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A NOTE ON THE ALGEBRAIC DE MORGAN'S LAW

by S. B. NIEFIELD and K. I. ROSENTHAL*

RÉSUMÉ. P. T. Johnstone a montré que le locale O(X) des ouverts d'un espace topologique X satisfait la deuxième loi de de Morgan (DML), $\neg(U \wedge V) = \neg U \vee \neg V$ ssi X est extrêmement disconnexe. L'analogue algébrique de la DML est l'équation des idéaux

$$\mathrm{Ann}(B \cap C) = \mathrm{Ann}(B) + \mathrm{Ann}(C)$$

dans un anneau commutatif R. Dans cet article, nous montrons que R satisfait la DML algébrique ssi R est un anneau de Baer. Si R n'a pas de nilpotents, ceci équivaut à la disconnexité extrémale de Spec(R). Enfin, nous montrons que si X est complètement régulier, C(X) satisfait la DML algébrique ssi X satisfait la DML topologique.

In [3], P.T. Johnstone showed that the locale O(X) of open subsets of a topological space X satisfies the second de Morgan's law

$$\neg(A \wedge B) = \neg A \vee \neg B$$

iff X is extremally disconnected. Furthermore, O(X) satisfies the logical principle

$$(A \Rightarrow B) \vee (B \Rightarrow A) = 1$$

(strong de Morgan's law) iff every closed subset of X is extremally disconnected. Motivated by this result and using the fact that a locale and the lattice of ideals of a commutative ring are both examples of closed posets, in [5] we characterized those commutative rings R such that the Zariski spectrum Spec(R) satisfies strong de Morgan's law. This was closely related to the ideal theoretic equation

$$A : B + B : A = R,$$

the algebraic analogue of this law. Using the techniques of [5], it is not difficult to show that if R has no nilpotents, Spec(R) is extremally disconnected iff

$$\mathrm{Ann}(A) + \mathrm{Ann}(B) = \mathrm{Ann}(A \cap B),$$

for all ideals A and B of R (the algebraic analogue of second de Morgan's law). In this paper, we present several equivalent characterizations of this class of rings in terms of ideal theoretic properties and obtain the extremally disconnectedness of the spectrum as a corollary. The rings under consideration turn out to be Baer rings, i.e., commutative

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rings R such that the annihilator Ann(A) of A is a principal ideal generated by an idempotent element of R, for all ideals A of R.

Let R be a commutative ring with identity. For any pair A, B of ideals, one can consider the following algebraic analogues of de Morgan's laws

(1) $$\mathrm{Ann}(A+B) = \mathrm{Ann}(A) \cap \mathrm{Ann}(B),$$

(2) $$\mathrm{Ann}(A \cap B) = \mathrm{Ann}(A) + \mathrm{Ann}(B).$$

As in the topological case, it is a straightforward exercise to show that (1) holds for all ideals A and B of R. On the other hand, although

$$\mathrm{Ann}(A) + \mathrm{Ann}(B) \subset \mathrm{Ann}(A \cap B),$$

the reverse containment does not hold in general. We shall refer to (2) as the *second de Morgan's law* (DML).

A related but weaker condition than (2) has also been considered, namely

(3) $$\mathrm{Ann}(a) + \mathrm{Ann}(b) = \mathrm{Ann}(ab)$$

where $a, b \in R$. We shall call (3): *weak de Morgan's law* (WDML). This condition was studied by Artico and Marconi in [1] where they showed that if R has no nilpotent elements, then (3) holds iff every prime ideal contains a unique minimal prime (possibly 0).

Recall that R is a *Baer ring* if for every ideal A, there exists an idempotent $e \in R$ such that $\mathrm{Ann}(A) = Re$. Since $Re = \mathrm{Ann}(1-e)$ and $1-e$ is idempotent whenever e is idempotent, it follows that R is Baer iff for every ideal A, there exists an idempotent $e \in R$ such that $\mathrm{Ann}(A) = \mathrm{Ann}(e)$. Now, if one requires that the above condition holds for principal ideals A only, or equivalently for annihilators of elements of A, such a ring is called a *weak Baer ring*.

There are several characterizations of weak Baer rings. In particular, Artico and Marconi also show in [1] that a ring with no nilpotents is weak Baer iff it satisfies weak de Morgan's law and its minimal spectrum is compact. Motivated by this result and our interest in second de Morgan's law, we began studying Baer rings.

We shall use the following three lemmas. The proofs are straightforward and therefore left to the reader.

Lemma 1. *If R satisfies WDML, then R has no nilpotents.*

Lemma 2. *If e and e' are idempotents, then*

$$\mathrm{Ann}(ee') = \mathrm{Ann}(e) + \mathrm{Ann}(e').$$

Lemma 3. *If* $\mathrm{Ann}(A) = \mathrm{Ann}(A')$ *and* $\mathrm{Ann}(B) = \mathrm{Ann}(B')$, *then*

$$\mathrm{Ann}(AB) = \mathrm{Ann}(A'B').$$

Theorem 1. *The following are equivalent for a commutative ring*

R *with identity.*

(a) Ann(AB) = Ann(A) + Ann(B), *for all ideals* A *and* B *of* R.
(b) R *satisfies DML and* R *has no nilpotents.*
(c) Ann(A) ⊕ Ann(Ann(A)) = R, *for every ideal* A *of* R.
(d) R *is a Baer ring.*
(e) R *satisfies WDML and* Ann(A) *is principal, for every ideal* A *of* R.

Proof. We shall show that (a) ⇒ (b) ⇒ (c) ⇒ (d) ⇒ (e) ⇒ (a).

(a) ⇒ (b) If (a) holds, then R has no nilpotents by Lemma 1, and R satisfies DML since

$$\mathrm{Ann}(A \cap B) \subset \mathrm{Ann}(AB) \subset \mathrm{Ann}(A) + \mathrm{Ann}(B) \subset \mathrm{Ann}(A \cap B).$$

(b) ⇒ (c) Suppose R satisfies (b). Since R has no nilpotents,

$$A \cap \mathrm{Ann}(A) = 0.$$

Applying DML, we get

$$\mathrm{Ann}(A) + \mathrm{Ann}(\mathrm{Ann}(A)) = \mathrm{Ann}(A \cap \mathrm{Ann}(A)) = \mathrm{Ann}(0) = R.$$

Therefore, Ann(A) ⊕ Ann(Ann(A)) = R.

(c) ⇒ (d) Suppose Ann(A) ⊕ B = R. Then $1 = x + y$, where $x \in A$ and $y \in B$. A straightforward calculation shows that $x^2 = x$ and Ann(A) = Rx.

(d) ⇒ (e) If R is a Baer ring, then clearly Ann(A) is principal for every A, and R satisfies WDML by Lemma 2.

(e) ⇒ (a) Suppose that (e) holds. Since R satisfies WDML, by Lemma 3, it suffices to show that for every ideal A, Ann(A) = Ann(a), for some $a \in R$. Since the annihilator of every ideal is principal, we can write Ann(A) = Rx and Ann(x) = Ra. Then

$$\mathrm{Ann}(a) = \mathrm{Ann}(Ra) = \mathrm{Ann}(\mathrm{Ann}(x)) = \mathrm{Ann}^3(A) = \mathrm{Ann}(A).$$

Using the above theorem, we shall obtain an elementary ring theoretic proof that Spec(R) is extremally disconnected iff R/N satisfies DML, where N denotes the nil radical of R. Note that this result can also be obtained via a lattice theoretic proof, using the techniques of [5], and Johnstone's characterization [3] of extremally disconnected spaces as those spaces X such that the topos Sh(X) of set-valued sheaves on X satisfies second de Morgan's law.

Recall that every open subset of Spec(R) is of the form

$$D(A) = \{P \mid A \not\subset P\},$$

where A is an ideal of R. The complement of D(A) is denoted by V(A).

Lemma 4. *If* R *has no nilpotents, then the closure of* D(A) *in* Spec(R) *is given by* $\overline{D(A)} = V(\mathrm{Ann}(A))$.

Proof. If P is prime, and $A \not\subset P$, then Ann(A) ⊂ P since A.Ann(A) = 0.

Hence, $D(A) \subset V(Ann(A))$, and so $\overline{D(A)} \subset V(Ann(A))$. For the reverse containment, we shall show that if $P \in V(Ann(A))$ (i.e., $Ann(A) \subset P$), then every open neighborhood of P meets D(A). Let D(B) be such an open set, i.e., $B \not\subset P$. Then $B \not\subset Ann(A)$, since $Ann(A) \subset P$. Hence, $AB \neq 0$. Since R has no nilpotents, $0 \subsetneq \sqrt{AB}$, where $\sqrt{\ }$ denoted the prime radical of an ideal. Therefore,

$$D(A) \cap D(B) = D(\sqrt{AB}) \neq D(\sqrt{0}) = \emptyset,$$

as desired.

Recall that a space X is *extremally disconnected* if the closure of every open set is open. These are the projective spaces in the category of compact topological spaces [6].

Theorem 2. *Suppose* R *has no nilpotents. Then* Spec(R) *is extremally disconnected iff* R *satisfies DML.*

Proof. It is well known that if R has no nilpotents, then A is a direct summand of R iff V(A) is an open subset of Spec(R). Using this and Theorem 1 (b) $\Leftrightarrow$ (c), R satisfies DML iff V(Ann A) is open, for all A. By Lemma 4, V(Ann A) is open, for all A, iff $\overline{D(A)}$ is open, for all A, iff Spec(R) is extremally disconnected.

Now, $Spec(R) \simeq Spec(R/N)$, where N denotes the nilradical of R. Thus, we obtain the following corollary.

Corollary 1. *The following are equivalent for a commutative ring* R *with identity.*

(a) Spec(R) *is extremally disconnected.*
(b) R/N *satisfies DML.*
(c) R/N *is a Baer ring.*

We conclude by examining the relationship between a space X and the ring C(X) of continuous real-valued functions on X. In the study of rings of continuous functions, it is shown that C(X) has nice completeness properties as a lattice if X is extremally disconnected (cf.[2] or [4]). In the following theorem, we show that a completely regular space X satisfies second de Morgan's law (i.e., is extremally disconnected) iff the ring C(X) satisfies second de Morgan's law (i.e., is a Baer ring).

Theorem 3. *Let* X *be a topological space.*

(1) If X *is extremally disconnected, then* C(X) *is a Baer ring.*
(2) If C(X) *is a Baer ring and* X *is completely regular, then* X *is extremally disconnected.*

Proof. (1) Let A be an ideal of C(X). We shall produce an idempotent $h \in C(X)$ that generates Ann(A).

If $f \in A$, let U_f denote the cozero set of f, i.e.,

$$U_f = \{ x \in X \mid f(x) \neq 0 \}.$$

Consider $U = \bigcup_{f \in A} U_f$. Then U is an open subset of X. Also, if $x \in U$, then $f(x) \neq 0$ for some $f \in A$, and hence $g(x) = 0$ for all $g \in \mathrm{Ann}(A)$. Thus, it follows that

$$\mathrm{Ann}(A) = \{g \in C(X) \mid g(\bar{U}) = 0\}.$$

Let $h: X \to R$ be defined by $h \equiv 0$ on $\bar{U}$ and $h \equiv 1$ on $X \setminus \bar{U}$. Since X is extremally disconnected, $\bar{U}$ is open, and therefore h is continuous. Then h is clearly idempotent, and it is not difficult to show that h generates Ann(A). Therefore, C(X) is a Baer ring.

(2) Let U be an open subset of X. To show that $\bar{U}$ is open, we shall show that $X \setminus \bar{U} = h^{-1}(1)$, for some $h \in C(X)$. Consider

$$A = \{f \in C(X) \mid f(X \setminus U) = 0\}.$$

Then A is an ideal of C(X). We claim that $U = \bigcup_{f \in A} U_f$, where U_f denotes the cozero set of f. Clearly, $\bigcup_{f \in A} U_f \subset U$. Conversely, if $x \in U$, then since X is completely regular, there is a continuous function

$$f : X \to R \quad \text{such that} \quad f(X \setminus U) = 0 \quad \text{and} \quad f(x) = 1.$$

Since $f \in A$, it follows that $x \in U_f$, as desired. Next, since C(X) is Baer, there exists $h \in C(X)$ such that $\mathrm{Ann}(A) = (h)$. Since

$$h \in \mathrm{Ann}(A) \quad \text{and} \quad U = \bigcup_{f \in A} U_f,$$

we must have $h(U) = 0$, and hence, $h(\bar{U}) = 0$. To see that $h^{-1}(1) = X \setminus \bar{U}$, it remains to show that $h(X \setminus \bar{U}) = 1$. If $x \notin \bar{U}$, then by complete regularity, we can find

$$g \in C(X) \quad \text{with} \quad g(\bar{U}) = 0 \quad \text{and} \quad g(x) = 1.$$

Since $g \in \mathrm{Ann}(A)$, it follows that $gh = g$, and so $h(x) = 1$. Thus, $h^{-1}(1) = X \setminus \bar{U}$, and so $\bar{U}$ is open. Therefore, X is extremally disconnected.

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