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# Proarrows II

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# PROARROWS II

by R. J. WOOD

**RÉSUMÉ.** On continue une étude récente (cf. "Cahiers" XXIII-3, 1982) sur les homomorphismes de bicatégories $(\ )_* : K \to M$ qui partagent certaines propriétés avec $(\ )_* : CAT \to PROF$. Ici on impose sur $(\ )_*$ des conditions d'exactitude sur des collages dans $M$, et la structure qui en résulte est assez riche. Notamment, on a un système de factorisation utile dans $K$, et les colimites locales existent dans $M$. Ces dernières donnent un calcul de matrices pour les morphismes de collages dans $M$. $(\ )_* : TOP \longrightarrow TOPLEX^{co}$, "oublier l'image inverse", est encore un modèle pour le système élargi d'axiomes. Quelques propositions élémentaires sur les morphismes géométriques de topoi se retrouvent par ce formalisme simple.

## 0. INTRODUCTION.

This paper is a sequel to [WD1]. The objects of study are homomorphisms of bicategories, $(\ )_* : K \to M$, which satisfy the following :

**Axiom 1.** The objects of $M$ are those of $K$ and $(\ )_*$ is the identity on objects.

**Axiom 2.** $(\ )_*$ is locally fully faithful.

**Axiom 3.** For every arrow $f$ in $K$, $f_*$ has a right adjoint $f^*$ in $M$ .

Such a homomorphism $(\ )_*$ is said to *equip $K$ with proarrows*, and is referred to as *proarrow equipment.*

There are many examples but two deserve particular attention. The paradigm is $(\ )_*$: $CAT \to PROF$ , where for $f: A \to B$,

$$f_*(b, a) = B(b, af).$$

A number of aspects of category theory are simplified by the introduction of profunctors - modulo their somewhat complicated composition. An axiomatic approach preserves the simplification while forgetting this complexity. When generalized category theory is contemplated in the form of V-*CAT* , S-*ind CAT*, or *cat*(S) for monoidal V or left exact S this advantage becomes quite significant. Motivated chiefly by this, [WD1] pursued a development of a fragment of formal category theory along the lines set out in [S&W] for "Yoneda structures", a related notion. Axioms 1, 2 and 3 do not take us as far as those for Yoneda structures. Missing is an ingredient which allows one to speak of "bijective on objects" arrows (here called "surjections", for reasons which will become clear shortly). In Section 2 we impose Axiom 5 which makes a strong claim about existence of Kleisli objects in *M*. Section 2 then quickly completes those aspects of the program in [S&W] which are compatible with proarrow equipment.

A second leading example is provided by "forget the inverse image", $(\ )_*: TOP \to TOPLEX^{co}$, where *TOP* is topoi and geometric morphisms and $TOPLEX^{co}$ is topoi and left exact functors. It does not follow from Axioms 1, 2 and 3 that $(\ )_*: K \to M$ enjoys a universal property. However *PROF* is equivalent to codiscrete cofibrations in *CAT* and it transpires that $TOPLEX^{co}$ is equivalent to codiscrete cofibrations in *TOP*. The latter result was the main focus of [RW1] and the example was pursued further in [RW2]. Similar results for abelian categories and geometric morphisms were presented in [RW3].

Section 2 is independent of Section 1, which deals briefly with finite sums in *K* and *M* and concludes with a simple Axiom (4) that is assumed together with Axiom 5 for Sections 3 through 6. Local colimits, as described in [ST1], are met in Section 1 and subsequently. We remind the reader that a local colimit in *M* is a colimit in a hom category *M*(B, A) that is preserved by composing with all B'→ B and all A → A'. Some of our results are similar to those in [ST1].

Section 3 establishes the existence of Artin glueing and a number of propositions that follow easily from it. A matrix calculus is employed and this is developed somewhat further in Section 5. Section 4, "Surjections Revisited", continues the theory of Section 2 using, by then available, local coequalizers in *M*. A monadicity theorem for *M* is the key result.

Section 6 deals exclusively and in some detail with the verification of Axioms 4 and 5 for the examples *CAT* and *TOP*. The reader may wish to consult it immediately after the introduction of Axiom 5. The latter is initially presented in a non-concise form that allows one to work with it immediately - at the expense of masking an overview structure. In Section 7 we introduce a formalism that reveals this structure and shows the common nature of Axioms 4 and 5.

Some remarks about notation and terminology are in order. $\Phi\Psi$ denotes the composite

$$\cdot \xrightarrow{\Phi} \cdot \xrightarrow{\Psi} \cdot \ .$$

We wrote $\Phi \otimes \Psi$ for it in [WD1]. Whenever possible we write $f$ for $f_*$ and refer to such (and their isomorphs) as *representables*. For the most part we ignore coherence and explicit descriptions of bi-phenomena, saying "pushout" etc. when we mean bi-pushout etc. We hope that the loss of precision is off-set by a gain in clarity. For $f : A \to B$ in $K$ we write $\tilde{f}$ for the unit and $\underset{\sim}{f}$ for the counit of the $f \dashv f^*$ adjunction in $M$. If $\tilde{f}$ is an isomorphism we say that $f$ is an *inclusion*. The term *fully faithful* was employed in [WD1].

The author wishes to thank R.D. Rosebrugh, R. Paré and F.W. Lawvere for useful discussions. Lawvere and Paré have suggested independently that Axiom 1 be dispensed with. The theory that results will be developed elsewhere ; however, some preliminary remarks as to a further unification of the two leading examples above seem appropriate.

Call a homomorphism satisfying Axioms 2 and 3 *proequipment*. Write *cat* for the bicategory of small categories and *TOT* for the bicategory of total objects in *CAT* (relative to $P = set^{(\ )^{op}}$, see [S&W] or [WD1]) and cocontinuous functors. Then $cat \to TOT$ given by

$$(f : A \to B) \mapsto (f_! : set^{A^{op}} \to set^{B^{op}}),$$

where $f_!$ is left Kan extension along $f$, is proequipment. In [WD1] we pointed out that taking the full image of a homomorphism satisfying Axioms 2 and 3 produces proarrow equipment. Applying this to $cat \to TOT$ we recover $(\ )_* : cat \to prof$. Any cocontinuous functor between total categories has a ($CAT$) right adjoint, so $TOT \to CAT$ is also proequipment.

Write *LEX* (respectively *lex* ) for the bicategory of left exact objects, left exact arrows and arbitrary transformations in *CAT* (resp. *cat*). Noting that the left Kan extension of a left exact functor is left exact and observing dualities, we see that $lex^{co} \to TOP$ given by

$$(f : A \to B) \mapsto (f^! : set^{A^{op}} \to set^{B^{op}}),$$

where $f^!$ is *right* Kan extension along $f$, is proequipment. Clearly $TOP \to LEX^{co}$ is proequipment too.

It is interesting to note these examples together :

$$cat \to TOT \to CAT, \qquad lex^{co} \to TOP \to LEX^{co}.$$

Certainly, general questions about change of universe are provoked. Moreover, the second row is essentially $lex^{co}$ of the first. This is because *TOP* could be replaced by Grothendieck topoi and Freyd has shown that the latter are precisely the total objects in *LEX*.

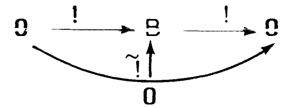
## 1. FINITE SUMS.

***Proposition 1.*** *If $K$ has an initial object, $\mathbf{0}$, which is preserved by $(\ )_* : K \to M$ and $(\ )^* : K^{coop} \to M$ preserves the terminal object of*

$K^{coop}(\mathbf{0})$, *then*

*(i) For all* B *in* $K$, $! : \mathbf{0} \to B$ *is an inclusion,*
*(ii)* $M$ *has initial objects locally.*

**Proof.** (i)




is an isomorphism since $\mathbf{0}$ is initial in $M$.

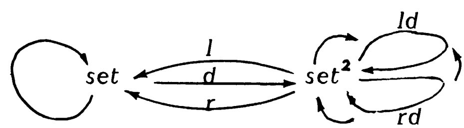
(ii) $(B \xrightarrow{0} A) \simeq (B \xrightarrow{!^*} \mathbf{0} \xrightarrow{!} A)$. For given $\Phi : B \to A$,

$$\frac{\dfrac{!^*! \to \Phi : B \to A}{! \to !\Phi : \mathbf{0} \to A}}{*}$$

Hence $0 : B \to A$ is initial in $M(B, A)$. For any $\Phi : A \to A'$, $0\Phi \simeq 0$ since $(!_A)\Phi \simeq !_{A'}$. (Observe that thus far we have not used the hypothesis concerning $(\ )^*$.) For any $\Phi : B' \to B$, $\mathbf{0}$ being terminal in $M$ yields

$$\Phi(!_B)^* \simeq (!_{B'})^* \quad \text{and hence} \quad \Phi 0 \simeq 0. \qquad \Diamond$$

**Counterexample 2.** If $K$ has an initial object $\mathbf{0}$ which is preserved by $(\ )_* : K \to M$, it does not follow that $\mathbf{0}$ is also terminal in $M$. Let $K$ be the subbicategory of $CAT$, $d : set \to set^2$, where $d$ is the diagonal. (So $K$ is isomorphic to the locally discrete bicategory **2** and *set* is initial in $K$.) Let $M$ be the subbicategory of $CAT$ :




where $l \dashv d \dashv r$ in $CAT$, and the non-identity transformations shown are those arising from the adjointness. Then, we still have $l \dashv d \dashv r$ in $M$, the inclusion of $K$ in $M$ is proarrow equipment, *set* is initial in $M$ but *set* is not terminal in $M$.

***Proposition 3.*** *If* $K$ *has finite sums which are preserved by* $(\ )_* : K \to M$ *and* $(\ )^* : K^{coop} \to M$ *preserves finite products, then*

*(i)* $M$ *has finite sums locally ;*

*for every sum diagram,* $i : A \to A{+}B \leftarrow B : j$, *in* $K$, *with* $A{+}B$ *denoted* $A \oplus B$ *when regarded as an object of* $M$,

*(ii) the injections are inclusions,*
*(iii)* $0 \xrightarrow{\sim} ij^*$ *and* $0 \xrightarrow{\sim} ji^*$ ,
*(iv)* $A \oplus B \simeq i^*i + j^*j$.

**Proof.** (i) If $\Phi$, $\Psi : B \to A$, then

$$(B \xrightarrow{\Phi + \Psi} A) \simeq (B \xrightarrow{c^*} B \oplus B \xrightarrow{\Phi \oplus \Psi} A \oplus A \xrightarrow{c} A),$$

where $c: A+A \to A$ is the codiagonal. For given $\Gamma: B \to A$ :

$$\frac{\dfrac{c^*(\Phi \oplus \Psi)c \to \Gamma: B \to A}{(\Phi \oplus \Psi)c \to c\Gamma: B \oplus B \to A}}{\dfrac{\begin{bmatrix}\Phi\\ \Psi\end{bmatrix} \to \begin{bmatrix}\Gamma\\ \Gamma\end{bmatrix} : B \oplus B \to A}{\Phi \to \Gamma, \quad \Psi \to \Gamma : B \to A}}$$

Hence $\mathcal{M}(B, A)$ has finite sums. For any $\Gamma: A \to A'$,

$$(\Phi \oplus \Psi)c_A\Gamma \simeq \begin{bmatrix}\Phi\Gamma\\ \Psi\Gamma\end{bmatrix} \simeq (\Phi\Gamma \oplus \Psi\Gamma)c_{A'} .$$

So $(\Phi + \Psi)\Gamma \simeq \Phi\Gamma + \Psi\Gamma$. (Again, observe that thus far we have not used the hypothesis concerning $(\ )^*$; it does not follow from the other hypotheses, as Counterexample 2 surely suggests.) $c^* : B \to B \oplus B$ is the product diagonal. For any $\Gamma: B' \to B$,

$$\Gamma(c_B)^*(\Phi \oplus \Psi) \simeq [\Gamma\Phi, \Gamma\Psi] \simeq (c_{B'})^*(\Gamma\Phi \oplus \Gamma\Psi).$$

So $\Gamma(\Phi + \Psi) \simeq \Gamma\Phi + \Gamma\Psi$.

(ii), (iii) and (iv). Consider the diagram

$$A \underset{\begin{bmatrix}A\\ 0\end{bmatrix}}{\overset{i}{\rightleftarrows}} A \oplus B \underset{\begin{bmatrix}0\\ B\end{bmatrix}}{\overset{j}{\leftrightarrows}} B .$$

For any $\Gamma: A \to C$ and any $\begin{bmatrix}\Phi\\ \Psi\end{bmatrix}: A \oplus B \to C$ we have :

$$\frac{\dfrac{\dfrac{\begin{bmatrix}A\\ 0\end{bmatrix}\Gamma \to \begin{bmatrix}\Phi\\ \Psi\end{bmatrix}}{\begin{bmatrix}\Gamma\\ 0\end{bmatrix} \to \begin{bmatrix}\Phi\\ \Psi\end{bmatrix}}}{\Gamma \to \Phi}}{\Gamma \to i\begin{bmatrix}\Phi\\ \Psi\end{bmatrix}} .$$

Hence $i \dashv \begin{bmatrix}A\\ 0\end{bmatrix}$ and thus $i^* \simeq \begin{bmatrix}A\\ 0\end{bmatrix}$. Similarly, $j^* \simeq \begin{bmatrix}0\\ B\end{bmatrix}$. So

$$A \xrightarrow{\sim} ii^*, \quad B \xrightarrow{\sim} jj^*, \quad 0 \xrightarrow{\sim} ij^* \quad \text{and} \quad 0 \xrightarrow{\sim} ji^* .$$

This proves (ii) and (iii). For any $\begin{bmatrix}\Phi\\ \Psi\end{bmatrix}: A \oplus B \to A \oplus B$ we have :

$$\frac{i^*i + j^*j \to \begin{bmatrix}\Phi\\ \Psi\end{bmatrix} : A \oplus B \to A \oplus B}{\begin{bmatrix}A\\ 0\end{bmatrix}i + \begin{bmatrix}0\\ B\end{bmatrix}j \to \begin{bmatrix}\Phi\\ \Psi\end{bmatrix}: A \oplus B \to A \oplus B}$$

$$\begin{array}{c}
\left[{i \atop 0}\right] + \left[{0 \atop j}\right] \to \left[{\Phi \atop \Psi}\right] : A \oplus B \to A \oplus B \\
\hline
\left[{i \atop 0}\right] \to \left[{\Phi \atop \Psi}\right],\ \left[{0 \atop j}\right] \to \left[{\Phi \atop \Psi}\right] : A \oplus B \to A \oplus B \\
\hline
i \to \Phi : A \to A \oplus B, \quad j \to \Psi : B \to A \oplus B \\
\hline
\left[{i \atop j}\right] \to \left[{\Phi \atop \Psi}\right] : A \oplus B \to A \oplus B \\
\hline
A \oplus B \to \left[{\Phi \atop \Psi}\right] : A \oplus B \to A \oplus B \ .
\end{array}$$

So $i^*i + j^*j \simeq A \oplus B$ as claimed. ◊

From $i^*i \simeq A$ and $ij^* \simeq 0$ we see that $i \simeq [A, 0]$. Similarly, $j \simeq [0, B]$. Indeed, if $(\ )_*$ and $(\ )^*$ satisfy the hypotheses of proposition 3, then $M$ admits a matrix calculus for proarrows

$$B_1 \oplus \dots \oplus B_m \longrightarrow A_1 \oplus \dots \oplus A_n$$

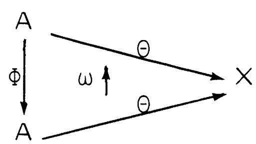
which is analogous to that available for a category with finite direct sums and abelian monoid valued homs. This will become most useful when we discuss glueing and local finite colimits. A transformation between matrices is a matrix of transformations and will be denoted with the help of brackets. On the other hand, transformations $\varphi : \Phi \to \Gamma$, $\psi : \Psi \to \Gamma : B \to A$ give rise to a transformation $\Phi + \Psi \to \Gamma : B \to A$ which will be denoted $\left({\Phi \atop \Psi}\right)$. This brackets-global versus parentheses-local convention will be extended without further ado to other tuples of transformations in the sequel.

Beginning in Section 3 we will assume :

**Axiom 4.** $K$ has finite sums which are preserved by $(\ )_*$ and $(\ )^*$ preserves finite products. ◊

## 2. THE KLEISLI CONSTRUCTION.

We recall some basic terminology concerning monads $(A, \Phi) = (A, \Phi, \eta, \mu)$ in a bicategory $M$. An *opalgebra* for $(A, \Phi)$ is an arrow $\Theta : A \to X$, together with a transformation



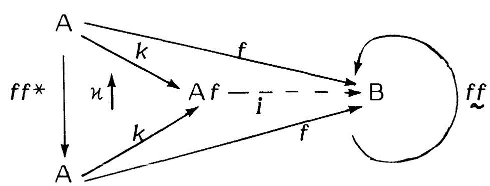

which satisfies the "equations" of a left $\Phi$-action. (As usual, "equations" are to be understood with the help of the structural isomorphisms for horizontal composition in $M$.) For opalgebras $\left[{\Theta \atop \omega}\right]$ and $\left[{\Theta' \atop \omega'}\right]$ with common domain, a *homomorphism of opalgebras* $\left[{\Theta \atop \omega}\right] \to \left[{\Theta' \atop \omega'}\right]$ is a transformation $\vartheta : \Theta \to \Theta'$ which is equivariant with respect to the $\Phi$-actions. Write $M(A, \Phi)$- $OPALG\,(X)$ for the resulting category. A *Kleisli opalagebra* for $(A, \Phi)$ is an opalgebra $\left[{K \atop \varkappa}\right]$, $K : A \to A_\Phi$, which is universal in that composing with it yields an equivalence of categories

$$M(A_\Phi, X) \sim M(A, \Phi)\text{- } OPALG\,(X).$$

Dually, one speaks of *algebras* $[\Omega : X \to A,\ \vartheta : \Omega\Phi \to \Omega]$ (right $\Phi$-actions) and a universal such is called an *Eilenberg-Moore algebra.* The main reference for such matters is Street's paper [ST2].

**Axiom 5.** Every monad $(A, \Phi)$ in $M$ has a representable Kleisli opalgebra $\left[{k \atop \varkappa}\right]$, $k : A \to A_\Phi$. The equivalences $M\,(A_\Phi, X) \sim M(A, \Phi)$- $OPALG$ $(X)$ respect representability and $[k^* : A_\Phi \to A,\ \varkappa^*]$ is an Eilenberg-Moore algebra for $(A, \Phi)$. ◊

Let $f : A \to B$ be an arrow in $K$. We have a monad $(A, ff^*)$ in $M$ for which $\left[{f \atop \underset{\sim}{f}f}\right]$ is an opalgebra. ($\underset{\sim}{f}$ is the counit for $f \dashv f^*$.) We will write $Af$ for the Kleisli object $A_{ff^*}$. By Axiom 5 we have a "commutative" diagram




$(\varkappa i = \underset{\sim}{ff})$

$f$ is said to be a *surjection* if $i$ in the diagram above is an equivalence.

***Proposition 4.*** *In the factorization $f \simeq ki$ above, $k$ is a surjection and $i$ is an inclusion.*

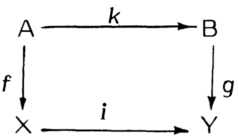
**Proof.** Since $\left[{k \atop \varkappa}\right]$ is a Kleisli opalgebra for $ff^*$, we have from [ST2] that $ff^*$ is generated by the adjunction $k \dashv k^*$. That is, $kk^* \simeq ff^*$. Hence $Ak \sim Af$ ; so $k$ is a surjection.

Explicitly, the isomorphism $kk^* \simeq ff^*$ is

$$k\tilde{i}k^* : kk^* \xrightarrow{\sim} kii^*k^*.$$

Since $[k^*, \varkappa^*]$ is Eilenberg-Moore, we have $\tilde{ki} : k \xrightarrow{\sim} kii^*$. Since $\left[{k \atop \varkappa}\right]$ is Kleisli we have $\tilde{i} : Af \xrightarrow{\sim} ii^*$ and $i$ is an inclusion. ◊

***Proposition 5.*** *If in a "commutative" diagram*




*in $K$, $k$ is a surjection and $i$ is an inclusion, then there exists an essentially unique arrow $d : B \to X$, in $K$, such that both triangles "commute".*

**Proof.** $f \simeq fii^* \simeq kgi^*$. According to Axiom 5 the equivalence mediated by composition with $k$ respects representability. Hence, there exists an essentially unique $d : B \to X$ in $K$ and an isomorphism $\bar{\tau} : d \tilde{\to} gi^*$. So $kd \simeq f$. By adjointness we have $\tau : di \to g$. But

$$(k\tau : kdi \to kg) = (kdi \tilde{\to} fi \tilde{\to} kg)$$

where the isomorphisms are as above. Since $k$ is Kleisli we have $\tau : di \tilde{\to} g$. Finally, such an isomorphism essentially determines $d$ since $d \simeq dii^* \simeq gi^*$. ◊

We will not assume for our axiomatic development that $K$ admits the construction of all Eilenberg-Moore algebras ; nevertheless, for those which exist we can establish two classical relationships between them and the Kleisli construction. Analogous synthetic results, in the context of Yoneda structures, were given first by Street and Walters [S&W]. (See especially Section 5 of that paper.)

***Proposition 6.*** *If $(X, t)$ is a monad in $K$ for which an Eilenberg-Moore algebra $[u : X^t \to X, \alpha : ut \to u]$ exists and if the left adjoint of $u$, call it $f$, factors as $f \simeq ki$ with $k$ a surjection and $i$ an inclusion, then $k$ is Kleisli for $t$.*

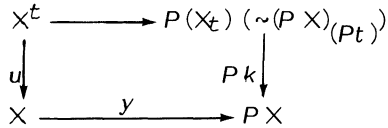
**Proof.** (We recall from [ST2] that universality for $[u, \alpha]$ guarantees the existence of $f \dashv u$ and, of course, that $t \simeq fu$.) If $f \simeq ki$, then $u \simeq i^*k^*$. Hence

$$t \simeq fu \simeq kii^*k^* \simeq kk^* .$$

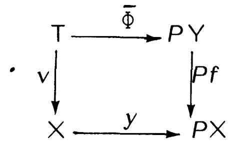
Since $k : X \to Xk = X_{kk^*} \sim X_t$, the conclusion follows immediately. ◊

In the opposite direction, recovering Eilenberg-Moore from Kleisli, we recall first the Yoneda structure version of Linton's Theorem [LIN]. If $k : X \to X_t$ is Kleisli for a monad $(X, t)$ in $K$, the latter equipped with a Yoneda structure for which $P$ has a left adjoint ; then Eilenberg-Moore $u : X^t \to X$ is given by the following pullback in $K$ :

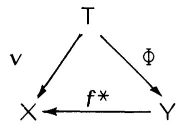
( $y$ is the Yoneda arrow)




To state and prove this in the context of proarrows, observe first that in the examples for which it makes sense

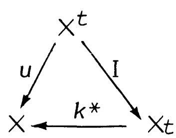



"commutes" given $f : X \to Y$ in $K$, iff

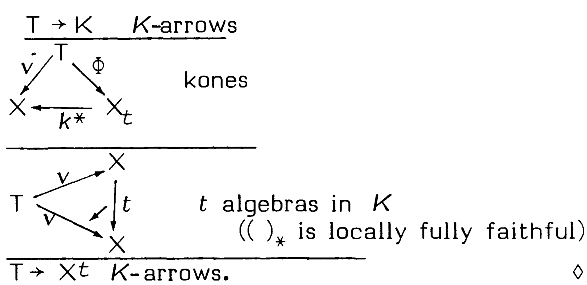



"commutes" in $M$. Here we intend that $\Phi : T \to Y$ in $M$ correspond to the $K$-arrow $\bar{\Phi} : T \to PY$. (The arrow v in both diagrams is in $K$.) For want of a better word, call such a diagram a *kone* for $f$. (Our need for a terminology does not extend beyond the next proposition.) Clearly, the pullback of $Pf$ along $y$ corresponds to a universal kone.

***Proposition 7.*** *If $k : X \to X_t$ is Kleisli for $(X, t)$ in $K$, then Eilenberg-Moore $u : X^t \to X$ is given by a universal kone*




**Proof.** $k$ Kleisli for $t$ in $K$ yields $k^*$ Eilenberg-Moore for $t$ in $M$. Let K denote the vertex of a universal kone.




◊

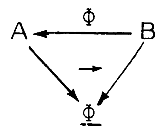
It is easy to identify I in the statement of proposition 7. $u$ has a left adjoint, $f$, which factors as

$$f \simeq ki \quad \text{with} \quad k : X \to X_t \text{ and } \quad i : X_t \to X^t$$

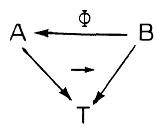
the comparison inclusion, $u \simeq f^* \simeq i^*k^*$. Since also $u \simeq Ik^*$ we have $I \simeq i^*$ by universality.

## 3. ARTIN GLUEING AND LOCAL FINITE COLIMITS.

We assume that $(\ )_*: K \to M$ satisfies Axioms 1 through 5. An arrow $A \leftarrow B : \Phi$ in $M$ (considered as a diagram in $M$) has a *collage* if there is a diagram




composing with which establishes an equivalence between $M(\underline{\Phi}, T)$ and the category of diagrams




for any T. General collages were discussed by Street in [ST1] and the term "collage" was credited to Walters. Presumably, the word was chosen because collages, in the everyday sense of the word, are usually obtained by glueing.

***Proposition 8.*** *Every* $A \leftarrow B : \Phi$ *in* $M$ *has a collage.*

**Proof.** Consider the following 2x2 matrix :

$$A \oplus B \xrightarrow{\begin{bmatrix} A & 0 \\ \Phi & B \end{bmatrix}} A \oplus B$$

It is a monad via

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \xrightarrow{\begin{bmatrix} A & 0 \\ \sigma & B \end{bmatrix}} \begin{bmatrix} A & 0 \\ \Phi & B \end{bmatrix} \xrightarrow{\begin{bmatrix} A & 0 \\ \gamma & B \end{bmatrix}} \begin{bmatrix} A & 0 \\ \Phi+\Phi & B \end{bmatrix}$$

where $\sigma$ and $\gamma$, as displayed below, are the canonical transformations

$$0 \xrightarrow{\ \sigma\ } \Phi \xleftarrow{\ \gamma\ } \Phi+\Phi$$

which uniquely exhibit $\Phi$ as a monoid in the monoidal category $(M(B, A), +, 0)$.

An opalgebra for $\begin{bmatrix} A & 0 \\ \Phi & B \end{bmatrix}$, with codomain T, amounts to a diagram of the kind preceeding the statement of this proposition. In particular the Kleisli opalgebra provides a collage for $A \leftarrow B : \Phi$. ◊

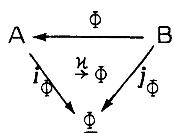
***Corollary 9.*** *The injections into the collage for* $\Phi$ *are representable and the equivalence mediated by the collage respects representability.*

**Proof.** By Axiom 5 the Kleisli arrow for $\begin{bmatrix} A & 0 \\ \Phi & B \end{bmatrix}$ is representable ; call it

$$k : A \oplus B \longrightarrow A \oplus B_{\begin{bmatrix} A & 0 \\ \Phi & B \end{bmatrix}} \sim \underline{\Phi} \ .$$

By Axiom 4, $k$ is necessarily of the form $k \simeq \begin{bmatrix} i \\ j \end{bmatrix}$, for representable $i$ and $j$. The second statement is similarly immediate from those axioms.◊

We will denote the collage for by




and suppress the subscripts "$\Phi$" whenever possible.

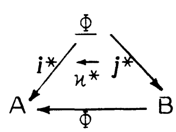
***Corollary 10.*** *The injections* $i$ *and* $j$ *are inclusions,* $\Phi \simeq ji^*$ *and* $ij^* \simeq 0$ .

**Proof.** We have

$$\begin{bmatrix} A & 0 \\ \Phi & B \end{bmatrix} \simeq kk^* \simeq \begin{bmatrix} i \\ j \end{bmatrix}\begin{bmatrix} i \\ j \end{bmatrix}^* \simeq \begin{bmatrix} i \\ j \end{bmatrix}[i^*, j^*] \simeq \begin{bmatrix} ii^* & ij^* \\ ji^* & jj^* \end{bmatrix}.$$

Now compare corresponding matrix entries. ◊

***Corollary 11.*** *The diagram*



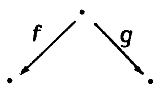

*is also universal in* $M$.

**Proof.** By Axiom 5, $[i^*, j^*] : \underline{\Phi} \to A \oplus B$ is Eilenberg-Moore. ◊

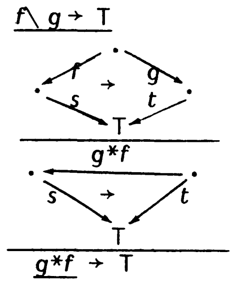
**Remark 12.** $\underline{\Phi}$ is the comma object $\Phi/A$. It is also the cocomma object $\Phi \backslash B$. (Generally, cocomma squares are not instances of collages.) In particular, for all A, both **2**.A and $\{$**2**, A$\}$ in $M$ are given by $\underline{A}$, the collage object for the identity, $A \leftarrow A : A$. $\underline{A}$ is also **2**.A in $K$. ◊

***Proposition 13.*** *K has all cocommas.*

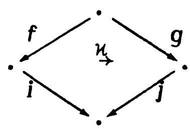
**Proof.** Given a span




$f \backslash g$ is given by $\underline{g^*f}$, for




Recall that a square



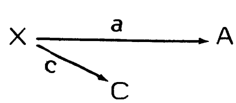

is said to be *exact* [GUI] (or satisfy the Beck-Chevalley condition) if $\varkappa^* : g^*f \to ji^*$ is an isomorphism. ◊

***Corollary 14.*** *Cocomma squares in K are exact.*

**Proof.** Note the construction of $f\backslash g$ in Proposition 13. The isomorphism $g^*f \simeq ji^*$ is then seen to be an instance of that in Corollary 10. ◊

It is quite helpful to know that every $\Phi$ in $M$ factors as $\Phi \simeq gf^*$ for respresentables $f$ and $g$. The next two propositions are included to illustrate this. The precise nature of the factorization is not required. As in [WD1] we use $\Psi \Rightarrow \Gamma$ to denote a right lifting of $\Gamma$ through $\Psi$ and $\Gamma \Leftarrow \Phi$ to denote a right extension of $\Gamma$ along $\Phi$.

***Pròposition 15.*** *M is biclosed iff for all*

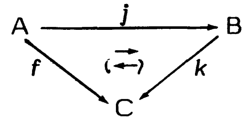



*in K,* $c \Leftarrow a$ *and* $a^* \Rightarrow c^*$ *exist in M. (We may take a and c to be inclusions.)*

**Proof.** $(cd^*) \Leftarrow (ab^*) \simeq b(c \Leftarrow a)d^*$ and $(ba^*) \Rightarrow (dc^*) \simeq d(a^* \Rightarrow c^*)b^*$. ◊

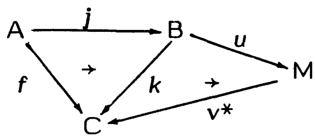
Paré [PAR] first proved that a colimit is absolute iff it is preserved by the Yoneda embedding. Proposition 15 is an abstract version of that fact. It should be compared with the definition of "pointwise extension" given in [WD1].

***Proposition 16.***



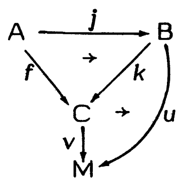
*is an absolute left (resp. right) extension diagram in $K$ iff $(\ )_*$ (resp. $(\ )^*$) applied to it yields a left extension (resp. lifting) diagram in $M$.*

**Proof.** (Only if) Consider the following diagram in $M$ :



$$\frac{\dfrac{\dfrac{f \to juv^*}{fv \to ju}}{kv \to u}}{k \to uv^*}\qquad \begin{matrix}2.\\1.\\4.\\3.\end{matrix}$$

(If) Consider the following diagram in $K$ :

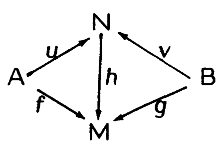


Now read the lines above as numbered. ◊

Provisionally, let $(COSPN\ K)(B, A)$ denote the *category* of cospans from B to A in $K$. Thus objects are typified by

$$f : A \to M \leftarrow B : g$$

and morphisms by "commutative" diagrams



We have a functor from this category to $M(B, A)$, which assigns to a morphism $h$ , as above,

$$vu^* \xrightarrow{v\tilde{h}u^*} vhh^*u^* \xrightarrow{\sim} gf^* .$$

An immediate consequence of Corollaries 9 and 10 is

**Proposition 17.** *The assignment*

$$(A \leftarrow B : \Phi) \mapsto (i : A \to \underline{\Phi} \leftarrow B : j)$$

*defines a fully faithful left adjoint to the functor above.* ◊

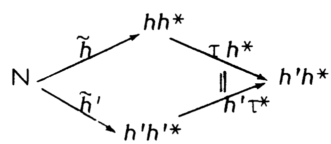
Thus Artin glueing, the $\underline{\Phi}$ construction, provides a two variable dual version of the comprehension schema studied by Lawvere in [LAW].

$M$(B, A) is by assumption just a category. ((*COSPN K*)(B, A), however, is naturally a *bicategory.*) If $h'$ is also a morphism from $\left[{u \atop v}\right]$ to $\left[{f \atop g}\right]$ in (*COSPN K*)(B, A), then a transformation from $h$ to $h'$ is a transformation $\tau : h \to h'$ in $K$ which also commutes with $\left[{u \atop v}\right]$ and $\left[{f \atop g}\right]$. The next two propositions show that the adjunction of Proposition 16 is tidy with respect to this extra structure.

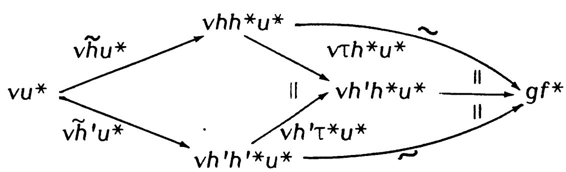
**Proposition 18.** *If $\tau : h \to h'$ is a transformation in (COSPN K)*(B, A)*, as above, then (COSPN K)* (B, A) $\to$ $M$(B, A) *identifies*

$$vu^* \to vhh^*u^* \xrightarrow{\sim} gf^* \quad \textit{and} \quad vu^* \to vh'h'^*u^* \xrightarrow{\sim} gf^*.$$

**Proof.** We have




and hence




◊

**Proposition 19.** *If* $A \leftarrow B : \Phi$ *is in* $M$(B, A), *then* $i : A \to \underline{\Phi} \leftarrow B : j$ *is codiscrete in (COSPN K)*(B, A).

**Proof.** From the defining universal property of $\underline{\Phi}$ we have that an arrow $h : \underline{\Phi} \to M$ is given by a tuple

$$\begin{bmatrix} f : A \to M \\ \sigma : \Phi f \to g \\ g : B \to M \end{bmatrix}$$

and a transformation $\tau : h \to h'$ is given by a tuple of transformations

$$\begin{bmatrix} f : A \to M \\ \sigma : \Phi f \to g \\ g : B \to M \end{bmatrix} \xrightarrow{\begin{bmatrix} \varphi : f \to f' \\ \gamma : g \to g' \end{bmatrix}} \begin{bmatrix} f' : A \to M \\ \sigma' : \Phi f' \to g' \\ g' : B \to M \end{bmatrix}$$

where

$$\begin{array}{ccc} \Phi f & \xrightarrow{\Phi\varphi} & \Phi f' \\ \sigma \downarrow & & \downarrow \sigma' \\ g & \xrightarrow[\gamma]{} & g' \end{array} \; .$$

(As remarked in 12, $\underline{\Phi}$ is a cocomma object in $M$.) Thus if $\tau : h \to h'$ is a transformation between arrows of cospans from $i : A \to \underline{\Phi} \leftarrow B : j$ to $f : A \to M \leftarrow B : g$, then this explicit description shows that

$$(i\,\tau : f \to f) = (f : f \to f) \quad \text{and} \quad (j\tau : g \to g) = (g : g \to g)$$

imply that $\tau$ is essentially an identity. ◊

If $K$ has pushouts, then cospans can be composed and *COSPN K* becomes a bicategory with bicategory valued homs. The functors

$$(COSPN\ K)(B, A) \to M(B, A)$$

then define a normal morphism of bicategories $COSPN\ K \to M$. In particular the functors $(COSPN\ K)(A, A) \to M(A, A)$ are normal monoidal functors between monoidal categories. The category of monoids in $(COSPN\ K)(A, A)$ is equivalent to $A/K$ and the forgetful functor is given by

$$(f : A \to X) \mapsto (f : A \to X \leftarrow A : f).$$

Write $MND\ M(A)$ for the category of monoids in $M(A, A)$. (So $MND\ M(A)$ is the category of monads on A in $M$.) It follows that we have à "commutative" diagram

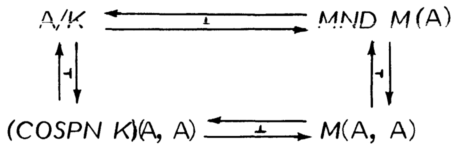
$$\begin{array}{ccc} A/K & \longrightarrow & MND\ M(A) \\ \downarrow & & \downarrow \\ (COSPN\ K)(A, A) & \longrightarrow & M(A, A) \end{array}$$

where $A/K \to MND(A)$ is given by $f \mapsto ff^*$. Its left adjoint is the Kleisli construction, $\Phi \mapsto k : A \to A_{\Phi}$.

We have an initial object in $K$, so the temporary assumption of pushouts gives us coequalizers. The left adjoint to $A/K \to (COSPN\ K)(A, A)$ is

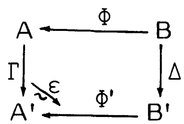
$$(f : A \to M \leftarrow A : g) \mapsto (A \to \mathrm{coeq}(f, g)).$$

If $K$ has countable sums locally, then $\Phi \mapsto \sum_{n \in \mathbf{N}} \Phi^n$ gives us a free monad construction. (Note that the full force. of "locally" is needed for this. Note too that the proof of proposition 3 is easily modified to handle the statement of that proposition with the words "finite" deleted. In short a global assumption about countable sums could be made.) Putting all these adjoints together we get,

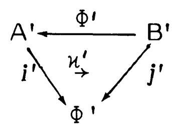


providing a relationship between dual structure-semantics (the top row adjunction) and dual comprehension.

We turn now to local finite colimits. A preliminary lemma will be useful. Consider the following square in $M$ :



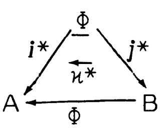
Pasting



to the square yields, via universality of $\underline{\Phi}$, an arrow $\Sigma : \underline{\Phi} \to \underline{\Phi}'$ in $M$. We can write

$$\Sigma = \begin{bmatrix} \Gamma i' \\ (\varepsilon i')(\Delta \varkappa') \\ \Delta j' \end{bmatrix}$$

Similarly, we can replace $\varepsilon$ in the square by $\varepsilon^{-1}$, paste



to the result and obtain, via universality of $\underline{\Phi}'$ (Corollary 11), an arrow

$$\Pi = [\, j^*\Delta,\ (j^*\varepsilon^{-1})(\varkappa^*\Gamma),\ i^*\Gamma\,] : \underline{\Phi} \to \underline{\Phi}'.$$

***Lemma 19.*** $\Sigma \simeq \Pi$ .

**Proof.** $\Sigma \simeq [\Sigma j^*,\ \Sigma\varkappa^*,\ \Sigma i^*]$. A direct calculation yields

$$\Sigma \simeq \begin{bmatrix} A \xrightarrow{0} B' & 0_{AA'} \to \Gamma & A \xrightarrow{\Gamma} A' \\ 0_{BB'} \to \Delta & & \Phi\Gamma \xrightarrow[\sim]{\varepsilon} \Delta\Phi \\ B \xrightarrow{\Delta} B' & \Delta\Phi \to \Delta\Phi & B \xrightarrow{\Delta\Phi} A' \end{bmatrix}$$

Similarly we have

$$\Pi \simeq \begin{bmatrix} i\,\Pi \\ \varkappa\,\Pi \\ j\,\Pi \end{bmatrix} \simeq \begin{bmatrix} A \xrightarrow{0} B' & 0_{AA'} \to \Gamma & A \xrightarrow{\Gamma} A' \\ 0_{BB'} \to \Delta & & \Phi\Gamma \to \Phi\Gamma \\ B \longrightarrow B' & \Delta\Phi \xrightarrow[\sim]{\varepsilon^{-1}} \Phi\Gamma & B \xrightarrow{\Phi\Gamma} A' \end{bmatrix}$$

Comparing entries shows $\Sigma = \Pi$ . ◊

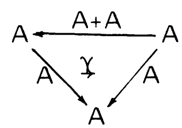
We will write $\underline{\Gamma, \Delta}$(= $\underline{\Gamma, \varepsilon, \Delta}$) for the "common value" of $\Sigma$ and $\Pi$ above.

***Proposition 20.*** *M has finite colimits locally.*

**Proof.** In virtue of Proposition 1 (or Propositions 1 and 3) it suffices to show that *M* has local pushouts (or local coequalizers). So let

$$\sigma : \Phi \leftarrow \Gamma \rightarrow \Psi : \tau$$

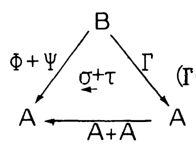
be a span in *M*(B, A). We have $A \leftarrow A : A{+}A$ (i.e., A+A is the sum in *M*(A, A) of the identity with itself). From



where $\gamma : A{+}A \to A$ is the codiagonal in *M*(A, A), we get a codiagonal arrow, $A \leftarrow \underline{A{+}A} : c$ , in *K*. Using Corollary 11 we define

$$\underline{A{+}A} \leftarrow B : [\Gamma,\ \sigma{+}\tau,\ \Phi{+}\Psi]$$

as in the following diagram :



($\Gamma(A{+}A) \simeq \Gamma{+}\Gamma$ by Proposition 3)

We show that the composite

$$A \xleftarrow{\ c\ } \underline{A{+}A} \xleftarrow{[\Gamma,\ \sigma+\tau,\ \Phi+\Psi]} B$$

is the required pushout.

Let $\Delta$ be an arbitrary object of *M*(B, A). Then

$$\frac{\dfrac{[\Gamma,\ \sigma{+}\tau,\ \Phi{+}\Psi]c \to \Delta : B \to A}{[\Gamma,\ \sigma{+}\tau,\ \Phi{+}\Psi] \to \Delta c^* : B \to \underline{A{+}A}} \quad (c^* = [A, \gamma, A])}{\dfrac{[\Gamma,\ \sigma{+}\tau,\ \Phi{+}\Psi] \to [\Delta, \Delta\gamma\ \Delta] : B \to \underline{A{+}A}}{\Gamma \xrightarrow{\zeta} \Delta,\ \ \Phi{+}\Psi \xrightarrow{\binom{\xi}{\eta}} \Delta : B \to A}}$$

such that

$$\begin{array}{ccc} \Gamma+\Gamma & \xrightarrow{\zeta+\zeta} & \Delta+\Delta \\ {\scriptstyle \sigma+\tau}\downarrow & \binom{\xi}{\eta} \quad /\!/ & \downarrow{\scriptstyle \Delta\gamma} \\ \Phi+\Psi & \longrightarrow & \Delta \end{array}$$

$$\overline{\Phi \xrightarrow{\xi} \Delta, \quad \Psi \xrightarrow{\eta} \Delta \ : B \to A}$$

such that

$$\begin{array}{ccc} \Gamma & \xrightarrow{\tau} & \Psi \\ {\scriptstyle \sigma}\downarrow & /\!/ & \downarrow{\scriptstyle \eta} \\ \Phi & \xrightarrow{\xi} & \Delta \end{array}$$

Hence $[\Gamma, \sigma+\tau, \Phi+\Psi]c$ is the pushout in $M(B, A)$. To see that it is preserved by precomposition with all $\Delta : B' \to B$, note that

$$\Delta[\Gamma, \sigma+\tau, \Phi+\Psi] \simeq [\Delta\Gamma, \Delta\sigma + \Delta\tau, \Delta\Phi+\Delta\Psi]$$

(where we again use Proposition 3). For preservation by postcomposition, consider $\Delta : A \to A'$ and

$$\begin{array}{ccc} A & \xleftarrow{A+A} & A \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta} \\ A' & \xleftarrow{A'+A'} & A' \end{array}$$

Define $\underline{\Delta, \Delta} : \underline{A+A} \to \underline{A'+A'}$ as after Lemma 19. From the description of $\underline{\Delta, \Delta}$ as an arrow into a comma object we get

$$[\Gamma, \sigma+\tau, \Phi+\Psi]\underline{\Delta,\Delta} \simeq [\Gamma\Delta, \sigma\Delta + \tau\Delta, \Phi\Delta+\Psi\Delta]$$

The proof is completed by showing that

$$\begin{array}{ccc} \underline{A+A} & \xrightarrow{\underline{\Delta, \Delta}} & \underline{A'+A'} \\ {\scriptstyle c}\downarrow & \cdot & \downarrow{\scriptstyle c'} \\ A & \xrightarrow{\Delta} & A' \end{array}$$

"commutes", where $c'$ is the corresponding codiagonal for $A'$. This follows from the description of $\underline{\Delta, \Delta}$ as an arrow out of a cocomma object. Explicitly :

$$\underline{\Delta,\Delta}c' \simeq \underline{\Delta,\Delta} \begin{bmatrix} A' \\ \gamma' \\ A' \end{bmatrix} \simeq \begin{bmatrix} \Delta \\ \Delta\gamma' \\ \Delta \end{bmatrix} \simeq \begin{bmatrix} \Delta \\ \gamma\Delta \\ \Delta \end{bmatrix} \simeq c\Delta .$$

We have chosen the above construction for pushouts in $M(B, A)$ because it invites comparison with the construction for sums given in Proposition 3. The matrix calculus permitted by the latter immediately gives another more symmetric sum construction. Indeed, if $\Phi, \Psi : B \to A$, then $\Phi + \Psi$ is given by the composite

$$B \xrightarrow{[\Phi,\, B]} A \oplus B \xrightarrow{\left[{A \atop \Psi}\right]} A.$$

It is natural to seek such a description for the pushout of $\sigma : \Phi \leftarrow \Gamma \rightarrow \Psi : \tau$ in $M(B, A)$. Experimentation with the examples, particularly

$$(\ )_* : TOP \rightarrow TOPLEX^{co}$$

suggests that such is provided by

$$B \xrightarrow{[B,\sigma,\, \Phi]} \underline{\Gamma} \xrightarrow{\left[\begin{smallmatrix} A \\ \tau \\ \Psi \end{smallmatrix}\right]} A.$$

This is a correct general construction. We postpone a proof until Section 5 (Corollary 31).

It should be clear from the proof of Proposition 20 that a slight modification of the construction gives local coequalizers.

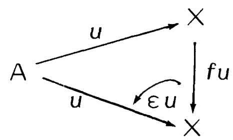
**Proposition 21.** *For* $\sigma, \tau : \Gamma \rightrightarrows \Phi$ *in* $M(B, A)$, *the coequalizer of* $\sigma$ *and* $\tau$ *is given by the following composite :*

$$A \xleftarrow{\ c\ } \underline{A+A} \xleftarrow{[\Gamma,\, ({\sigma \atop \tau}),\, \Phi]} B\ . \qquad \Diamond$$

## 4. SURJECTIONS REVISITED.

Our results in Section 2 suggest that surjections-inclusions constitute a bicategorical E-M factorization system for $K$, when $(\ )_* : K \rightarrow M$ is proarrow equipment satisfying Axioms 1 through 5. However, we should have each kind of arrow closed under composition. For inclusions this is trivial but a moment's reflection shows that for surjections the situation is more subtle. After all, PTT functors in $CAT$ are not composition-closed. To show that surjections are in fact closed under composition we find it convenient to establish first a general monadicity theorem for arbitrary bicategories, Proposition 22. The experienced reader will find nothing particularly new in either the statement or the proof of Proposition 22, however, we need precisely the version there and a published account of it does not seem to have been given. It is reminiscent of some of the results in W. Butler's unpublished, but well-circulated, tripleability theorems. We respectfully suggest that Proposition 22 be called the FTT (formal tripleability Theorem). Temporarily, $K$ is just an arbitrary category.

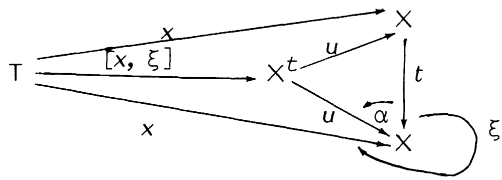
Consider an adjunction $f : X \rightleftarrows A : u$ in $K$, with counit $\varepsilon$. Consider the following diagram




It is an $fu$ algebra. A right adjoint $u$, as above, is said to be *monadic* if $[u, \varepsilon u]$ is an Eilenberg-Moore algebra. For a monad $(X, t, \eta, \mu)$ with Eilenberg-Moore algebra

$$[u : X^t \to X,\ \alpha : ut \to u],$$

arrows into $X^t$ may be denoted by pairs as in :




In particular, the identity on $X^t$ is $[u, \alpha]$ and $[t, \mu] = f \dashv u$. The unit for the adjunction is $\eta$. The counit

$$uf = u\,[t, \mu] = [ut, u\mu] \to [u, \alpha] = X^t$$

is $\alpha$ regarded as a homomorphism. Given an adjunction $f : X \rightleftarrows A : u$ and an $fu$ algebra $[x : T \to X, \xi : xfu \to x]$, note that we have a parallel pair, $\xi f$, $xf\varepsilon : xfuf \rightrightarrows xf$, in $K(T, A)$. Recall that a colimit (or limit) in $K(B, A)$ is said to exist *pointwise* if it is preserved by $K(b, A)$ for all $b : T \to B$ (for all T).

**Proposition 22.** *For an adjunction $f : X \rightleftarrows A : u$, with counit $\varepsilon$, in an arbitrary bicategory $K$, $u$ is monadic iff :*

*(i) for every algebra $[x, \xi]$, the coequalizer of $\xi f$ and $xf\varepsilon$ exists and is preserved by composition with $u$,*

*(ii) for the algebra $[u, \varepsilon u]$, the coequalizer of $\varepsilon uf$ and $uf\varepsilon$ exists pointwise,*

*(iii)*
$$ufuf \underset{uf\varepsilon}{\overset{\varepsilon uf}{\rightrightarrows}} uf \xrightarrow{\ \varepsilon\ } A.$$

*is a coequalizer.*

**Proof.** Assume $u$ is monadic. Write $t = fu$ and write

$$[u : X^t \to X,\ \alpha : ut \to u] \qquad \text{for} \qquad [u : A \to X,\ \varepsilon u : ufu \to u]\ .$$

Let $[x, \xi] : T \to X^t$ be given and compose it with the commuting fork displayed in (iii) above to obtain, in algebra notation, the commuting fork

$$[xtt, xt\mu] \underset{x\mu}{\overset{\xi t}{\rightrightarrows}} [xt, x\mu] \xrightarrow{\ \ \xi\ \ } [x, \xi].$$

For any $[b, \beta] : T \to X^t$ and commuting fork

$$[xtt,\ xt\mu] \underset{x\mu}{\overset{\xi t}{\rightrightarrows}} [xt,\ x\mu] \xrightarrow{\ \vartheta\ } [b,\ \beta],$$

it is easy to verify that $(x\eta)\vartheta$: $[x, \xi] \to [b, \beta]$ is the unique homomorphism that commutes with $\xi$ and $\vartheta$. This verifies the existence of the coequalizer required in (i). Composing it with $u$ yields

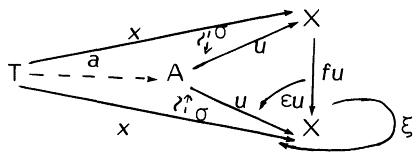
$$xtt \underset{x\mu}{\overset{\xi t}{\rightrightarrows}} xt \xrightarrow{\ \xi\ } x,$$

which is absolutely a coequalizer. Now set

$$[x, \xi] = [u, \alpha] \ (= X^t = A).$$

This verifies (iii). It also verifies (ii) after one notes how the coequalizer required in (i) was constructed.

Conversely, assume (i), (ii and (iii). Consider




where $[x, \xi]$ is an $fu$ algebra. We have to show that there is an essentially unique $a$ and, for such, a unique isomorphism $\sigma$ as shown, satisfying $(\sigma fu)(a\varepsilon u)\sigma^{-1} = \xi$.

Assume that a solution $a$, $\sigma$ exists. Consider

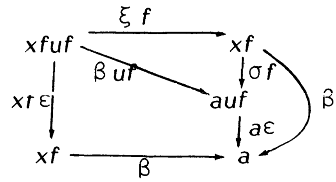
$$\begin{array}{ccccc}
xfuf & \underset{xf\varepsilon}{\overset{\xi f}{\rightrightarrows}} & xf & & \\
{\scriptstyle \sigma fuf}\downarrow\wr & & \wr\downarrow{\scriptstyle \sigma f} & & \\
aufuf & \underset{auf\varepsilon}{\overset{a\varepsilon uf}{\rightrightarrows}} & auf & \xrightarrow{\ a\varepsilon\ } & a
\end{array}$$

By (iii) and (ii) the bottom row is a coequalizer. The bottom square commutes "naturally". The top square is seen to commute by applying $f$ to the assumed equation. Thus $a$ is essentially uniquely determined as a coequalizer of $\xi f$ and $xf\varepsilon$. Moreover, for the coequalizer transformation $xf \to a$, $\sigma : x \overset{\sim}{\to} au$ is the correspondent via the adjointness $f \dashv u$. So both aspects of uniqueness have been verified.

For existence use (i) and define $\beta : xf \to a$ to be a coequalizer of $\xi f$ and $xf\varepsilon$. Also by (i), $\beta u : xfu \to au$ is a coequalizer of $\xi fu$ and $xf\varepsilon u$. However, $\xi$ is a coequalizer (absolutely) of $\xi fu$ and $xf\varepsilon u$ so we have an isomorphism

$$\begin{array}{ccc}
xfu & \xrightarrow{\ \xi\ } & x \\
 & {\scriptstyle \beta u}\searrow & \wr\downarrow{\scriptstyle \sigma} \\
 & & au
\end{array}$$

The coequalizer $\xi : xfu \to x$ is further preserved by applying the left adjoint $f$. Hence we have




where the rightmost region commutes by uniqueness of transformations out of the coequalizer $xf$. Hence

$$(\sigma fu)(a\,\varepsilon u) = \beta u = (\xi)(\sigma). \qquad \Diamond$$

Proposition 22 is easily dualized. For an adjunction $f: X \rightleftarrows A : u$ say that $f$ is *opmonadic* if $[{f \atop f\varepsilon}]$ is a Kleisli opalgebra for $fu$. Note that for an $fu$ opalgebra $[{x \atop \xi}]$, $x: X \to T$, we have a parallel pair

$$\varepsilon ux,\ u\,\xi : ufux \rightrightarrows ux \qquad \text{in } K(A, T).$$

***Proposition 22<sup></sup>*** 

**Proposition $22^{op}$.** *For an adjunction $f : X \rightleftarrows A : u$, with counit $\varepsilon$, in an arbitrary bicategory $K$ ; $f$ is opmonadic iff :*

*($i^{op}$) for every opalgebra $[{x \atop \xi}]$, the coequalizer of $\varepsilon ux$ and $u\xi$ exists and is preserved by (pre)composition with f,*

*($ii^{op}$) for the opalgebra $[{f \atop f\varepsilon}]$, the coequalizer of $\varepsilon uf$ and $uf\varepsilon$ exists aboslutely,*

*($iii^{op}$)*
$$ufuf \underset{uf\varepsilon}{\overset{\varepsilon uf}{\rightrightarrows}} uf \longrightarrow A$$

*is a coequalizer.* $\Diamond$

**Corollary 23.** *For an adjunction $\Phi : X \rightleftarrows A : \Phi^*$, with counit $\underset{\sim}{\Phi}$, in a bicategory M with local coequalizers, the following are equivalent :*

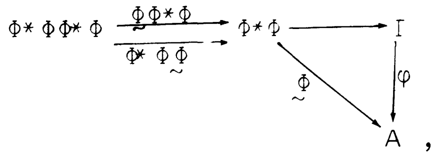
*(i) $\Phi^*$ is monadic,*
*(ii) $\Phi^*$ reflects isomorphisms in M,*

*(iii)*
$$\Phi^*\Phi\Phi^*\Phi \underset{\Phi^*\Phi\underset{\sim}{\Phi}}{\overset{\underset{\sim}{\Phi}\Phi^*\Phi}{\rightrightarrows}} \Phi^*\Phi \xrightarrow{\underset{\sim}{\Phi}} A$$

*is a coequalizer,*
*(iv) $\Phi$ opreflects isomorphisms in M,*
*(v) $\Phi$ is opmonadic.*

**Proof.** Conditions (i) and (ii) of proposition 22 and ($i^{op}$) and ($ii^{op}$) of Proposition $22^{op}$ are automatically satisfied in a bicategory with local coequalizers, so the equivalence of (i), (iii) and (v) above is immediate.

In any bicategory, $\Phi^*$ monadic implies $\Phi^*$ reflects isomorphisms. In the present situation we have



where the top fork is a coequalizer and the bottom fork, and hence the triangle, commute. Applying $\Phi^*$ makes both forks coequalizers and hence $\varphi\Phi^*$ is an isomorphism. Therefore, if $\Phi^*$ reflects isomorphisms, condition (iii) above is satisfied.

By " $\Phi$ opreflects isomorphisms" we mean "for all composable transformations $\tau$, $\Phi\tau$ an isomorphism implies $\tau$ an isomorphism". The equivalence of (iv) with the other statements is just dual to that in the paragraph above. ◊

**Corollary 24.** *For any arrow $f : A \to B$ in a bicategory $K$, with proarrow equipment $(\;)_* : K \to M$ satisfying Axioms 1 through 5, the following are equivalent :*

*(i) $f^*$ reflects isomorphisms in $M$,*
*(ii) $f$ is a surjection,*
*(iii) $f$ opreflects isomorphisms in $M$.* ◊

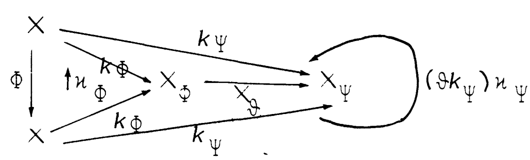
Z If $f^*$ above is also in $K$ and reflects isomorphisms in $K$, it does not follow that $f^*$ reflects isomorphisms in $M$.

**Corollary 25.** *In any bicategory $K$, with proarrow equipment $(\;)_* : K \to M$ satisfying Axioms 1 through 5, surjections are closed under composition.*

**Proof.** Conditions (i) and (iii) above are clearly stable under composition.◊

**Corollary 26.** *If $fk$ is a surjection, then so is $k$.* ◊

If $\Phi$ and $\Psi$ are monads on $X$ in $M$ and $\vartheta : \Phi \to \Psi$ is a morphism of monads, then for general reasons we have an arrow $X_\vartheta : X_\Phi \to X_\Psi$ as below :



For any transformation $\tau : \Phi \to \Psi : B \to A$ ($\Phi$ and $\Psi$ arbitrary) we have, in Proposition 16, implicitly mentioned an arrow $\underline{\tau} : \underline{\Phi} \to \underline{\Psi}$ which is a morphism of cospans from B to A. It is a special case of $X_\vartheta$. Indeed

$$\begin{bmatrix} A & 0 \\ \Phi & B \end{bmatrix} \xrightarrow{\begin{bmatrix} A & 0 \\ \tau & B \end{bmatrix}} \begin{bmatrix} A & 0 \\ \Psi & B \end{bmatrix}$$

is a morphism of monads. Thus an immediate consequence of Corollary 26 is :

***Corollary 27.*** *The arrows* $X_\vartheta$ *and* $\underline{\tau}$ *are surjections.* ◊

## 5. MATRIX ARITHMETIC EXTENDED.

Consider monads $(A, \Phi)$ and $(B, \Psi)$ in $M$. An arrow $A_\Phi \to X$ "is" a $\Phi$-opalgebra

$$\begin{bmatrix}\Gamma\\ \gamma\end{bmatrix} \quad (\Gamma\colon A \to X, \ \ \gamma : \Phi\Gamma \to \Gamma)$$

or *left* $\Phi$*-module*. An arrow $X \to B_\Psi$ "is" a $\Psi$-algebra

$$[\Delta, \delta] \ \ (\Delta\colon X \to B, \ \ \delta : \Delta\Psi \to \Delta)$$

or *right* $\Psi$*-module*. The composite

$$\begin{bmatrix}\Gamma\\ \gamma\end{bmatrix} [\Delta, \delta] : A_\Phi \to B_\Psi$$

"is" a *left* $\Phi$, ***right*** $\Psi$*-module*. (The actions associate.) Indeed, any arrow $A_\Phi \to B_\Psi$ "is" an arrow $A \to B$ together with a pair of associating actions. Somewhat more interesting is a composite of the forme $Y \to A_\Phi \to X$.

***Proposition 28.*** *If* $[\Delta, \delta] : Y \to A_\Phi$ *and* $\begin{bmatrix}\Gamma\\ \gamma\end{bmatrix} ; A_\Phi \to X$ *then* $[\Delta, \delta]\begin{bmatrix}\Gamma\\ \gamma\end{bmatrix} \simeq \Delta_\Phi \Gamma$ *where*

$$\Delta\Phi\Gamma \underset{\Delta\gamma}{\overset{\delta\Gamma}{\rightrightarrows}} \Delta\Gamma \longrightarrow \Delta_\Phi\Gamma$$

*is a coequalizer in* $M(Y, X)$.

**Proof.** Write $k : A \to A_\Phi$ for the Kleisli opalgebra for $\Phi$. By the FTT we have a coequalizer, composition stable by Proposition 20,

$$k^*kk^*k \underset{k^*k\underset{\sim}{k}}{\overset{\underset{\sim}{k}k^*k}{\rightrightarrows}} k^*k \xrightarrow{\ k\ } A_\Phi \ .$$

Applying $[\Delta, \delta] - \begin{bmatrix}\Gamma\\ \gamma\end{bmatrix}$ to this diagram (i.e. precomposing with $[\Delta, \delta]$ and postcomposing $\begin{bmatrix}\Gamma\\ \gamma\end{bmatrix}$) and noting the remarks about such composites in Section 4 (preceding Proposition 22) we obtain

$$\Delta\Phi\Gamma \underset{\Delta\delta}{\overset{\delta\Gamma}{\rightrightarrows}} \Delta\Gamma \longrightarrow [\Gamma, \gamma]\begin{bmatrix}\Delta\\ \delta\end{bmatrix}$$

a coequalizer. ◊

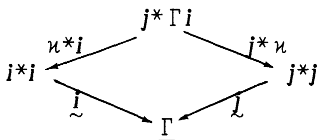
So in general, composites $A_\Phi \to B_\Psi \to C_\Gamma$ are "tensor products" of modules. Note that the full force of local coequalizer is required to equip such a composite with the requisite left $\Phi$, right $\Gamma$-structure.

The title of this section is admittedly overly ambitious. The point is to encourage comparison of composites such as

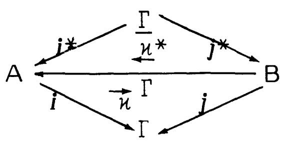
$$(Y \xrightarrow{[\Phi,\ \Psi]} A \oplus B \xrightarrow{\left[\begin{smallmatrix}\Gamma\\ \Delta\end{smallmatrix}\right]} X) \simeq (Y \xrightarrow{\Phi\Gamma + \Psi\Delta} X)$$

with composites such as in Proposition 28. In each case the intermediate object is simultaneously a "limit" and a "colimit" and the result is a local colimit. A simple combination of these cases is given in Proposition 30.

**Lemma 29.** For $A \leftarrow B : \Gamma$ in $M(B, A)$, the following diagram is a pushout in $M(\underline{\Gamma}, \underline{\Gamma})$ :




where, as usual, we have




exhibiting $\underline{\Gamma}$ as both a "limit" and a "colimit".

**Proof.** (Sketch) Since $\left[\begin{smallmatrix}i\\ j\end{smallmatrix}\right] : A \oplus B \to \underline{\Gamma}$ is Kleisli (for the monad $\left[\begin{smallmatrix}A & 0\\ \Gamma & B\end{smallmatrix}\right]$) we have a local coequalizer

$$[i^*,\ j^*]\left[\begin{smallmatrix}i\\ j\end{smallmatrix}\right][i^*,\ j^*]\left[\begin{smallmatrix}i\\ j\end{smallmatrix}\right] \rightrightarrows [i^*,\ j^*]\left[\begin{smallmatrix}i\\ j\end{smallmatrix}\right] \longrightarrow \underline{\Gamma}.$$

Expanding and noting that $ij^* \simeq 0$, $ji^* \simeq \Gamma$ this becomes

$$i^*ii^*i + j^*\Gamma i + j^*jj^*j \rightrightarrows i^*i + j^*j \longrightarrow \underline{\Gamma}.$$

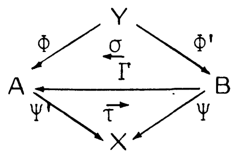
The diagram restricted to $j^*\Gamma i$ then yields the required pushout. ◊

***Proposition 30.*** *For* $A \leftarrow B : \Gamma$, *consider the following composite :*

$$Y \xrightarrow{[\Phi',\ \sigma,\ \Phi]} \underline{\Gamma} \xrightarrow{\left[\begin{smallmatrix}\Psi'\\ \tau\\ \Psi\end{smallmatrix}\right]} X.$$

*It is given by* $\Phi\Psi' +_{\Phi'\Gamma\Psi'} \Phi'\Psi$, *the pushout of* $\sigma\Psi' : \Phi\Psi' \leftarrow \Phi'\Gamma\Psi' \rightarrow \Phi'\Psi : \Phi'\tau$ *in* $M(B, A)$.

**Proof.** The given data is




Apply the technique in the proof of proposition 28 using Lemma 29. ◊

**Corollary 31.** *For* $\sigma : \Phi \leftarrow \Gamma \rightarrow \Psi : \tau$ *in* $M(B, A)$ *, the pushout is given by*

$$B \xrightarrow{[B,\ \sigma,\ \Phi]} \underline{\Gamma} \xrightarrow{\begin{bmatrix} A \\ \tau \\ \Psi \end{bmatrix}} A \quad (\simeq B \xrightarrow{[B,\ \tau,\ \Psi]} \underline{\Gamma} \xrightarrow{\begin{bmatrix} A \\ \sigma \\ \Phi \end{bmatrix}} A).$$

**Proof.** Take

$$(\Phi' : Y \rightarrow B) = (B : B \rightarrow B) \quad \text{and} \quad (\Psi' : A \rightarrow X) = (A : A \rightarrow A)$$

in Propoisition 30. ◊

On the other hand, any arrow $\underline{\Phi} \rightarrow \underline{\Psi}$ is a "matrix" consisting of four arrows and four transformations as in the proof of Lemma 19. A good notation would be helpful. We conclude this section with some simple results, stated without proofs.

**Proposition 32.** *For* $A \leftarrow B : \Phi$

$$i^* \simeq \begin{bmatrix} A \\ \Phi \\ \Phi \end{bmatrix}, \quad j^* \simeq \begin{bmatrix} 0 \\ \sigma \\ B \end{bmatrix}, \quad j^*\Phi \simeq \begin{bmatrix} 0 \\ \sigma \\ \Phi \end{bmatrix} \xrightarrow{\varkappa^* = \left[\begin{smallmatrix} \sigma \\ \Phi \end{smallmatrix}\right]} \begin{bmatrix} A \\ \Phi \\ \Phi \end{bmatrix}$$

*and*

$$i \simeq [0,\ \sigma,\ A], \quad j \simeq [B,\ \Phi,\ \Phi], \quad \Phi i \simeq [0,\ \sigma,\ \Phi] \xrightarrow{\varkappa\ =\ [\sigma,\ \Phi]} [B,\ \Phi,\ \Phi]$$

*where all* $\sigma$*'s are transformations out of* 0*'s.* ◊

**Corollary 33.** *For the codiagonal* $c : \underline{A} \rightarrow A$ *we have* $i \dashv c \dashv j$. ◊

**Proposition 34.** *For* $\tau : \Gamma \rightarrow \Psi : B \longrightarrow A$,

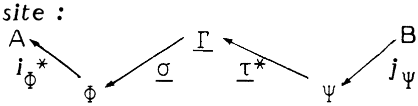
$$\underline{\tau} = \begin{bmatrix} i_\Psi \\ (\tau i_\Psi)\varkappa_\Psi \\ j\Psi \end{bmatrix} : \ \underline{\Gamma} \longrightarrow \underline{\Psi}$$

*satisfies*

$$\underline{\tau} i_\Psi^* = \begin{bmatrix} A \\ \sigma \\ \Psi \end{bmatrix} \quad \text{and} \quad j_\Psi \underline{\tau}^* = [B,\ \sigma,\ \Psi]$$

*where the* $\sigma$*'s are transformations out of* 0*'s.* ◊

**Corollary 35.** *For* $\sigma : \Phi \leftarrow \Gamma \rightarrow \Psi : \tau$ *in* $M(B, A)$ *the pushout is given by the following composite :*




$(\simeq j_\Phi \underline{\sigma}^* \underline{\tau} i_\Phi^*)$.

Write $\Sigma$ for the pushout above. By Proposition 17 we have :

***Proposition 36.***

$$\begin{array}{ccc} & \underline{\Gamma} & \\ {}^{\underline{\sigma}}\swarrow & & \searrow^{\underline{\tau}} \\ \underline{\Phi} & & \underline{\Psi} \\ \searrow & & \swarrow \\ & \underline{\Sigma} & \end{array}$$

*is a "pushout" in (COSPN K)*(B, A) *and hence also a "pushout" in K.* ◊

**Lemma 37.** Let $K$ be an arbitrary bicategory. Let $(A, t, \eta, \mu)$ and $(A, t', \eta', \mu')$ be monads in $K$ such that

$$\begin{array}{ccc} A & \xrightarrow{\ \eta\ } & t \\ {\scriptstyle \eta'}\downarrow & & \downarrow{\scriptstyle \eta' t} \\ t' & \xrightarrow[\ ]{\ t'\eta\ } & t't \end{array}$$

is a local pushout. Then $\begin{pmatrix} \eta t' \\ t\ \eta' \end{pmatrix} = \delta : t't \to tt'$ is a distributive law with the property that an opalgebra for the composite monad $t\,{}_{\delta}t'$ "is" an arrow $h : A \to X$ together with a $t$-opalgebra structure and a $t'$-opalgebra structure. (I.e., the usual distibutivity requirement for such structures is automatically satisfied in this situation.) If Kleisli objects exist in $K$, the following square is a "pushout" :

$$\begin{array}{ccc} & A & \\ \swarrow & & \searrow \\ A_t & & A_{t'} \\ \searrow & & \swarrow \\ & A_{t\delta t'} & \end{array}$$

◊

By Corollary 27, $\underline{\sigma}$ and $\underline{\tau}$ of Proposition 36 are universal opalgebras for the monads $\underline{\sigma\sigma}^*$ and $\underline{\tau\tau}^*$. It can be shown that

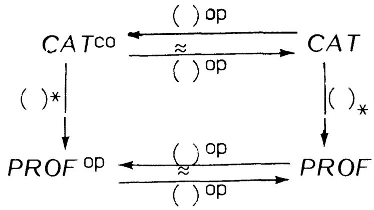
$$\begin{array}{ccc} \underline{\Gamma} & \xrightarrow{\ \tilde{\underline{\sigma}}\ } & \underline{\sigma\sigma}^* \\ {\scriptstyle \tilde{\underline{\tau}}}\downarrow & & \downarrow \\ \underline{\tau\tau}^* & \longrightarrow & \underline{\tau\tau}^*\underline{\sigma\sigma}^* \quad (\simeq \underline{\sigma\sigma}^*\underline{\tau\tau}^*) \end{array}$$

is a pushout in $M(\underline{\Gamma}, \underline{\Gamma})$, hence Lemma 37 explains the existence of the global "pushout" $\underline{\Sigma}$. It is a Kleisli object for a composite monad.

## 6. THE EXAMPLES.

***Proposition 38.*** *$(\ )_*$: CAT $\to$ PROF satisfies Axiom 4.*

**Proof.** *CAT* duality, $(\ )^{op} : CAT^{co} \xrightarrow{\ \approx\ } CAT$ extends to *PROF*, as in commutative diagram below.




In particular, any colimit diagram in *PROF* consisting entirely of profunctors of the form $f_*$ becomes a limit diagram in *PROF* when each $f_*$ is replaced by $f^*$. So we have only to show that $(\ )_*$ preserves the finite sums of *CAT* .

$$\frac{\dfrac{\dfrac{\mathbf{0} \to A \text{ in } PROF}{A^{op}\times\mathbf{0} \to SET \quad \text{functors}}}{\mathbf{0} \to SET \quad \text{functors}}}{\mathbf{1}}$$

$$\frac{\dfrac{\dfrac{\dfrac{B + C \to A \quad \text{in } PROF}{A^{op}\times(B+C) \to SET \quad \text{functors}}}{(A^{op}\times B)+(A^{op}\times C) \to SET \quad \text{functors}}}{A^{op}\times B \to SET, \quad A^{op}\times C \to SET \quad \text{functors}}}{B \to A, \quad C \to A \quad \text{in} \quad PROF\ .}$$

◊

***Proposition 39.*** *$(\ )_*$ : CAT → PROF satisfies Axiom 5.*

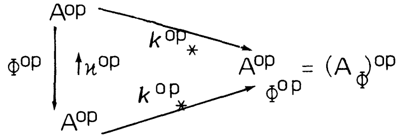
**Proof.** Let $(A, \Phi, \eta, \mu)$ be a monad in *PROF* . $A_\Phi$ is the category whose objects are those of A with $A_\Phi(a, b) = \Phi(a, b)$. Composition is given by $\mu : \Phi\Phi \to \Phi$ and a (bijective on objects) functor $k : A \to A_\Phi$ is obtained from $\eta : A \to \Phi$ . The opalgebra structure, $\varkappa : \Phi k \to k$ is constructed from $\mu$. Explicitly :

$$\Phi k_*(a, c) \simeq \int^b \Phi(b, c)\times k_*(a, b) \simeq \int^b \Phi(b, c)\times A_\Phi(a, bk)$$

$$\simeq \int^b \Phi(b, c)\times \Phi(a, b) \simeq \Phi\Phi(a, c) \xrightarrow{\mu} \Phi(a, c) \simeq A_\Phi(a, ck) \simeq k_*(a, c).$$

It is easy to check that this opalgebra, $[{}^{k}_{\varkappa}]$, is universal amongst opalgebras $[{}^{f}_{\varphi}]$ with $f$ in *CAT* . It continues to be universal if $f$ is a functor $A \to SET^{X^{op}}$ ($SET \notin CAT$) but the latter is a profunctor $A \to X$. So $A_\Phi$ is Kleisli in *PROF* and representable opalgebras give rise to representables out of $A_\Phi$.

Finally, note that $[{}^{k}_{\varkappa}]$ Kleisli for $(A, \Phi)$ implies



is a Kleisli diagram for the monad $(A^{op}, \Phi^{op})$. Applying $(\ )^{op}$ to this diagram yields an Eilenberg-Moore diagram for the monad $(A, \Phi)$. (Recall the duality diagram of the proof of Proposition 38 and note that ${k^{op}}_*{}^{op} = k^*$. ) ◊

The preceding two propositions can of course be established without the use of functors that suggest a change of universe. The proofs are tedious but suitably modified they allow *CAT* to be replaced by V- *CAT*, S- *ind CAT*, *cat*(S), etc.

***Proposition 40.*** *$(\ )_*$: TOP → $TOPLEX^{co}$ satisfies Axiom 4.*

***Proposition 41.*** *$(\ )_*$ : TOP → $TOPLEX^{co}$ satisfies Axiom 5.*

**Proof.** A monad in $TOPLEX^{co}$ is a cotriple, on a topos, whose underlying functor is left exact. If $(E, g)$ is such it is well-known that the category of Eilenberg-Moore coalgebras, $E_g$, is again a topos and that the cofree functor $k_*: E \to E_g$ is the direct image part of a geometric morphism. Necessarily, $k_*$ is an opcoalgebra for $g$, via $\delta : g \to gg$ since $ek_* = (eg, e\delta)$. Write $\varkappa : k_* \to gk_*$. Let $\left[\begin{smallmatrix} h \\ \omega \end{smallmatrix}\right]$ be any opcoalgebra with $h : E \to F$ in *TOPLEX* . For all $(e, \alpha)$ in $E_g$ , we have

$$(e,\ \alpha) \xrightarrow{\ \alpha\ } (eg,\ e\delta) \underset{e\delta}{\overset{\alpha g}{\rightrightarrows}} (egg,\ eg\delta)$$

an equalizer in $E_g$. So a left exact functor $\bar{h}: E_g \to F$ satisfying $k_*\bar{h} = h$ and $\varkappa\bar{h} = \omega$ must be given by

$$(e,\ \alpha)\bar{h} \longrightarrow eh \underset{e\omega}{\overset{\alpha h}{\rightrightarrows}} egh\ ,$$

an equalizer in F. Hence there exists an essentially unique such $\bar{h}$. $\bar{h}$ has a left exact left adjoint (thus making it an arrow in *TOP* ) iff $h$ does. This verifies all the Kleisli aspects of Axiom 5.

Finally, the forgetful functor $k^*: E_g \to E$ exhibits $E_g$ as an Eilenberg-Moore object in $TOPLEX^{co}$. ◊

It is clear that Propositions 40 and 41 apply equally well to

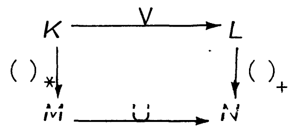
$(\ )_*$ : *ABEL* → $ABELLEX^{co}$ and $(\ )_*$: *GEOM* → $LEX^{co}$ ,

where *ABEL* is abelian categories and geometric morphisms (see [RW3]) and *GEOM* is left exact categories and geometric morphisms.

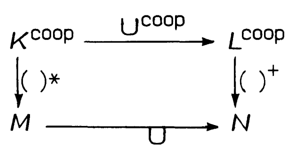
## THE AXIOMS.

We begin with a notion of morphism of proarrow equipments. Only Axioms 1, 2 and 3 are assumed. A *strong morphism from $(\ )_*$ : K → M to*

$(\ )_+ : L \to N$ is a pair of homomorphisms $V : K \to L$, $U : M \to N$ such that




commutes up to (specified) equivalence. (The equivalence should really be regarded as part of the data.) We write U for both U and V above and say that $U : M \to N$ preserves representability. Since homomorphisms preserve adjunctions we have also that

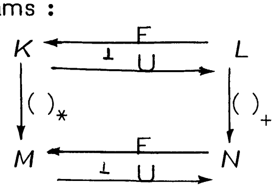


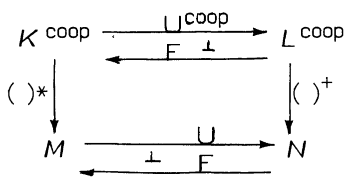

"commutes". We will write $U : (\ )_* \to (\ )_+$. An example is provided in the duality diagram of Proposition 38. (If $(\ )_* : K \to M$ is proarrow equipment then so is $(\ )^* : K^{co} \to M^{op}$.)

Now suppose that $U : M \to N$ has a left adjoint $F : N \to M$. (I.e., we have equivalences of hom categories, $M(\mathrm{N}F, \mathrm{M}) \sim N(\mathrm{N}, \mathrm{M}U)$, "natural" in M and N.) If F is also a strong homomorphism of proarrow equipments then the components of the unit, $\mathrm{N}n : \mathrm{N} \to \mathrm{N}FU$ and the counit $\mathrm{M}e : \mathrm{M}UF \to \mathrm{M}$ are representables. Their right adjoints

$$\mathrm{N}n^+ : \mathrm{N}FU \to \mathrm{N} \quad \text{and} \quad \mathrm{M}e^* : \mathrm{M} \to \mathrm{M}UF$$

tend to give $U : M \underset{\bot}{\rightleftarrows} N : F$ in addition to the assumed $F \dashv U$. Indeed, the triangle identities hold. However, while $n$ and $e$ are necessarily "natural", in N and M respectively, $n^+$ and $e^*$ need not be. A very special instance of this is provided by Counterexample 2. This leads us to say that $U : (\ )_* \to (\ )_+$ *has a left adjoint* if $U : M \to N$ has a left adjoint which preserves representables and the right adjoints of the unit and counit are "natural". Applying $(\ )^{coop}$ to an adjunction reverses the sense of it so in this situation we get a pair of "commutative" diagrams :




Write $I$ for the bicategory with one object, 0, and $I(0, 0) = \mathbf{1}$. (It is the terminal bicategory!) $I \to I$ is proarrow equipment. If $(\ )_* : K \to M$ is proarrow equipment then so is $(\ )_* \times (\ )_* : K \times K \longrightarrow M \times M$. Clearly,

$! : M \to I$ and $\Delta : M \to M \times M$ define strong homomorphisms of proarrow equipments, $! : (\ )_* \to I$ and $\Delta : (\ )_* \to (\ )_* \times (\ )_*$.

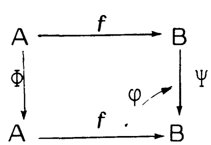
***Proposition 42.*** *Axiom 4 is equivalent to* $! : (\ )_* \to I$ *and* $\Delta : (\ )_* \to (\ )_* \times (\ )_*$ *having left adjoints.* ◊

We will use the machinery of this section to give an analogous formulation of Axiom 5. The situation is considerably more complex. In [ST2] Street defined, for a bicategory $K$, a bicategory *MND(K)* whose objects are monads $(X, t, \eta, \mu)$ in $K$. There is a diagonal homomorphism $I : K \to MND(K)$, a right adjoint for which is equivalent to the existence of Eilenberg-Moore objects in $K$. However, for any $K$, I has a left adjoint. It is a forgetful homomorphism. Existence of Kleisli objects in $K$ is equivalent to the existence of a left adjoint to a similar diagonal, $K \to MND(K^{oppp})$ (which always has a forgetful right adjoint).

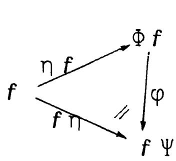
Henceforth, assume that $M$ has local coequalizers. *MOD(M)* is defined to be the bicategory whose objects are monads $(A, \Phi, \eta, \mu) = (A, \Phi)$ in $M$, whose arrows $(A, \Phi) \to (B, \Psi)$ are left $\Phi$, right $\Psi$ modules and whose transformations are equivariant transformations between modules. More explicitly, an arrow $(A, \Phi) \to (B, \Psi)$ is a triple $(\gamma, \Gamma, \delta)$, where $\Gamma : A \to B$ is an arrow in $M$ and $\gamma : \Phi\Gamma \to \Gamma \leftarrow \Gamma\Psi : \delta$ are assciating actions. Modules are composed using the coequalizer formula of Proposition 28. We have a diagonal $\Delta : M \to MOD(M)$ given by $X \to (X, 1_X)$.

**Lemma 43.** $\Delta$ has a right adjoint iff $M$ has Eilenberg-Moore objects. $\Delta$ has a left adjoint iff $M$ has Kleisli objects. ◊

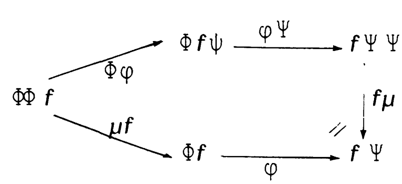
For proarrow equipment $(\ )_* : K \to M$ we define a bicategory $HOM(_*)$ as follows : The objects of $HOM(_*)$ are those of *MOD(M)* , monads $(A, \Phi)$ in $M$. An arrow $(A, \Phi) \to (B, \Psi)$ is a pair $(f, \varphi)$ where $f : A \to B$ is in $K$ and



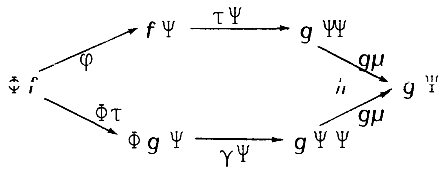

is a transformation in $M$ satisfying




and



For $(f, \varphi), (g, \gamma) : (A, \Phi) \to (B, \Psi)$, a transformation $(f, \varphi) \to (g, \gamma)$ is a transformation $\tau : f \to g\Psi$ satisfying




Arrows in $HOM(_*)$ are composed by pasting squares. If also

$$\sigma : (g, \gamma) \to (h, \vartheta) : (A, \Phi) \to (B, \Psi),$$

the composite $\tau.\sigma$ is given by

$$f \xrightarrow{\tau} g\Psi \xrightarrow{\sigma\Psi} h\Psi\Psi \xrightarrow{h\mu} h\Psi.$$

**Example 44.** Let $(\ )_+ : SET \to MAT$ denote the proarrow equipment obtained by restricting $(\ )_* : CAT \to PROF$ to the discrete objects of $CAT$. (So $MAT$ is equivalent as a bicategory to the bicategory of spans, $SPN(SET)$.) Then $HOM(+)$ is equivalent as a bicategory to $CAT$. ◊

For $(f, \varphi) : (A, \Phi) \to (B, \Psi)$ as above $f\Psi$ becomes a $\Phi, \Psi$ module, free on the right via $f\mu : f\Psi\Psi \to f\Psi$ with left $\Phi$ action given by

$$\Phi f\Psi \xrightarrow{\varphi\Psi} f\Psi\Psi \xrightarrow{f\mu} f\Psi.$$

A transformation $\tau : (f, \varphi) \to (g, \gamma)$ gives rise to an equivariant transformation $f\Psi \to g\Psi$ via

$$f\Psi \xrightarrow{\tau\Psi} g\Psi\Psi \xrightarrow{g\mu} g\Psi.$$

The assignments define a homomorphism $MON(_*) : HOM(_*) \to MOD(M)$. Every transformation $f\Psi \to g\Psi$ in $MOD(M)$ is obtained as above from a unique transformation $(f, \varphi) \to (g, \gamma)$ in $HOM(_*)$. (Indeed, it is obtained by composing $f\eta : f \to f\Psi$ and $f\Psi \to g\Psi$.) Furthermore, $\Psi f^*$ is $\Psi, \Phi$ module from $(B, \Psi)$ to $(A, \Phi)$ in $MOD(M)$. The right $\Phi$ action is constructed using $\varphi^* : f^*\Phi \to \Psi f^*$. We have $f\Psi \dashv \Psi f^*$ in $MOD(M)$. Hence :

**Lemma 45.** $MON(_*) : HOM(_*) \to MOD(M)$ is proarrow equipment. ◊

These and related matters will be dealt with in further detail in a forthcoming paper by Rosebrugh and Wood [RW4].

**Example 46.** For $(\ )_+ : SET \to MAT$ in Exemple 44,

$$MON(_+) : HOM(_+) \to MOD(MAT)$$

is equivalent to $(\ )_* : CAT \to PROF$. ◊

The homomorphism $\Delta: M \to MOD(M)$ defines a strong homomorphism of proarrow equipments, $\Delta: (\ )_* \to MON(_*)$.

***Proposition* 47.** *Axiom 5 is equivalent to* $\Delta: (\ )_* \to MON(_*)$ *having a left adjoint.* ◊

Artin glueing can be described globally in this manner too. The bicategories required for the description are related to the bicategory of glueing data in [NIE] in much the same way that *HOM* and *MOD* are related to *MND*.

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