TIMOTHY PORTER

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CROSSED MODULES IN Cat AND A BROWN-SPENCER THEOREM FOR 2-CATEGORIES

by Timothy PORTER

RÉSUMÉ. Si C et H sont des catégories, on appelle C-structure sur H une extension de catégories $H \hookrightarrow K \rightarrow C$ (au sens de Hoff) qui est scindée. La catégorie des C-structures est équivalente à la catégorie $\mathrm{Groupes}^C$. Un module croisé est une C-structure particulière. La catégorie des modules croisés est équivalente à une sous-catégorie de 2-Cat qu'on caractérise. Comme corollaires, on retrouve deux théorèmes de Brown-Spencer reliant les modules croisés aux catégories internes.

Recently there has been renewed interest in crossed modules. The new results obtained have been in two main areas. There have been a series of papers linking crossed modules, their higher dimensional analogues, crossed extensions, and the cohomology of groups (Holt [9], Huebschmann [10] and also MacLane's historical appendix to [9]). Other results have identified the category of crossed modules with the category of internal categories in the category of groups. (Here the principal paper is Brown-Spencer [4], but one should also note Brown-Spencer [5] and Brown-Higgins [1, 2].)

Aspects of the cohomology of categories - for instance the theory of derived functors of lim (cf. Jensen [12]) - have proved to be of great use in homological algebra. Considering the power of crossed module techniques in combinatorial group theory and group cohomology (cf. Brown-Huebschmann [3] and Huebschmann [11] for instance) it may be worth while developing a similar theory to that of Holt and Huebschmann for the cohomology of categories. In low dimensions an approach related to this, based on ideas of Ehresmann (cf. [6]), has been made by Hoff [8].

In [7], Gerstenhaber says of algebra cohomology — "Fixing a suitable concept of extension normalises H^2 and therewith the theory". Thus in the initial development of a crossed extension interpretation of the cohomology of categories one needs a "good" notion of extension of categories. To the author's knowledge three such exist - Nico [14], Wells [16] and Hoff [8], the paper already cited. Although more restrictive the extensions of Hoff seem to be the most convenient for this work. Using them we develop the elementary theory of crossed modules in the category of small categories with fixed object set. Some of the results are quite well known so proofs are often sketched or omitted altogether.

Since the "crossed complexes over a groupoid" of Brown-Higgins [1] seemed to be related to some concept of crossed complex in the category of groupoids (with fixed base) it seemed useful to clarify this connection by proving a version of the first Brown-Spencer Theorem [4] linking crossed modules with internal categories. Using a result of Spencer [15], this theorem has as corollary the second Brown-Spencer Theorem [5] on crossed modules and special double groupoids with connection. This link is not at all surprising and indicates that the notion of crossed module introduced here is a reasonably good one.

The development of the cohomology theory must wait whilst several problems are resolved.

I would like to thank R. Brown for helpful comments on an earlier version of this work.

1. EXTENSIONS OF CATEGORIES.

Let O-Cat denote the category of all (small) categories having the set O as their set of objects and all functors which are the identities on objects. We call such categories, O-categories. The following definition is due to Hoff [8].

A sequence in O-Cat

$$E : H \xrightarrow{i} K \xrightarrow{\pi} C$$

is an *extension*, if i identifies H to a subcategory of K and C is a quotient category of K, the projection functor being π, such that for all $k_1, k_2 \in K$,

(*) $\pi(k_1) = \pi(k_2) \Leftrightarrow$ there is a unique $h \in H$ such that $k_2 = k_1 \circ i(h)$.

Remarks. a) We write composition, within the O-categories, on the right - if $k_1 : x \to y$, $k_2 : y \to z$, then $k_1 \circ k_2 : x \to z$ - but all other compositions, e.g. of functors, on the left.

b) $\pi i(H) = O$.

c) If h, k are such that $i(h) \circ k$ is defined then there is a unique $h_1 \in H$ such that

$$i(h) \circ k = k \circ i(h_1).$$

Proposition 1. *If*

$$E : H \xrightarrow{i} K \xrightarrow{\pi} C$$

is an extension of O-categories then H *is a disjoint union of an O-indexed family of groups.*

Proof. πi (H) = O implies that H is a disjoint union of monoids, whilst (*) gives immediately that each $H\{x\}$ is a group. ◊

The extension E is *split* if there is a functor

$$s : C \to K \quad \text{with} \quad \pi s = id_C .$$

We shall say, in this case, that H has a C-*structure*.

A C-structure on H is an "action" of C on H and corresponds to a functor from C to Groups in the following way.

Proposition 2. *If*

$$E : H \xrightarrow{i} K \underset{s}{\overset{\pi}{\rightleftarrows}} C$$

is a C-structure on H, then for $h \in H\{x\}$, $c \in C(x, y)$, *there is a unique* ${}^c h \in H\{y\}$ *satisfying*

(i) $\qquad i(h) \circ s(c) = s(c) \circ i({}^c h).$

(ii) $h \mapsto {}^c h$ *is a morphism of groups.*

The assignment

$$F_E(x) = H\{x\} , \qquad F_E(c)(h) = {}^c h$$

is functorial, $F_E : C \to$ Groups.

Varying E within the category of C-structures (which is defined in the obvious way) one gets an equivalence of categories

$$F_{(\ )} : \text{C-Structures} \to \text{Groups}^{\hat{C}} .$$

Proof. Much of this is routine checking, using (*). The fact that $F_{(\ )}$ is an equivalence of categories is best proved by using a version of Ehresmann's semi-direct product construction [6] to produce a quasi-inverse. As we will need this construction later we shall sketch the construction.

Let $F : C \to$ Groups be a functor. Let H_F be the category coproduct (disjoint union) of the groups

$$\{ F(x) : x \in O\} .$$

If $h \in F(x)$ and $c : x \to y$ in C then it is convenient to write $F(c)(h) = {}^c h$. Now let $C \tilde{\times} H_F$ be the O-category with

$$(C \tilde{\times} H_F)(x, y) = C(x, y) \times H_F\{y\} \qquad (\text{or } C(x, y) \times F(y))$$

and composition defined by

$$(c, h) \circ (c', h') = (c \circ c', {}^{c'}h \circ h')$$

($C \tilde{\times} H_F$ is the semi-direct product of C with H_F). The sequence

$$E_F : H_F \xrightarrow{i} C \tilde{\times} H_F \underset{s}{\overset{\pi}{\rightleftarrows}} C$$

with

$$i(h) = (e_y\, ,\, h),\ \pi(c,\, h) = c\ ,\quad s(c) = (c,\, e_y),$$

is a split extension. The functor $F \mapsto E_F$ is quasi-inverse to $F_{(\ )}$, as is easily checked. ◊

2. CROSSED MODULES AND INTERNAL GROUPOIDS IN O-Cat.

Given a C-structure on A and a functor $\partial : A \to C$ we say (A, ∂) is a *crossed module* if ∂ satisfies the conditions :

(i) $\quad \partial(a) \circ c = c \circ \partial({}^c a) \quad$ for $\quad c \in C(x, y),\ a \in A\{x\}$.

(ii) $\quad a' \circ ({}^{\partial(a')}a) = a \circ a' \quad$ for $\quad x \in O,\ a', a \in A\{x\}$.

If C is a group (so O has only one element) then this notion reduces to the classical one. If C is a groupoid then it corresponds to a "crossed module over a groupoid", the low dimensional case of the crossed complexes over a groupoid of Brown-Higgins (cf. [1] and its reference list).

As in these cases, if $a \in A\{x\}$ satisfies $\partial(a) = e_x$, the identity at x in C, then for any $a' \in A\{x\}$, $a' \circ a = a \circ a'$. Also for any $c : x \to y$, ${}^c a$ satisfies $\partial({}^c a) = e_y$ because of the uniqueness clause in condition (*).

Summing up we have :

Proposition 3. *If $\partial : A \to C$ is a crossed module in* O-Cat, $\operatorname{Ker} \partial = K$, *say, corresponds to a functor*

$$F_K : C \to \text{Abelian Groups}.$$

Thus the kernel of ∂ is a C-module in the usual sense. The following lemma is easily proved.

Lemma. *If $\partial : A \to C$ is a crossed module, then*

$$d_1(c,\, a) = c \circ \partial(a)$$

defines a functor

$$d_{\_} : C \tilde{\times} A \to C\ .$$

Using $d_0 = \pi$, d_1 as above and $s : C \to C \tilde{\times} A$ as in the proof of Proposition 2 we get a diagram

$$C(\partial) : C \tilde{\times} A \;\underset{s}{\overset{d_0,\ d_1}{\rightrightarrows}}\; C\ .$$

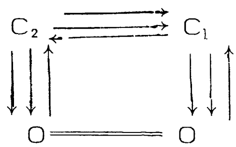
Proposition 4. *Defining* $*$ *by*

$$(c, a) * (c \circ \partial(a), a') = (c, a \circ a')$$

gives $C(\partial)$ *the structure of an internal groupoid in* O-Cat.

Proof. The verification of the category axioms is simple. The inverse for $*$ of (c, a) is $(c \circ \partial(a), a^{-1})$ where a^{-1} is the inverse in $A\{y\}$. ◊

Thus we can represent $C(\partial)$ by a diagram (in Sets)



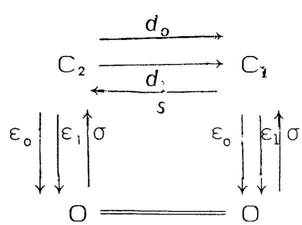
and consider it as a 2-category with object set O and with all its 2-cells invertible.

Writing 2-O-Cat for the category of 2-categories with O as their set of objects, the above construction is easily checked to give a functor

$$C(\ ) : X\ \mathrm{Mod}(O) \to \text{2-O-Cat},$$

where X Mod(O) is the category of crossed modules in O-Cat.

Of course since, in each $C(\partial)$, the 2-cells are invertible $C(\ )$ is not an equivalence. However even restricting to 2-categories with this property does not seem to be sufficient. In fact 2-categories of the form $C(\partial)$ satisfy the following two conditions, the notation used being relative to the diagram



(A) For each $x \in O$, the set

$$\{c \in C_2 \mid d_0(c) = \sigma(x)\} = K\{x\},$$

say, is a group for both induced compositions $\circ$ and $*$ and for each $k \in K\{x\}$ the inverses for $\circ$ and $*$ are related by

$$k_*^{-1} = sd_1(k) \circ k_\circ^{-1}.$$

(B) Given $c \in C_2$, then it can be uniquely written as

$$c = s(d_0(c)) \circ k \quad \text{with} \quad k \in K\{\varepsilon_1(c)\} .$$

These two properties are found by mere observation and are not very elegant. The following proposition connects them with other known properties.

Proposition 5. *(a) If (A) and (B) are satisfied then the 2-cells of* C *are all invertible (i.e.* $(C_2, d_0, d_1, s, *)$ *is a groupoid).*

(b) If $(C_2, \varepsilon_0, \varepsilon_1, \sigma, \circ)$ *is a groupoid then (A) and (B) are satisfied.*

(c) If $(C_1, \varepsilon_0, \varepsilon_1, \sigma, \circ)$ *is a groupoid, then (B) is satisfied.*

Proof. (a) Given $c \in C_2$, $c = s(d_0(c)) \circ k$ by (B),

$$c_*^{-1} = s(d_1(c)) \circ k_\circ^{-1}$$

as is easily checked by using (A).

(b) (A) is obvious. For (B) set $k = s(d_0(c)) \circ c$ and check $k \in K\{\varepsilon_1(c)\}$.

(c) If $(C_1, \varepsilon_0, \varepsilon_1, \sigma, \circ)$ is a groupoid we can, using the inverse in C, we get $a = s(d_0(c)^{-1}) \circ c$. ◊

Theorem. *The functor*

$$C(\ ) : X\,\mathrm{Mod}(O) \to \text{2-O-Cat}$$

induces an equivalence between X Mod(O) *and the full subcategory of* 2-O-Cat *determined by those 2-categories satisfying (A) and (B).*

Proof. The quasi-inverse for C( ) is constructed as follows. Given a 2-category as in the above diagram, which satisfies (A) and (B), then the extension

$$K \longrightarrow C_2 \xrightarrow{d_0} C_1$$

is split by s and $\partial : K \to C_1$ is given by $d_1|K$.

The verification that ∂ is a crossed module is quite long but is a straightforward use of the interchange law in the 2-category, i.e.,

$$(a * b) \circ (c * d) = (a \circ c) * (b \circ d)$$

whenever one side is defined. ◊

Corollary 1. *C( ) restricts to give an equivalence of categories between the category of crossed modules over* O*-groupoids and the category* 2-O-Groupoids, *considered as a full subcategory of* 2-O-Cat.

Corollary 2 *(case : O, a singleton).* C( ) *restricts to give an equivalence of categories between the category of crossed modules and the category of internal groupoids in the category of Groups.*

This is essentially the statement of the first Brown-Spencer Theorem [4]. Corollary 1 is a generalisation of the second Brown-Spencer Theorem [5], the connection being given by Spencer's results on 2-categories and double categories with connection, [15]. Corollary 1 can also be considered to be a special case of one of the equivalences given by Brown-Higgins.

Restricting the general theorem to the singleton case gives, of course, a theorem on crossed modules in the category of monoids. It should be pointed out that these do not seem to be the same concept as that considered by Lavendhomme and Roisin [13] in this context.

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Department of Pure Mathematics
University College of North Wales
BANGOR, Gwynedd, LL57 2UW. U.K.