(A/T)$. So (i) implies (ii) by Theorem 5.3. To give a discrete fibration $T : K \to A$ with small fibres is equally to give a functor $U : A^{op} \to Set$, whereupon $\pi(A/T)$ is the small set UA ; so (ii) implies (iii). In these circumstances $U * 1_A$ = colim T by (3.34) of [12]; so that (iii) implies (i) . ◊

We now turn to limits in A, and begin by introducing some nomenclature. Extending to V-categories the term of Isbell [11], we call A *compact* if $U : A^{op} \to V$ is representable whenever it preserves (as every representable must) all those limits that exist in A^{op} . Recall from the last paragraph of Section 3 that we consider only those limits in A^{op} indexed by some $J : K \to V$ with K a V-category; so that, taking $V = Set$, a locally-small ordinary category A is compact in our sense if $U : A^{op} \to Set$ is representable whenever it preserves all those lim T that exist in A^{op} where $T : K \to A^{op}$ has locally-small domain K . This restriction to locally-small K is absent in Isbell's original definition; yet our definition is in fact equivalent to his. If K is not locally small, we can factorize T into a $P : K \to K'$ that is bijective on objects and surjective on maps, and a $T' : K' \to A^{op}$ that is faithful. Now K' is locally small since A^{op} is so, while T and T' admit the same cones, giving lim T = lim T' if either exists.

Adapting to V-categories the language of Börger *et al.* in [2], we call A *hypercomplete* if it admits the limit $\{J, T\}$ whenever the necessary condition, that the right side

$$[K, V](J, A(A, T-)) \tag{5.2}$$

of (2.2) exists in V for each $A \in A$, is satisfied. That is to say, A admits all such limits as are not excluded by size-considerations. Compare this with the in-some-sense-dual (ii) of Theorem 5.3; while the latter is equivalent to totality, we shall see that hypercompleteness is strictly weaker. Note that, when $V = Set$, hypercompleteness of A becomes the condition that $T : K \to A$ have a limit whenever K is locally small and Cone(A, T) is small for each $A \in A$. Again, the restriction to locally-small K is missing in the original definition; but, for reasons given at the end of the last paragraph, this does not change the scope of the definition.

Our result on the existence of limits in total categories is the implication (i) => (iii) of the following theorem. The implication (i) => (ii) is in [19], while the other implications, for *ordinary* categories, are in [2].

Theorem 5.6. *For a V-category A, each of the assertions below implies the next :*

(i) A is total.

(ii) A is compact.

(iii) A is hypercomplete.

(iv) A is complete and admits arbitrary intersections of monomorphisms.

(v) A is complete.

Proof. Given $U : A^{op} \to V$, the colimit $U * 1_A$ that exists by Definition 5.1 when A is total is equally the limit $\{U, 1_{A^{op}}\}$ in A^{op}. If U preserves all limits that exist in A^{op}, it preserves this limit and is therefore representable by Theorem 4.80 of [12]. Thus a total A is compact.

Given $J : K \to V$ and $T : K \to A$ such that (5.2) exists in V for each $A \in A$, write $U : A^{op} \to V$ for the functor

$$\int_K [JK, A(-, TK)]$$

sending A to (5.2). Since the representable $A(-, TK) : A^{op} \to V$ preserves whatever limits exist, as does the representable $[JK, -] : V \to V$, and since $\int_K$ commutes with limits (see (3.20) of [12]), U preserves whatever limits exist. If A is compact, U is therefore representable, which is to say that $\{J, T\}$ exists; thus a compact A is hypercomplete.

A hypercomplete A is certainly complete, since (5.2) surely exists in V when K is small. It remains to show that a hypercomplete A admits arbitrary intersections of monomorphisms. Regard a family

$$(f_\lambda : B_\lambda \to C)_{\lambda \in \Lambda}$$

as in (4.1) as a functor $P : L \to A_0$, where ob $L = \Lambda + 1$ and L has, besides identity maps, one map $\lambda \to 1$ for each $\lambda \in \Lambda$; clearly L is locally small. We saw in the last paragraph of Section 2 how to express as an indexed limit $\{J, T\}$ the conical limit of P in A ; for this J and T, it follows from (3.50) of [12] that (5.2) is the limit of

$$A(A, P-)_0 : L \to V_0 .$$

When the f_λ are monomorphisms in A, (5.2) does indeed exist in V, being the intersection in V_0 of the monomorphisms $A(A, f_\lambda)$ — recall from Section 2 that we are supposing V_0 to admit all intersections of monomorphisms. So the hypercomplete A does admit the conical limit $\{J, T\}$ of P, namely the intersection in A of the monomorphisms f_λ. ◊

Remark 5.7. None of the implications in Theorem 5.6 can be reversed, even for $V = Set$. (i) The following example of a compact A, with a strong generator, that does not admit coequalizers and thus is certainly not total, was pointed out by Tholen [22]. Adámek [1] exhibits a monadic functor $A \to B$, where B is the category of graphs, such that A does not admit coequalizers. Since B has a strong generator, so has A. Moreover B, being locally finitely presentable, is total (as we shall see below) and hence compact. The compactness of B and the monadicity of $A \to B$ imply, as was shown by Rattray [15], the compactness of A. (ii) Adámek has communicated to the author the following example of a hypercomplete A with a strong generator that is not compact. $\mathrm{P} : Set \to Set$ is the functor sending A to its set of non-empty subsets, and an object of A is a set A with an action $\alpha : \mathrm{PA} \to \mathrm{A}$ satisfying $\alpha(\{a\}) = a$. (iii) The dual Grp^{op} of the category of small groups is complete and (being wellpowered) admits all intersections of monomorphisms, but is not hypercomplete. Let $\mathrm{T} : K \to Grp^{op}$ be the inclusion of the *discrete* subcategory of all simple groups, and let $\mathrm{J} = \Delta 1 : K \to Set$. Then (5.2) is the set of inductive cones over T in Grp with vertex A, which is clearly a small set; yet $\{\mathrm{J}, \mathrm{T}\}$, which would be the coproduct $\Sigma_{K \in K} K$ in Grp, does not exist. (iv) If ∞ is the inaccessible cardinal associated to our universe, seen as the ordered set (and hence the category) of all small ordinals, ∞^{op} is complete but does not admit arbitrary intersections of monomorphisms.

Remark 5.8. When $V = Set$, the Rattray-Adámek-Tholen result of Remark 5.7 shows that a compact category, unlike a total one, need not be even *finitely* cocomplete. Isbell gives an example in 2.8 of [11] of a compact category that *is* cocomplete but lacks large co-intersections even of *strong* epimorphisms.

Remark 5.9. A total category need not be hypercocomplete, even when it has a small dense subcategory; for we shall see below that Grp, which by Remark 5.7 is not hypercocomplete, is total. The author does not know whether a total category need admit arbitrary cointersections of epimorphisms, or even of strong ones.

6. SUFFICIENT CONDITIONS FOR TOTALITY.

In accordance with the notation of [12], given $\mathrm{T} : K \to A$ we write $\tilde{\mathrm{T}} : A \to [K^{op}, V]$ for the functor given by $\tilde{\mathrm{T}}\mathrm{A} = A(\mathrm{T}-, \mathrm{A})$.

Theorem 6.1. *The V-category A is total if and only if, for some V-category B and some extension V' of V (in the sense of Section 3 above) such that* $[B^{op}, V]$ *exists as a V'-category, A is equivalent to a full reflective subcategory of* $[B^{op}, V]$.

Proof. "Only if" being immediate from Theorem 5.2, we turn to the converse. Replace V' if necessary by a further extension of V' such that $[A^{op}, V]$ too exists as a V'-category. We have a fully-faithful

$P : A \to [B^{op}, V]$ with a left adjoint Q. Write $y : B \to [B^{op}, V]$ for the Yoneda embedding of B, retaining $Y : A \to [A^{op}, V]$ for that of A, and set $G = Qy : B \to A$. Then

$$PA \simeq [B^{op}, V](y\text{-}, PA) \simeq A(Qy\text{-}, A) = A(G\text{-}, A),$$

so that

$$\tilde{P} : [B^{op}, V] \to [A^{op}, V]$$

is given by

$$(\tilde{P}H)A = [B^{op}, V](PA, H) \simeq [B^{op}, V](A(G\text{-}, A), H).$$

Now we have

$$[A^{op}, V](U, \tilde{P}H) = \int_A [UA, (\tilde{P}H)A]$$

$$\simeq \int_A [UA, [B^{op}, V](A(G\text{-}, A), H)] = \int_A [UA, \int_B [A(GB, A), HB]]$$

$$\simeq \int_B \int_A [A(GB, A), [UA\ \ HB]] \underset{\text{(by Yoneda)}}{\simeq} \int_B [UGB, HB]$$

$$= [B^{op}, V](UG, H) = [B^{op}, V]([G^{op}, 1]U, H) ;$$

so that $\tilde{P}$ has the left adjoint $[G^{op}, 1]$. Since both P and $\tilde{P}$ have left adjoints, so does $\tilde{P}P$; but $\tilde{P}P \simeq Y$ since P is fully faithful, so that Y has a left adjoint and A is total. ◊

Corollary 6.2. (See Walters [23].) *Any full reflective subcategory of a total category is total.* ◊

Corollary 6.3. *If A is total, so is $[C, A]$ for any small C.*

Proof. $[C, A]$ is a full reflective subcategory of

$$[C, [A^{op}, V]] \simeq [C \otimes A^{op}, V]. \qquad ◊$$

Corollary 6.4. *The V-category V is total.*

Proof. Take $A = [B^{op}, V]$ in Theorem 6.1 where B is the unit V-category I, so that $[I^{op}, V] \simeq V$. ◊

Corollary 6.5. *A cocomplete A with a small dense subcategory B is total.*

Proof. If $K : B \to A$ is the inclusion, $\tilde{K} : A \to [B^{op}, V]$ is fully faithful since K is dense, and has the left adjoint $- * K$. ◊

Remark 6.6. Those total categories that are full reflective subcategories of some $[B^{op}, V]$ with *B small* are, by Propositions 5.15 and 5.11 of [12], precisely the cocomplete categories that have a small dense subcategory. Since all locally-presentable categories satisfy these conditions, both (see [8]) when $V = Set$ and (see [13]) when V is locally finitely presentable as a closed category, such categories are total. Certainly all categories of finitary algebras, such as *Grp*, are total.

There are of course reflective full subcategories of $[B^{op}, Set]$, with B small, that are *not* locally presentable: see Example 5.2.3 of [7].

Whether a total category need have a generator seems to be unknown; even for a lex-total category when $V = Set$ (see [17]). At any rate, we now turn to sufficient conditions for totality involving the existence of a generator. First, Proposition 4.4 gives :

Proposition 6.7. *A cocomplete category that admits arbitrary cointersections of epimorphisms and has a generator is compact.* ◊

Whether such a category need be total is unknown; we do, however, get totality if we strengthen the cocompleteness conditions to hypercocompleteness, provided that V_0 is hypercomplete. First:

Lemma 6.8. *For any* $U : A^{op} \to V$ *and* $B \in A$, *the set* $[A^{op}, V]_0(U, YB)$ *is small if* A *has a generator* G.

Proof. If $\alpha : U \to YB$ is a V-natural transformation, write

$$\beta_A : UA \to \Pi_G[A(G, A), A(G, B)]$$

for the composite of its component $\alpha_A : UA \to A(A, B)$ with the monomorphism φ of (4.2), and

$$\beta_{GA} : UA \to [A(G, A), A(G, B)]$$

for the G-component of β_A, noting that β_{GA} is still V-natural in A. To give the V-natural β_G is equivalently to give maps

$$\gamma_{GA} : A(G, A) \to [UA, A(G, B)]$$

V-natural in A, and hence by Yoneda to give a map $\delta_G : UG \to A(G, B)$. Since V_0 is locally small, there is only a small set of possibilities for δ_G and hence for β_G; since G is small there is only a small set of possibilities for β; since φ is monomorphic, α is determined by β, and there is only a small set of possibilities for α . ◊

Theorem 6.9. *A hypercomplete category with a generator is total, provided that* V_0 *is hypercomplete.*

Proof. By Definition 5.1, we need the existence in A of $U * 1_A$ for each $U : A^{op} \to V$; since A is hypercomplete, we have this if each $[A^{op}, V](U, YB)$ exists in V ; which is to say that the end

$$\int_A [UA, A(A, B)]$$

exists in V_0. By Section 2.1 of [12], this end is just a (large but locally small) classical limit in V_0, representing the set of V-natural transformations

$$\rho_A \colon X \to [UA, A(A, B)];$$

so that, by the hypercompleteness of V_0, it does exist if this set is small for each $X \in V$. But to give ρ is equally to give a V-natural

$$\sigma_A : X \otimes UA \to A(A, B),$$

or an element σ of $[A^{op}, V]_0(X \otimes U, YB)$ — which by Lemma 6.8 is a small set. ◊

Remark 6.10. For $V = Set$, this was observed by Tholen [22], but with "cocompact" rather than "hypercocomplete". The hypercompleteness of V_0 seems to be no real restriction in practice: we observed in Remark 4.3 that V_0 admits a generator in the cases of practical interest and since it also admits in these cases all cointersections of epimorphisms, it is compact by Proposition 6.7. In fact, as we shall observe below, we know of no practical case where V_0 is not total.

In the case $V = Set$, various partial versions of the following go back to Isbell [11], Börger *et al.* [2], and Tholen [22] :

Theorem 6.11. *Let V_0 be hypercomplete. Then if a complete A admits all intersections of monomorphisms and has both a generator and a cogenerator, it is total and cototal.*

Proof. Proposition 6.7 shows A to be cocompact; then Theorem 5.6 shows it to be hypercocomplete, and thus Theorem 6.9 shows it to be total. Now Theorem 5.6 shows it to be hypercomplete, and so Theorem 6.9 shows it to be cototal. ◊

Remark 6.12. This theorem provides us with many more examples of total categories, going beyond those of Remark 6.6 where there was a small dense subcategory. If $V = Set$ and A is the category of sets, or of topological spaces, or of compactly-generated topological spaces (with no separation axioms), or of Spanier's quasi-topological spaces, 1 is a generator and 2, with its chaotic structure, is a cogenerator; so that both A and A^{op} are total. The category of Hausdorff spaces has no cogenerator, but is total by Corollary 6.2, being reflective in the category of topological spaces. Again, the category of compact Hausdorff spaces has a generator and a cogenerator, as does that of Banach spaces and norm-decreasing maps, so that these too are total and cototal. So is the category of *pointed* compactly-generated topological spaces; here the discrete 2 is a generator, while the disjoint sum of 1 (as base-point) and the chaotic 2 is a cogenerator. Moreover, any complete lattice (large or small, but if large admitting *large* infima) is total : the empty G is both a generator and a cogenerator.

Taking $V = Set$, recall that a *topological* functor $P : A \to B$ is a faithful one that admits *initial structures* in the following sense. Given any family $g_\lambda : B \to PA_\lambda$ of maps in B, there is a family $f_\lambda : A \to A_\lambda$ of maps in A with

$$PA = B \quad \text{and} \quad Pf_\lambda = g_\lambda$$

with the property that, whenever $y : PC \to B$ is such that each $g_\lambda y$ is of the form Ph_λ for some $h_\lambda : C \to A_\lambda$, we have $y = Px$ for some $x : C \to A$. Then it is easy to see that P admits *final structures* as well, so that $P^{op} : A^{op} \to B^{op}$ too is topological; that P admits left and right adjoints given by the discrete and the chaotic structures; and that P creates limits and colimits. A category A is said to be *topological* if it admits a topological functor $P : A \to Set$; the first four categories mentioned in Remark 6.12 are topological. The following is a simple proof of a special case of a result of Tholen [22] ; for the general case see Theorem 6.15 below.

Theorem 6.13. *Take $V = Set$. If $P : A \to B$ is topological and B is total, so is A.*

Proof. We use the criterion of Theorem 5.5. Let $T : K \to A$ be such that each $\pi(A/T)$ is small; we are to show that colim T exists, and it suffices, since P creates colimits, to show that colim PT exists. Since B is total, we have only to show that each $\pi(B/PT)$ is small. Consider the family $(g : B \to PTK_g)$ of all maps with domain B and codomain of the form PTK. Since P is topological, there are maps

$$f_g : A \to TK_g \quad \text{with} \quad PA = B \quad \text{and} \quad P f_g = g \ .$$

It is clear that, if f_g and f_h lie in the same component of A/T, then g and h lie in the same component of B/PT ; so that $g \mapsto f_g$ induces an injection $\pi(B/PT) \to \pi(A/T)$, and $\pi(B/PT)$ like $\pi(A/T)$ is small. ◊

Remark 6.14. This result gives still more examples of totality. The category of topological groups (with no separation axioms) has neither a small dense subcategory, nor a cogenerator; yet it is total, since its forgetful functor to the category of groups is topological. Now the category of Hausdorff topological groups, as a reflective subcategory, is also total.

Tholen [21] defines the notion of *semi-topological functor* $Q : C \to B$ and shows in Theorem 8.3 of [21] that Q is semi-topological if and only if it is the restriction of some topological $P : A \to B$ to a full reflective subcategory C of A. Accordingly Corollary 6.2 combined with Theorem 6.13 gives the full result of Tholen [22] :

Theorem 6.15. *Let $V = Set$. If $Q : C \to B$ is semi-topological and B is total, so is C.* ◊

Remark 6.16. There is a different generalization of Theorem 6.13 due to Wood [24], where the semi-topological Q is replaced by what he calls a "total op-fibration" — a concept which, so far as the author knows, has not been investigated further. By Example 4.4 of Tholen [21], any monadic $Q : A \to Set$ is semi-topological, so that, as Tholen points out in [22],

Theorem 6.17. *The category of algebras for any monad on Set is total.* ◊

Remark 6.18. The last result goes beyond Remark 6.6, since the category of algebras need not be locally presentable when the monad lacks a rank (that is, fails for each small regular cardinal α to preserve α-filtered colimits).

Remark 6.19. Even in the case $V = Set$, the author does not know whether, when T is a small finitely-complete category and A is total, the full subcategory Lex $[T, A]$ of $[T, A]$ given by the left-exact functors is reflective and hence, by Corollaries 6.2 and 6.3, total. It is certainly so if A has a small dense full subcategory G, for then A is a full reflective subcategory of $[G^{op}, Set]$, and we apply the usual arguments (say of [7]) to get reflectivity in

$$[T, [G^{op}, Set]] \simeq [T \times G^{op}, Set];$$

but these arguments seem to need the smallness of $T \times G^{op}$.

Remark 6.20. For each of the examples of a (symmetric monoidal) closed category V given in Section 1.1 of [12], V_0 is total by Remark 6.6 or by Remark 6.12.

7. THE RELATION BETWEEN TOTALITY OF A AND THAT OF A_0.

If the ordinary category A is to be total whenever the V-category A is so, it follows from Corollary 6.4 that V_0 must be total. In fact this totality of V_0, which by Remark 6.20 is common in practice, also suffices.

Theorem 7.1. *Let V_0 be total. Then A_0 is total whenever the V-category*

A is total.

Proof. By Theorem 5.5 we are to show that $P : L \to A_0$ admits a colimit in A whenever L is a locally-small ordinary category and each $\pi(A/P)$ is small. The following argument relies heavily on the ideas in the last paragraph of Section 2. It more than suffices to show that P admits a (conical) limit in A. Let K be the free V-category on L, and let $T : K \to A$ and $J : K^{op} \to V$ correspond respectively to $P : L \to A_0$ and to $\Delta I : L^{op} \to V_0$; so that the colimit of P in A is the same thing as $J * T$, either existing if the other does. By Theorem 5.3, the existence of $J * T$ follows from that of each $J * A(A, T-)$: which is the same thing as the colimit in V of $A(A, P-)_0 : L \to V_0$. Since V is cotensored, the colimit of $A(A, P-)_0$ exists in V if it exists in V_0. Because V_0 is total, this colimit does exist in V_0 by Theorem 5.5, if each $\pi(X/A(A, P-)_0)$ is small. The total and hence cocomplete A admitting tensor products, to give a map $X \to A(A, PL)$ in V_0 is to give a map $X \otimes A \to PL$ in A_0; so that $\pi(X/A(A, P-)_0)$ is isomorphic to $\pi((X \otimes A)/P)$, which is small by our hypothesis on P. ◊

It is certainly not the case in general that A is total when A_0 is so. Let V be the closed category of small categories, so that V-categories are 2-categories, and take for A the 2-category with two objects A and B, with just one map $f : A \to B$ apart from identities, and with $A(A, B)(f, f)$ the infinite cyclic monoid, all other 2-cells being identities. Then A_0 is the category **2**, which as a complete lattice is total and cototal; yet A is not even complete, since the cotensor product $\mathbf{2} \pitchfork B$ fails to exist.

It is otherwise when each component $\varepsilon_X : FVX \to X$ of the counit of the adjunction $F \dashv V$ is a coequalizer in V_0. (To require V to be conservative is to ask somewhat less, namely that each ε_X be a strong epimorphism; but in every example in Section 1.1 of [12] where V is conservative, the ε_X are in fact coequalizers.)

We use the following "adjoint-triangle Theorem", due independently to Dubuc [5] and Huq [9]; we provide a proof because the published ones (see also Theorem 21.5.3 of Schubert [16] and Corollary 7 of Tholen [20]) seem to be either unnecessarily complicated or incomplete.

Lemma 7.2. *Let P, Q, R be ordinary categories and $W : P \to Q$ and $V : Q \to R$ functors such that V has a left adjoint F, each component $\varepsilon_Q : FVQ \to Q$ of the counit for which is a coequalizer, and such that VW has a left adjoint G. Then W has a left adjoint if P admits coequalizers.*

Proof. Write Q' for the full subcategory of Q determined by the FR for $R \in R$. Then Q is the closure of Q' under coequalizers, in the sense of Section 3.5 of [12], since for any $Q \in Q$ the map $\varepsilon_Q : FVQ \to Q$

is the coequalizer of some pair $x, y : S \to FVQ$ and hence of

$$x\,\varepsilon_S,\ y\,\varepsilon_S : FVS \to FVQ .$$

The functor

$$Q(FR, W\text{-}) : P \to Set'$$

(where Set' is the category of sets in a suitably large universe) is representable, since

$$Q\,(FR,\ W\text{-}) \simeq R\,(R,\ VW\text{-}) \simeq P\,(GR,\ \text{-}) .$$

Hence each $Q\,(Q,\ W\text{-})$ is representable, by Propositions 3.36 and 3.37 of [12], if P admits coequalizers. ◊

Theorem 7.3. *If each $\varepsilon_X : FVX \to X$ is a coequalizer in V_o, the V-category A is total whenever A_o is so.*

Proof. Write $W : A_o \to [A_o^{op}, V_o]$ for the functor given by

$$WA = (YA)_o = A(\text{-}, A)_o .$$

By Section 1.6 of [12], the composite of W with

$$[1, V] : [A_o^{op}, V_o] \to [A_o^{op}, Set]$$

is the Yoneda embedding $y : A_o \to [A_o^{op}, Set]$, which has a left adjoint by hypothesis. Moreover $[1, V]$ has the left adjoint

$$[1, F] : [A_o^{op}, Set] \to [A_o^{op}, V_o],$$

with counit $[1,\varepsilon]$. The component of $[1, \varepsilon]$ at $N : A_o^{op} \to V_o$ is $\varepsilon N : FVN \to N$. Since small limits and colimits in $[A_o^{op}, V_o]$ are formed pointwise, εN is the coequalizer of its kernel-pair; for the same is true of each $(\varepsilon N)_A = \varepsilon_{NA}$. So, by Lemma 7.2, W has a left adjoint H, and we have an isomorphism

$$A_o(HN, A) \simeq [A_o^{op}, V_o](N, (YA)_o)$$

natural in A. Now take $N = U_o$ where $U : A^{op} \to V$. Since each ε_X is epimorphic, V is faithful; so that, by Section 1.3 of [12], a V-natural $\alpha : U \to YA$ is the same thing as a natural $\alpha : U_o \to (YA)_o$, giving

$$[A^{op}, V]_o(U, YA) = [A_o^{op}, V_o](U_o, (YA)_o) .$$

Recalling that $Y_o : A_o \to [A^{op}, V]_o$ has $Y_oA = YA$, we see that we have an isomorphism

$$A_o(HU_o, A) \simeq [A^{op}, V]_o(U, Y_oA)$$

natural in A ; that is, that Y_0 has a left adjoint. Since V is conservative, it now follows from Section 1.11 of [12] that $Y : A \to [A^{op}, V]$ has a left adjoint, as desired. ◊

Among examples of such V are pointed sets, abelian groups, R-modules, and Banach spaces. The assertion of the Introduction, that an additive A is total precisely when A_0 is so, now follows from Theorems 7.1 and 7.3.

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