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TORSORS AND SPECIAL EXTENSIONS

by María J. VALE

Résumé. Le but de cet article est de donner une interprétation des torseurs de Duskin en termes d'extensions spéciales dans une catégorie d'intérêt. Ceci conduit à une classification des groupes de cohomologie H^1 et H^2, et de H^n pour $n \geqq 3$ si les groupes de cohomologie s'annulent sur les modules injectifs.

Introduction.

Duskin's theory of torsors [9] provides an internal simplicial interpretation of the cohomology groups relative to a cotriple (defined by a tripleable adjoint pair) in a category with finite limits. This theory is similar to that of Yoneda [34], where the role of extensions is played by certain simplicial objects called torsors.

The concept of crossed module, which was introduced by Whitehead [33], leads naturally to the definition of special extension. Special extensions have been studied by Holt [18], Huebschmann [19], Ratcliffe [29], Lue [24], etc., in order to obtain a classification of the Eilenberg-MacLane cohomology groups. The notion of crossed module, not only for groups but also for Lie algebras, appears in publications by Loday and Kassels [20, 22]. They include an interpretation of the third relative cohomology groups of Eilenberg-MacLane and Chevalley-Eilenberg and their application to prove the exactness of the eight-term sequences in K-theory.

Lichtenbaum and Schlessinger [21] define homology and cohomology groups T_i and T^i ($i = 0, 1, 2$), making use of a suitable special extension of commutative algebras, and they point out the close relation of certain (co-)homological conditions and the corresponding conditions in algebraic geometry. These cohomology groups coincide with the groups D^i ($i = 0, 1, 2$) defined by André [1] and Quillen [28], and with Harrison's groups H^{i+1} ($i = 0, 1$) for algebras over a field [17].

The framework of the present paper is a category of interest (Orzech [26]), which is a certain variety of groups with multiple operators. Cohomology theories for a category of groups with operators have been developed by Fröhlich [11], Gerstenhaber [14], and Lue [23], and more recently, by various authors ([7, 12, 13, 16, 30], etc.).

Our purpose is to provide an interpretation of torsors in terms of special extensions, thus obtaining a classification of the cotriple cohomology groups H^1 and H^2 and, in special cases, of H^n for $n \geqq 3$.

First, we give an approach to torsor theory, and special extensions in categories of interest. Then, we show that the group of connected components of $K(\Pi, 1)$ and $K(\Pi, 2)$-torsors is isomorphic to the group of equivalence classes of singular extensions and 2-fold special extensions, respectively. For $n \geq 3$, we prove the analogous result if the cohomology groups vanish on injective modules. Finally, we apply these results to obtain Glenn's long exact sequence of torsors [15], and to give an interpretation of the Eilenberg-MacLane and Chevalley-Eilenberg cohomology.

This paper is based on my thesis [32], written under the direction of Prof. A.R. Grandjean to whom I would like to express my thanks for his help and constant encouragement.

1. Preliminaries.

1.1. *Torsors.*

A simplicial object in a category C is a system

$$X_\bullet = ((X_n)_{n\geqq 0}, d_i, s_i)$$

of objects X_n together with maps

$$d_i : X_{n+1} \to X_n \text{ (called the } \textit{face operators)}$$

and

$$s_i : X_{n+1} \to X_{n+2} \text{ (called the } \textit{degeneracy operators)},$$

$0 \leqq i \leqq n+1$, which satisfy the following (*simplicial)*) identities :

$$d_i d_j = d_{j-1} d_i \text{ if } i < j, \quad d_i s_j = s_{j-1} d_i \text{ if } i < j,$$
$$s_i s_j = s_{j+1} s_i \text{ if } i \leqq j, \quad d_i s_j = s_j d_{i-1} \text{ if } i > j+1,$$
$$d_i s_i = d_{i+1} s_i = 1.$$

A simplicial map $f_\bullet : X_\bullet \to Y_\bullet$ is a family of maps $f_n : X_n \to Y_n$ $(n \geqq 0)$ which commute with all the face and degeneracy operators. The category of simplicial objects in C is denoted Simpl(C). An *augmented simplicial object over* X, denoted $X_\bullet \to X$, is a simplicial object with a map

$$d_{-1} : X_0 \to X \quad \text{such that} \quad d_{-1} d_0 = d_{-1} d_1 .$$

An X-*map of augmented simplicial objects* $X_\bullet \longrightarrow X$ and $Y_\bullet \longrightarrow X$ is a simplicial map

$$f_\bullet : X_\bullet \to Y_\bullet \quad \text{such that} \quad d'_{-1} f_0 = d_{-1} .$$

The *simplicial kernel* of the family of morphisms $f_i : X \to Y$,

$0 \leqq i \leqq n$, is an object K with maps

$$p_i : K \to X, \quad 0 \leqq i \leqq n+1,$$

and such that

(1) $$f_i p_j = f_{j-1} p_i \quad \text{for all} \quad i < j,$$

(2) for any family of morphisms $q_i: Z \to X$, $0 \leqq i \leqq n+1$, such that

$$f_i q_j = f_{j-1} q_i \quad \text{for all} \quad i < j,,$$

there exists a unique morphism

$$q = \langle q_0, ..., q_{n+1} \rangle : Z \longrightarrow K \quad \text{satisfying} \quad p_i q = q_i \ , \quad 0 \leqq i \leqq n+1.$$

An *n-truncated simplicial object* is a system

$$X_\bullet, \mathrm{tr} = ((X_k)_{0 \leqq k \leqq n}, d_i, s_i)$$

such that the d_i and s_i verify the simplicial identities whenever they are defined. The process of n -truncating is a functor, denoted tr^n . If C has finite limits then tr^n admits a right adjoint $cosk^n$, called *n-coskeleton functor*. We may use the following construction of the coskeleton of an n-truncated simplicial object $X_\bullet$, *tr*. Let

$$p_i : K_{n+1} \to X_n \ , \quad 0 \leqq i \leqq n+1,$$

be the simplicial kernel of the family

$$d_i : X_n \to X_{n-1} \ , \quad 0 \leqq i \leqq n \ .$$

One may define maps $s_j : X_n \to K_{n+1}$, by $s_j = \langle \alpha_{0j} , ..., \alpha_{nj} \rangle$, where

$$\alpha_{ij} = s_{j-1} d_i \text{ if } i < j, \quad \alpha_{ij} = s_j d_{i-1} \quad \text{if} \quad i > j+1$$
$$\text{and} \quad \alpha_{j\,j+1} = \alpha_{jj} = 1.$$

Thus one may build up the n -coskeleton of $X_\bullet$, *tr* by iterating simplicial kernels. The functor *Simpl(C)* $\to$ *Simpl(C)* obtained by truncating to dimension n and then applying $cosk^n$ is denoted $Cosk^n$. If $X_\bullet, tr \to X$ is an n-truncated augmented simplicial object, $n \geqq -1$, we may build up its *augmented coskeleton* denoted $cosk^n_{aug}(X_\bullet, tr \longrightarrow X)$ again iterating simplicial kernels.

If $X_\bullet \to X$ is an augmented simplicial object, then $X_\bullet \to X$ is said to be *split* if there is a family of maps

$$\{ s_{n+1} : X_n \to X_{n+1} \} \qquad \text{where} \quad X_{-1} = X,$$

called a *contraction*, satisfying the relations :

$$d_i s_n = s_{n-1} d_i \quad \text{for} \quad 0 \leqq i < n \quad \text{and} \quad s_i s_n = s_{n+1} s_i \quad \text{for} \quad 0 \leqq i \leqq n$$

$$\text{and} \quad d_n s_n = 1.$$

If $U : C \to B$ is a functor, $X_\bullet \to X$ is said to be *U-split* if the underlying augmented simplicial object $U(X_\bullet) \to UX$ is split. If $X_\bullet, tr \to X$ is split (U-split) then $cosk^n_{aug}(X_\bullet, tr \to X)$ is split (U-split, if U is left exact).

Given $X_\bullet$, one can form a split augmented simplicial object denoted $Dec(X_\bullet)$, where $Dec(X_\bullet)_n = X_{n+1}$ and where the face and degeneracy operators are those of $X_\bullet$ except that $d_n : X_n \to X_{n+1}$ is omitted for each n.

Let C be a category with zero object and finite limits, and $X_\bullet$ a simplicial object in C. One defines the *Moore complex* of $X_\bullet$ as the complex :

$$(MX_\bullet)_0 = X_0\,, \quad (MX_\bullet)_n = \bigcap_{i=1}^{n} ker\ d_i \quad (n \geqq 1)$$

with differential

$$\delta_n = d_0 |_{(MX_\bullet)_n} : (MX_\bullet)_n \to (MX_\bullet)_{n-1}\,.$$

If C is an interest category (1.2) and $X_\bullet \to X$ is a U-split simplicial object (U being the underlying functor to the category of pointed sets or modules), since the homotopy of $X_\bullet$ coincides with the homology of the Moore complex, this Moore complex is exact.

I.1.1. Definition. Let C be a category with finite limits and Π and abelian group object in C [32]. For any $n \geqq 1$, one defines the simplicial object $K(\Pi, n)$ as the $(n+1)$-coskeleton of the $(n+1)$-truncated simplicial object

$$\begin{tikzcd}
K(\Pi, n) : \Pi^{n+1} \arrow[r, shift left=4, "k_{n+1}"] \arrow[r, "pr_n"'] \arrow[r, phantom, shift right=2, "\vdots"] \arrow[r, shift right=4, "pr_0"'] & \Pi \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & 1 \quad \ldots \quad 1 \arrow[r] & 1
\end{tikzcd}$$

where

$$k_{n+1} = (-1)^n \sum_{i=0}^{n} (-1)^i pr_i$$

(i.e. the signed alternating sum of all projections). If $X_\bullet$ is any simplicial object in C, the function

$$Pr_n : Simpl\ (X_\bullet, K(\Pi, n)) \to Z^n(X_\bullet, \Pi)$$

given by

$$Pr_n(f_\bullet) = f_n$$

defines a functorial isomorphism of the group of simplicial mappings onto the group of normalized cocycles (i.e. the group of cocycles

$$c : X_n \to \Pi \qquad \text{such that} \qquad cs_i = 0 \quad \text{for} \quad 0 \leqq i < n \ ).$$

I.1.2. Definition. Given $X_\bullet$, $n \geqq 0$ and $0 \leqq i \leqq n+1$, denote by

$$pr_j : \Lambda^i(n)\,(X_\bullet) \to X_n\,, \qquad 0 \leqq j \leqq n+1, \quad i \neq j,$$

the family of maps satisfying :

(1) $$d_j pr_k = d_{k-1}\, pr_j\ , \quad j < k, \quad j,\ k \neq i,$$

(2) if $q_j : Y \to X_n$, $0 \leqq j \leqq n+1$, $j \neq i$, is any family of maps such that

$$d_j q_k = d_{k-1}\, q_j \text{ for all } j < k, \quad j,\ k \neq i,$$

there exists a unique map

$$q = < q_0, \dots, \overset{i}{-}, \dots, q_{n+1}> : \ Y \to \Lambda^i(n)(X_\bullet)$$

with : $pr_j\, q = q_j$ for all $j \neq i$.

For instance, in the category of sets,

$$\Lambda^i(n)\ (X_\bullet) =$$
$$= \{ (x_0, \dots, x_{i-1}, x_{i+1}\ , \dots, x_{n+1}\ ) \in X_n^{n+1} \mid d_j x_k = d_{k-1}\, x_j, \ j < k, \ j,\ k \neq i \} \ .$$

Let B and C be categories with finite limits and $U : C \to B$ a left exact functor (i.e., finite limits preserving). Let $n \geqq 1$ be an integer, X an object in C and Π an abelian group object.

I.1.3. **Definition** [10]. A K(Π, n)-*torsor* over X *relative to* U is a triplet $(X_\bullet,\ s_\bullet\ ,\ \chi_\bullet)$ where

(a) $X_\bullet \to X$ is a U-split augmented simplicial object, with contraction $s_\bullet$.

(b) $\chi_\bullet : X_\bullet \to K(\Pi\ ,\ n)$ is a simplicial map such that the following squares are pullbacks, for each $m \geqq n$ and $0 \leqq i \leqq m$,

$$\begin{tikzcd}
X_m \arrow[r, "\chi_m"] \arrow[d, "{<d_0, \dots, \overset{i}{-}, \dots, d_m>}"'] & K(\Pi, n)_m \arrow[d, "{< k_0, \dots, \overset{i}{-}, \dots, k_m>}"] \\
\Lambda^i(m-1)(X_\bullet) \arrow[r] & \Lambda^i(m-1)\,(K(\Pi,\ n))
\end{tikzcd}$$

(c) the canonical map $d : X_\bullet \to Cosk^{n-1}_{aug}(X_\bullet)$ is an isomorphism.

For a fixed Π, n and X, we define a *morphism of the* K (Π, n)-*torsor* $(X_\bullet,\ s_\bullet,\ \chi_\bullet)$ *into the* K(Π , n)-*torsor* $(Y_\bullet,\ t_\bullet\ ,\ \chi'_\bullet)$ to be an X-map of augmented simplicial objects $f_\bullet : X_\bullet \to Y_\bullet$ such that $\chi'_\bullet\ f_\bullet = \chi_\bullet$.

The category of K(Π, n)-torsors over X relative to U is denoted $\mathrm{TORS}_U(X, \Pi)$ and its class of connected components $\mathrm{TORS}_U[X,\ \Pi]$.

I.1.4. ***Lemma.*** *Let B be a category with finite limits and $U : C \to B$ a tripleable functor. If $(X_\bullet,\ s_\bullet,\ \chi_\bullet)$ is a* K(Π, n)-*torsor over* X *relative to*

U, $Ud : U(X_\bullet) \to \mathrm{Cosk}^{n-2}(U(X_\bullet))$ is the canonical map and

$$s^*(U\chi_\bullet) = (U\chi_n s_n)_{n \geqq 0} : U(X_\bullet) \to \mathrm{Dec}(K(U\Pi, n))$$

then the map

$$Ud,\ s^*(U\chi_\bullet) \ : U(X_\bullet) \to \mathrm{Cosk}^{n-2}_{aug}(U(X_\bullet)) \times \mathrm{Dec}(K(U\Pi, n))$$

is an isomorphism of augmented simplicial objects.

Proof. Let

$$p_i : K_{n-1} \to X_{n-2}, \quad 0 \leqq i \leqq n-1,$$

be the simplicial kernel of the family

$$d_i : X_{n-2} \to X_{n-3}, \quad 0 \leqq i \leqq n-2.$$

It is sufficient to show that the square

$$\begin{tikzcd}
UX_{n-1} \arrow[r, "U\chi_n s_n"] \arrow[d, "{\langle Ud_0, \ldots, Ud_{n-1} \rangle}"'] & U\Pi \arrow[d] \\
UK_{n-1} \arrow[r] & U1
\end{tikzcd}$$

is a pullback [9]. But this square is composite of the squares

$$\begin{tikzcd}
UX_{n-1} \arrow[r, "s_n"] \arrow[d, "{\langle Ud_0, \ldots, Ud_{n-1} \rangle}"'] & UX_n \arrow[r, "U\chi_n"] \arrow[d, "{\langle Ud_0, \ldots, Ud_{n-1}, - \rangle}"'] & U\Pi \arrow[d] \\
UK_{n-1} \arrow[r, "h"] & U\Lambda^n(n-1)(X_\bullet) \arrow[r] & U1
\end{tikzcd}$$

where

$$h = \langle s_{n-1} Up_0, \ldots, s_{n-1} Up_{n-1}, - \rangle,$$

and thus it is a pullback iff the left hand square is a pullback. Now, the result follows from the fact that the composite of the squares

$$\begin{tikzcd}
UX_{n-1} \arrow[r, "s_n"] \arrow[d, "{\langle Ud_0, \ldots, Ud_{n-1} \rangle}"'] & UX_n \arrow[r, "Ud_n"] \arrow[d, "{\langle Ud_0, \ldots, Ud_{n-1}, - \rangle}"'] & UX_{n-1} \arrow[d, "{\langle Ud_0, \ldots, Ud_{n-1} \rangle}"'] \\
UK_{n-1} \arrow[r, "h"] & U\Lambda^n(n-1)(X_\bullet) \arrow[r, "g"] & UK_{n-1}
\end{tikzcd}$$

where

$$g = \langle U(d_{n-1} pr_0), \ldots, U(d_{n-1} pr_{n-1}) \rangle,$$

is a pullback.

From (1.1.3) and [32] it follows that for U tripleable the above definition of $K(\Pi, n)$-torsor over X relative to U is equivalent to that of Duskin in [9].

1.1.5. ***Theorem (Duskin*** [9]***).*** *If B is a category with finite limits and $U : C \to B$ is a tripleable functor, there exists a bijection*

$$Z^n_{\mathbf{G}}\colon \mathrm{TORS}_U[X, \Pi] \to H^n(X, \Pi)_{\mathbf{G}}$$

where $H^n(X, \Pi)_{\mathbf{G}}$ is the nth cotriple cohomology group of X with coefficients in Π, $\mathbf{G}$ being the cotriple defined by the pair of adjoint functors (F, U).

The map $Z^n_{\mathbf{G}}$ is defined by sending a connected component $[(X_\bullet, s_\bullet, \chi_\bullet)]$ to the class of the *characteristic cocycle* $Z^n_{\mathbf{G}}(X_0)$ of the torsor $(X_\bullet, s_\bullet, \chi_\bullet)$.

The inverse map of Z^n is S^n given by

$$S^n[k] = [S^n(k')]$$

where k is a normalized n-cocycle, k' the non-homogeneous form of k and $S^n(k')$ the *standard* $K(\Pi, n)$*-torsor over* X *defined by* k'.

1.1.6. **Proposition.** *Let B be a category with finite limits and $U : C \to B$ a tripleable functor. If*

$$\Pi' \rightarrowtail{f} \Pi \xrightarrow{g} \Pi''$$

is a short U-exact sequence of abelian group objects in C (i.e., for each object X we have exactness of

$$0 \to B(X, U\Pi') \xrightarrow{B(X,\, Uf)} B(X, U\Pi) \xrightarrow{B(X,\, Ug)} B(X, U\Pi'') \to 0\ )$$

then there are connecting homomorphisms such that

$$0 \to C(X, \Pi') \to C(X, \Pi) \to C(X, \Pi'') \xrightarrow{\bar\partial_0} \mathrm{TORS}^1_U[X, \Pi'] \xrightarrow{\bar f_1}$$
$$\mathrm{TORS}^1_U[X, \Pi] \xrightarrow{\bar g_1} \mathrm{TORS}^1_U[X, \Pi''] \xrightarrow{\bar\partial_2} \mathrm{TORS}^2_U[X, \Pi'] \xrightarrow{\bar f_2} \ldots$$

is exact, where for each abelian group object Π the group structure of $\mathrm{TORS}^n_U[X, \Pi]$ is given by

$$[(X_\bullet, s_\bullet, \chi_\bullet)] \oplus [(X'_\bullet, s'_\bullet, \chi'_\bullet)] = S^n[Z^n(X'_\bullet) + Z^n(X'_\bullet)]$$

and

$$\bar f_n[(X_\bullet, s_\bullet, \chi_\bullet)] = S^n[Z^n(X_\bullet)], \quad \bar g_n[(X'_\bullet, s'_\bullet, \chi'_\bullet)] = S^n[Z^n(X'_\bullet)]$$
$$\bar\partial_n[(X''_\bullet, s''_\bullet, \chi''_\bullet)] = S^{n+1}(\partial_n[Z^n(X''_\bullet)]).$$

Proof. It follows from Theorem 1.1.5, making use of the long exact sequence in cohomology [5]

$$\dots \to H^n(X, \Pi)_G \to H^n(X, \Pi'')_G \xrightarrow{\partial_n} H^{n+1}(X, \Pi')_G \to \dots$$

1.1.7. **Definition.** A *groupoid object in C* is a 1-truncated simplicial object in C

$$\begin{tikzcd} A_1 \arrow[r, shift left, "d_0"'] \arrow[r, shift right=3, "d_1"'] & A_0 \arrow[l, shift right=4, "s"'] \end{tikzcd}$$

together with maps

$$\mu : A_1 \times_{A_0} A_1 \to A_1 , \qquad \sigma : A_1 \to A_1 ,$$

where the square

$$\begin{tikzcd} A_1 \times_{A_0} A_1 \arrow[r, "p_2"] \arrow[d, "p_1"'] & A_1 \arrow[d, "d_0"] \\ A_1 \arrow[r, "d_1"] & A_0 \end{tikzcd}$$

is a pullback and such that

$$d_0\mu = d_0 p_1, \quad d_1\mu = d_1 p_2, \quad \mu\langle s_0 d_0 , 1\rangle = \mu\langle 1, s_0 d_1\rangle = 1,$$

$$\mu(\mu \times 1) = \mu(1 \times \mu), \quad \mu\langle 1, \sigma\rangle = s_0 d_0 , \quad \mu\langle \sigma , 1\rangle = s_0 d_1 .$$

1.1.8. **Definition.** Let $(X_\bullet, s_\bullet, \chi_\bullet)$ be a $K(\Pi, n)$-torsor over X relative to U. The *fiber* of $X_\bullet$ is the pullback simplicial object

$$\begin{tikzcd} F_\bullet(X_\bullet) \arrow[r] \arrow[d] & X_\bullet \arrow[d, "\chi_\bullet"] \\ 1 \arrow[r] & K(\Pi, n) \end{tikzcd}$$

If $(X_\bullet, s_\bullet, \chi_\bullet)$ is a $K(\Pi, 2)$-torsor over X relative to U, the 1-truncation of $F_\bullet(X_\bullet)$ is a groupoid [32].

1.2. Categories of interest.

A category of interest is a category C, with the following axioms [26] :

(1) There is a triple $\mathbf{T} = (T, \eta, \mu)$ on *Set*, such that $T(\emptyset) = \{\cdot\}$ (a one-point set) and C is equivalent to $Set^{\mathbf{T}}$.

(2) $U : C \to Set^{\bullet}$ (the category of pointed sets) factors through the category of groups.
(3) All operations in C are finitary.
(4) There is a generator set Ω for the operations in C and

$$\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$$

(Ω_n is the set of n-ary operations in Ω) such that Ω contains identity, inverse and $+$ operations associated with the group structure, and if

$$* \in \Omega_2' = \Omega_2 - \{+\} \ ,$$

then $*^{\circ}$, defined by

$$x*^{\circ}y = y*x,$$

is also in Ω_2'.
(5) $a * (b + c) = a*b + a * c$, for $* \in \Omega_2'$.

(6) $\omega (a + b) = \omega(a) + \omega(b)$ and $\omega (a*b) = \omega(a)*b$,
for $\omega \in \Omega_1' = \Omega_1 - \{-\}$ and $* \in \Omega_2'$.

(7) $a + (b*c) = (b*c) + a$, for $* \in \Omega_2'$.

(8) For each ordered pair $(*, *') \in \Omega_2' \times \Omega_2'$ there is a word w, satisfying

$$(a*b)*'c = w(a(bc), a(cb), (bc)a, (cb)a, b(ac), b(ca), (ac)b, (ca)b)$$

where juxtaposition represents an operation in Ω_2'.

An object A in C is *singular* if it is abelian as a group and

$$A * A = 0, \quad \text{for each} \quad * \in \Omega_2' .$$

Let A and X be objects in C. A is an X-*structure* in C if there is a right-split extension of X by A

$$\begin{tikzcd} A \arrow[r, tail, "i"] & E \arrow[r, two heads, shift left, "p"] & X \arrow[l, shift left, "s"] \end{tikzcd} \qquad (ps = 1_X).$$

Moreover, if A is singular, we call it an X-*module*.

An X-structure induces actions of X on A by

$$x.a = i^{-1}(sx + ia - sx) \quad \text{and} \quad x*a = i^{-1}(sx * a)$$

for $* \in \Omega_2'$.
These actions determine the object E, which can be considered to be the cartesian product A x X with the following operations

$$(a, x) + (a', x') = (a + x.a', x + x'),$$

$$(a, x) * (a', x') = (a*a' + x*a' + a*x', x*x')$$

for $* \in \Omega_2'$,

$$\omega (a, x) = (\omega(a), \omega (x))$$

for $\omega \in \Omega_2'$.

It is denoted $E = A \top X$ and called *semidirect-product of* X *by* A .

A morphism of X *-structures* is a morphism $g : A \to A'$ such that

$$g(x.a) = x.g(a) \quad \text{and} \quad g(x*a) = x*g(a)$$

for $* \in \Omega'_2$. The categories of X-structures and X-modules will be denoted by X- *Str* and X- *Mod*, respectiveley.

We shall denoted by Der(X, A) the set of all *derivations* of X by A, i.e., the set of all functions $d : X \to A$ verifying

$$d(x + y) = dx + x.dy,$$

$$d(x*y) = dx * dy + dx * y + x * dy,$$

for $* \in \Omega'_2$,

$$d(\omega(x)) = \omega(dx), \quad \text{for} \quad \omega \in \Omega'_1 .$$

If $f : Y \to X$ is a morphism in C an *f-derivation* is a derivation $d : Y \to A$ where A is an Y-structure *via f* (i.e.,

$$y.a = f(y).a, \quad y*a = f(y)*a \;).$$

The set of all f-derivations of Y by A will be denoted by $\mathrm{Der}(X, A)_f$. There is a natural equivalence

$$(C, X)(-, A \top X \xrightarrow{p_X} X) \simeq \mathrm{Der}(-, A)_f$$

where (C, X) is the category of objects in C over a fixed object X [32].

If *Ab* (C, X) is the category of abelian group objects in (C, X), then there is an equivalence $Ab(C, X) \simeq X\text{-}Mod$ [26].

Since C is an exact category [3], so is (C, X). Thus X-*Mod* is an abelian category.

We denote by E (resp. E_L) the class of all surjective epimorphisms in C (resp. epimorphisms in C, which split as epimorphisms in a category L of modules over a ring). The class of all epimorphisms in X-*Mod* (resp. epimorphisms in X-*Mod* which split in L) is denoted by EM (resp. EM_L).

I.2.1. Definition. An *(L-)n-fold special extension,* $n \geqq 1$, *of* X *by an* X *-module* A is an exact sequence in C

$$\begin{tikzcd}
E^n : & A \arrow[r, tail, "\partial_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] & \cdots \arrow[r] & C_2 \arrow[r, "\partial_2"] & C_1 \arrow[rr, "\partial_1"] \arrow[dr, "\partial_2^C"'] & & C_0 \arrow[r, two heads, "p"] & X \\
& & & & & & L \arrow[ur] & &
\end{tikzcd}$$

satisfying :

(1) $p, \partial_2^C \in E_{(L)}$.

(2) C_1 is a C_0-structure and the morphism $\partial_1 : C_1 \to C_0$ is a morphism of C_0-structures, where C_0 acts on itself by conjugation, and such that

$$\partial_1(c_1).c_1' = c_1 + c_1' - c_1 \quad \text{and} \quad \partial_1(c_1)*c_1' = c_1 * c_1'$$

for $* \in \Omega_2'$, and $c_1, c_1' \in C_1$.

(3) for $k \geqq 2$, C_k is an X-module.

(4) $\partial_2 : C_2 \to C_1$ is a morphism of C_0-structures, where C_0 acts on C_2 via $p \colon C_0 \to X$.

(5) for $k \geqq 3$, ∂_k is a morphism of X-modules with $\partial_k^{C} \in EM_{(L)}$.

We call E^1 a(n) *(L-)singular extension,* if A is considered a C_0-structure via $p : C_0 \to X$.

Two (*L* -) *n*-fold special extensions E^n and E'^n of X by A are *related* if there is a family of morphisms $\alpha = \{\alpha\}_{0 \leqq k \leqq n-1}$ such that

(a) the diagram

$$\begin{tikzcd}
E^n\colon A \arrow[r, tail] \arrow[d, equal] & C_{n-1} \arrow[r] \arrow[d, "\alpha_{n-1}"] & \cdots \arrow[r] & C_2 \arrow[r] \arrow[d, "\alpha_2"'] & C_1 \arrow[r] \arrow[d, "\alpha_1"'] & C_0 \arrow[r, two heads] \arrow[d, "\alpha_0"] & X \arrow[d, equal] \\
E'^n\colon A \arrow[r, tail] & C'_{n-1} \arrow[r] & \cdots \arrow[r] & C'_2 \arrow[r] & C'_1 \arrow[r] & C'_0 \arrow[r, two heads] & X
\end{tikzcd}$$

is commutative,

(b) α_1 is a morphism of C_0-structures, where C_0 acts on C'_1 via α_0,

(c) α_k is a morphism of X-modules, for $k \geqq 2$.

This relation generates an equivalence relation. We write $[E^n]$ for the equivalence class of E^n and $S^n(X, A)_{(L)}$ for the quotient set.

If $f \colon A \to B$ is a morphism of X-modules, the map

$$f_n \colon S^n(X, A)_{(L)} \longrightarrow S^n(X, B)_{(L)}$$

given by

$$f_1[E^1] = [B \rightarrowtail (B \top C_0)/\{(fa, -ia) \mid a \in A\} \twoheadrightarrow X]$$

$$f_2[E^2] = [B \rightarrowtail (B \times C_1)/\{(fa, -ia) \mid a \in A\} \to C_0 \twoheadrightarrow X]$$

$$f_n[E^n] = [B \rightarrowtail (B \times C_{n-1})/\{(fa, -ia) \mid a \in A\} \to \dots \to C_0 \twoheadrightarrow X], \quad n \geqq 3,$$

is a homomorphism of groups.

$S^n(X, A)_{(L)}$ is an abelian group with the Baer sum, the zero element being the class

$$[A \rightarrowtail A \top X \twoheadrightarrow X], \quad \text{for} \quad n = 1,$$

and

$$[A = A \to 0 \to \ldots \to 0 \to X = X]\,, \qquad \text{for } n \geqq 2.$$

1.2.2. **Theorem** [32]. *If*

$$A \rightarrowtail^{f} B \xrightarrow{g} C$$

is a short exact sequence of X*-modules, there are connecting homomorphisms such that*

$$0 \to \mathrm{Der}(X, A) \to \mathrm{Der}(X, B) \to \mathrm{Der}(X, C) \xrightarrow{\delta_0} S^1(X, A) \xrightarrow{f^1}$$
$$S^1(X, B) \xrightarrow{g_1} S^1(X, C) \xrightarrow{\delta_1} S^2(X, A) \to \ldots$$

is exact, where

$$\delta_0(d) = [A \rightarrowtail BT_C X \xrightarrow{p_X} X\,] \qquad \textit{with} \quad BT_C X = \{(b, x) \in BT\,X \mid gb = dx\},$$

$$\delta_1[\,C \rightarrowtail C_0 \xrightarrow{p} X\,] = [A \xrightarrow{f} B \longrightarrow C_0 \xrightarrow{p} X\,],$$

$$\delta_n[\,A \rightarrowtail C_{n-1} \to \ldots \to C_0 \xrightarrow{p} X\,] =$$
$$[\,A \xrightarrow{f} B \to C_{n-1} \to \ldots \to C_0 \xrightarrow{p} X\,]\,, \quad n \geqq 2.$$

Moreover if

$$\begin{tikzcd}
A \arrow[r, tail, "f"] \arrow[d, "\nu"'] & B \arrow[r, two heads, "g"] \arrow[d, "\sigma"] & C \arrow[d, "\mu"] \\
A' \arrow[r, tail, "f'"] & B' \arrow[r, two heads, "g'"] & C'
\end{tikzcd}$$

is a commutative diagram in X*-Mod, with short exact rows, then the following diagram is commutative*

$$\begin{tikzcd}
\ldots \arrow[r] & S^n(X, A) \arrow[r, "f_n"] \arrow[d, "\nu_n"'] & S^n(X, B) \arrow[r, "g_n"] \arrow[d, "\sigma_n"] & S^n(X, C) \arrow[r, "\delta_n"] \arrow[d, "\mu_n"] & S^{n+1}(X, A) \arrow[r] \arrow[d, "\nu_{n+1}"'] & \ldots \\
\ldots \arrow[r] & S^n(X, A') \arrow[r, "f'_n"] & S^n(X, B) \arrow[r, "g'_n"] & S^n(X, C) \arrow[r, "\delta'_n"] & S^{n+1}(X, A) \arrow[r] & \ldots
\end{tikzcd}$$

for $n \geqq 0$, $S^0 = \mathrm{Der}$.

The following lemma is immediate.

1.2.3. **Lemma.** *If* I *is an injective* X*-module,* $S^n(X, I) = 0$, *for* $n \geqq 3$.

1.2.4. **Lemma.** *If*

$$\begin{tikzcd}
A_1 \arrow[r, shift left=4, leftarrow, "s"] \arrow[r, "d_1"] \arrow[r, shift right=4, "d_0"'] & A_0
\end{tikzcd}$$

is a groupoid object in C and $H = \ker d_1$ *, then* H *is an* A_0 *-structure*

$$\begin{tikzcd}(H \arrow[r, tail] & A_1 \arrow[r, two heads, shift left, "d_1"] & A_0) \arrow[l, shift left, "s"]\end{tikzcd}$$

satisfying

$$d_0a.b = a + b - a, \quad d_0a * b = a * b, \quad * \in \Omega_2', \quad a, b \in H .$$

Proof. The elements of A_1 can be represented uniquely as $a + sx$ for $a \in H$, $x \in A_0$. Since $\mu : A_1 \times_{A_0} A_1 \to A_1$ is a homomorphism of groups, we have

$$(a + s(d_0b + y), b + sy) = a + b + sy.$$

Then,

$$a + b = \mu(sd_0a, a) + \mu(b, o) = \mu(d_0a.b + sd_0a, a) = d_0a.b + a$$

and thus,

$$d_0a.b = a + b - a.$$

On the other hand,

$$d_0a * b = \mu(d_0a * b, o) = \mu(sd_0a, a) * \mu(b, o) = a * b.$$

2. Singular extensions and *n*-fold special extensions.

If X is an object of *C* and *L* a category of modules over a ring, we denote by

$$U_X : (C, X) \to (Set^\bullet, X)$$

the functor induced by the underlying functor *U* to $Set^\bullet$, and by

$$U_{L\,X} : (C, X) \to (L, X)$$

the functor induced by the underlying functor to *L* (if *U* factors through *L*). Let A be an X-module and

$$\begin{tikzcd}\Pi = (A \top X \arrow[r, shift left, "p_X"] & X). \arrow[l, shift left, "s"]\end{tikzcd}$$

2.1. K(Π , 1) *-torsors and singular extensions.*

2.1.1. ***Proposition.*** *If* $(X_\bullet , s_\bullet, \chi_\bullet)$ *is a* K(Π , 1)*-torsors over* $1 : X \to X$ *relative to* $U_{(L)\,X}$ *in (C, X), then the Moore complex of* $X_\bullet$ *is a(n) (L-) singular extension.*

Proof. Since $X_\bullet$ is $U_{(L)X}$ -split and

$$Cosk^0_{aug}(X_\bullet \to X) \simeq (X_\bullet \to X),$$

then $MX_\bullet$ is exact and $(MX_\bullet)_m = 0$, for all $m \geq 2$. Thus

$$\begin{tikzcd}MX_\bullet : \ker d_1 \arrow[r, tail, "d_0"] & X_0 \arrow[r, two heads, "d_1"] & X .\end{tikzcd}$$

Now, Definition 1.1.2 (b) yields the isomorphism $\chi_1 : ker\ d_1 \simeq A$ and the existence of a morphism

$$s' : X_0 \longrightarrow X_1 \qquad \text{such that} \qquad sd_{-1} = \chi_1 s', \quad d_1 s' = 1.$$

Furthermore there exists a map $r : X_0 \to X_1$, satisfying

$$d_0 r = s_0 d_{-1} \ , \quad d_1 r = 1 \ .$$

The short exact sequence

$$A \overset{d_0 \chi_1^{-1}}{\rightarrowtail} X_0 \overset{d_1}{\twoheadrightarrow} X$$

is a(n) (L -)singular extension of X by A. In fact,

$$(d_0 \chi_1^{-1})^{-1} (s_0 x + d_0 \chi_1^{-1} a - s_0 x) = (d_0 \chi_1^{-1})^{-1} d_0 (r s_0 x + \chi_1^{-1} a - r s_0 x) =$$

$$= (d_0 \chi_1^{-1})^{-1} d_0 (s' s_0 x + \chi_1^{-1} a - s' s_0 x) = sx + a - sx$$

and

$$(d_0 \chi_1^{-1})^{-1} (s_0 x * d_0 \chi_1^{-1} a) = (d_0 \chi_1^{-1})^{-1} d_0 (r s_0 x * \chi_1^{-1} a) = sx * a.$$

2.1.2. **Proposition.** *Let*

$$E^1 : A \overset{i}{\rightarrowtail} X_0 \overset{p}{\twoheadrightarrow} X$$

be a(n) (L-)singular extension of X *by* A. *There is a* $K(\Pi, 1)$- *torsor* $(X_{\bullet}, s_{\bullet}, \chi_{\bullet})$ *over* $1 : X \to X$ *in* (C, X) *relative to* $U_{(L)X}$, *such that*

$$MX_{\bullet} = E^1 .$$

Proof. Let s_0 be a section of p in *Set*$^{\bullet}$ *(L)*. Since

$$A \xrightarrow{i} X_0 \xrightarrow{p} X$$

is a(n) (L -)singular extension, we have

$$x.a = i^{-1} (s_0 x + ia - s_0 x), \quad x * a = i^{-1} (s_0 x * ia).$$

The simplicial object

$$X_{\bullet} = cosk^0_{aug}((X_0 \xrightarrow{p} X) \overset{p}{\to} (X \xrightarrow{1} X))$$

has a $U_{(L)X}$-contraction $s_{\bullet} = \{s_n\}_{n \geqq 0}$ induced by s_0 . We define

$$\begin{tikzcd}
\delta_1 : X_0 \times_X X_0 \arrow[rr] \arrow[dr, "pd_0"'] & & A\,\mathrm{T}X \arrow[dl, "p_X"] \\
& X &
\end{tikzcd}$$

by

$$\delta_1(x_0, y_0) = (i^{-1}(x_0 - y_0), px_0).$$

Since

$$\delta_1 d_0 - \delta_1 d_1 + \delta_1 d_2 = 0\,, \qquad \delta_1 s_0 = 0,$$

δ_1 is a 1-normalized cocycle in (C, X).

Let $\delta_\bullet : X_\bullet \to K(\Pi, 1)$ be the morphism of simplicial objects in (C, X) obtained from δ_1 (see 1.1.1). Since the squares

$$\begin{tikzcd} X_1 \arrow[r, "\delta_1"] \arrow[d, "d_0"'] & A\,TX \arrow[d, "p_X"] \\ X_0 \arrow[r, "p"] & X \end{tikzcd} \qquad \begin{tikzcd} X_1 \arrow[r, "\delta_1"] \arrow[d, "d_1"'] & A\,TX \arrow[d, "p_X"] \\ X_0 \arrow[r, "p"] & X \end{tikzcd}$$

are pullbacks, then $TE^1 = (X_\bullet, s_\bullet, \delta_\bullet)$ is a $K(\Pi, 1)$-torsor over $1 : X \to X$. (Notice that in the condition (b) of Definition 1.1.3 it is sufficient to consider $m = n$.) It is immediate to see that $MX_\bullet = E^1$.

Now, we easily obtain :

2.1.3. *Theorem.* *There is a bijection*

$$\bar{M} : TORS^1_{U(L)X}\,[X \xrightarrow{1} X,\ \Pi] \longrightarrow S^1(X, A)_{(L)}$$

given by

$$\bar{M}\,[(X_\bullet, s_\bullet, \chi_\bullet)] = [MX_\bullet].$$

In [32] we have proved the following propositions.

2.1.4. *Proposition.* *If $(X_\bullet, s_\bullet, \chi_\bullet)$ is a $K(\Pi, 1)$-torsor over $1 : X \to X$ relative to $U_{(L)X}$ and $f : A \to B$ is a morphism of X-modules, then*

$$\bar{f}_1[(X_\bullet, s_\bullet, \chi_\bullet)] = \bar{T}(f_1[MX_\bullet])$$

where $\bar{f}_1[(X_\bullet, s_\bullet, \chi_\bullet)]$ is the class of $K(BTX \xrightarrow{p_X} X,\ 1)$-torsors defined in Proposition 1.1.6, $f_1[MX_\bullet]$ is the class of (L-)singular extensions of X by B (1.2.1), and $\bar{T}$ the inverse map of $\bar{M}$.

2.1.5. *Proposition.* *The map*

$$\bar{M} : TORS^1_{U(L)X}\,[X \xrightarrow{1} X, \Pi] \to S^1(X, A)_{(L)}$$

is an isomorphism of abelian groups.

2.2. *$K(\Pi, 2)$-torsors and 2-fold special extensions.*

2.2.1. *Proposition.* *If $(X_\bullet, s_\bullet, \chi_\bullet)$ is a $K(\Pi, 2)$-torsor over $1 : X \to X$*

relative to $U_{(L)X}$ *in (C, X), then the Moore complex of* X *is a(n) (L-) 2-fold special extension of* X *by* A.

Proof. Since $X_\bullet$ is $U_{(L)X}$ -split and $Cosk^1(X_\bullet) \simeq X_\bullet$, then $MX_\bullet$ is exact and

$$(MX_\bullet)_m = 0 \qquad \text{for} \quad m \geqq 3.$$

Thus

$$MX_\bullet : \ker d_1 \cap \ker d_2 \xrightarrow{d_0} \ker d_1 \xrightarrow{d_0} X_0 \xrightarrow{d_{-1}} X.$$

Definition 1.1.2, (b) yields the isomorphism

$$\ker d_1 \cap \ker d_2 \underset{\chi_2}{\simeq} A$$

and the existence of a morphism $s' : \Lambda^2(1)(X_\bullet) \to X_2$ such that

$$\chi_2 s' = sd_{-1}\ d_0\ pr_0\ , \qquad \langle d_0, d_1, -\rangle s' = 1.$$

The exact sequence

$$A \xrightarrow{d_0 \chi_2^{-1}} \ker d_1 \xrightarrow{d_0} X \xrightarrow{d_{-1}} X$$

is a(n) (L-)2-fold special extension. In fact,

(1) If $\sigma : \ker d_1 \to \ker d_0$ is the cokernel morphism of $d_0\chi_2^{-1}$, the map

$$\nu : \ker d_0 \to \ker d_1\ , \qquad \nu(x_0) = s_0x_0 - s_1x_0$$

is a section of σ in *Set*$^\bullet$ *(L)*.

(2) $\ker d_1$ is an X_0-structure

$$\begin{tikzcd} \ker d_1 \arrow[r, tail] & X_1 \arrow[r, two heads, shift left, "d_1"] & X_0) \arrow[l, shift left, "s_0"] \end{tikzcd}$$

and the morphism $d_0 | \ker d_1$ is a morphism of X_0-structures, where X_0 acts on itself by conjugation, satisfying

$$d_0x_1.y_1 = x_1 + y_1 - x_1, \quad d_0x_1 * y_1 = x_1 * y_1 = x_1 * y_1, \quad x_1, y_1 \in \ker d_1\ ,$$

by Lemma 1.2.3.

(3) $d_0\chi_2^{-1}$ is a morphism of X_0-structures :

$$d_0\chi_2^{-1}(x_0.a) = d_0\chi_2^{-1}(sd_{-1}x_0 + a - sd_{-1}\ x_0) =$$

$$= d_0\chi_2^{-1}(sd_{-1}d_0\ pr_0\ j_0 s_0\ x_0 + a - sd_{-1}d_0 pr_0\ j_0 s_0 x_0) =$$

$$= d_0 s' j_0 s_0 x_0 + d_0\chi_2^{-1}\ a - d_0 s' j_0 s_0 x_0 = s_0x_0 + d_0\chi_2^{-1}\ a - s_0x_0$$

and

$$d_0\chi_2^{-1}(x_0 * a) = d_0\chi_2^{-1}(sd_{-1}x_0 * a) = d_0\chi_2^{-1}(sd_{-1}\ d_0 pr_0 j_0 s_0 x_0 * a) =$$

$$= d_0 s' j_0 s_0 x_0 * d_0 \chi_2^{-1} a = s_0 x_0 * d_0 \chi_2^{-1} a$$

where $j_0 = <1, s_0 d_0, - >$.

2.2.2. **Proposition.** *If*

$$E^2 : A \xrightarrow{\quad i \quad} X_1 \xrightarrow{\rho} X_0 \xrightarrow{\quad p \quad} X$$

is a(n) (L-)2-fold special exension of X *by* A*, there is a* $K(\Pi, 2)$ *- torsor* $(X_\bullet, s_\bullet, \delta_\bullet)$ *over* $1 : X \to X$ *relative to* $U_{(L)X}$ *such that* $MX_\bullet = E^2$ *.*

Proof. Let $Y_\bullet, tr$ be the 1-truncated augmented simplicial object in (C, X)

$$\begin{tikzcd}
Y_\bullet, tr : X_1 T X_0 \arrow[r, shift left=3, "d_1"] \arrow[r, shift right=3, "d_0"'] & X_0 \arrow[l, shift right=9, "s_0"'] \arrow[l, shift left=9, "s_1"] \arrow[r, "p"] & X
\end{tikzcd}$$

with face and degeneracy operators d_0, d_1, s_0 given by

$$d_0(x_1, x_0) = \rho x_1 + x_0, \quad d_1(x_1, x_0) = x_0, \quad s_0(x_0) = (o, x_0).$$

$Y_\bullet, tr$ has a $U_{(L)}{}_X$-contraction $\{ s_0, s_1 \}$, where s_0 is a section of d_0 in *Set*$^\bullet$ *(L)* and

$$s_1(x) = (\nu(s_0 p x_0 - x_0), x_0), \quad \nu : ker\ p \to ker\ d_1$$

being a section of the cokernel morphism of i in *Set*$^\bullet$ *(L)*.

Let $X_\bullet = cosk^1 (Y_\bullet, tr)$ and $s_\bullet$ the $U_{(L)X}$-contraction induced by $\{s_0, s_1\}$. We define a morphism $\delta_2 : X_2 \to A T X$ in (C, X) by

$$\delta_2((x_1, \rho z_1 + y_0), (y_1, y_0), (z_1, y_0)) = (i^{-1} (x_1 + z_1 - y_1), p y_0).$$

We can show that

$$\delta_2 d_0 - \delta_2 d_1 + \delta_2 d_2 - \delta_2 d_3 = 0, \qquad \delta_2 s_0 = \delta_2 s_1 = 0.$$

Thus δ_2 is a 2-normalized cocycle.

If $\delta_\bullet : X_\bullet \to K(\Pi, 2)$ is the morphism of simplicial objects obtained from δ_2(see 1.1.1), then $TE^2 = (X_\bullet, s_\bullet, \delta_\bullet)$ is a $K(\Pi, 2)$-torsor over $1 : X \to X$, since the squares

$$\begin{tikzcd}
X_2 \arrow[r, "\delta_2"] \arrow[d] & A T X \arrow[d, "p_X"] \\
\Lambda^i(1)(X_\bullet) \arrow[r] & X
\end{tikzcd}$$

are pullbacks, for $0 \leqq i \leqq 2$.

It is immediate that $MX_\bullet = E^2$.

One verifies easily now that

2.2.3. Theorem. *The map*

$$\bar{M} : \mathrm{TORS}^2_{U_{(L)X}} [X \xrightarrow{1} X, \Pi] \longrightarrow S^2(X, A)_{(L)}$$

given by

$$\bar{M}[(X_\bullet, s_\bullet, \chi_\bullet)] = [MX_\bullet]$$

is a bijection.

In [32] we have proved :

2.2.4. Proposition. *If* $(X_\bullet, s_\bullet, \chi_\bullet)$ *is a* $K(\Pi, 2)$*-torsor over* $1 : X \to X$ *relative to* $U_{(L)X}$ *and* $f : A \to B$ *is a morphism of* X*-modules, then*

$$\bar{f}_2[(X_\bullet, s_\bullet, \chi_\bullet)] = \bar{T}(f_2[MX_\bullet]).$$

2.2.5. Proposition. *The map*

$$\bar{M} : \mathrm{TORS}^2_{U_{(L)X}} [X \xrightarrow{1} X, \Pi] \to S^2(X, A)_{(L)}$$

is an isomorphism of abelian groups.

2.3. Balanced cohomology. Interpretation by torsors.

2.3.1. Definition. An interest category C is said to have *balanced cohomology* if

$$H^n(X \xrightarrow{1} X, ITX \xrightarrow{p_X} X)_{G_X} = 0 \quad \text{for} \quad n > 0,$$

for each object X and each injective X-module I, G_X being the cotriple induced in (C , X) by the adjoint pair (F_X, U_X).

2.3.2. Proposition. *If C has balanced cohomology and I is an injective X-module, then* $S^n(X, I) = 0$ *for* $n \geq 2$.

Proof. It follows from Lemma 1.2.3 and from

$$S^2(X, A) \simeq \mathrm{TORS}^2_{U_X} [X \xrightarrow{1} X, ITX \xrightarrow{p_X} X] \simeq$$
$$\simeq H^2(X \xrightarrow{1} X, ITX \to X)_{G_X} = 0.$$

2.3.3. Theorem. *If C has balanced cohomology and A is an X-module, then there are isomorphisms of abelian groups*

$$W_A : \mathrm{TORS}^n_{U_X}[X \xrightarrow{1} X, A\,TX \to X] \to S^n(X, A), \qquad n \geq 3.$$

Proof. Let

$$A \rightarrowtail I_1 \to \dots \to I_{n-2} \twoheadrightarrow D$$

be an exact sequence of X-modules where I_k , $1 \leqq k \leqq n-2$, is an injective X-module. By Proposition 1.1.6, we have an isomorphism of abelian groups

$$\psi_A = (\partial_{n-1} \dots \partial_2) : \mathrm{TORS}^n_{\mathcal{U}(L)X}[X \xrightarrow{1} X,\ A \top X \to X] \to$$
$$\xrightarrow{\sim} \mathrm{TORS}^2_{\mathcal{U}(L)X}[X \xrightarrow{1} X,\ D \top X \to X\ ].$$

On the other hand, Proposition 1.2.2 gives an isomorphism

$$\Phi_A = \delta_{n-1} \dots \delta_2 : S^2(X, D) \xrightarrow{\sim} S^n(X, A).$$

Let now $W_A = \Phi_A . \bar{M} . \psi_A$, $\bar{M}$ being the isomorphism of Theorem 2.2.3. The isomorphism W_A is independent from the choice of the exact sequence

$$A \rightarrowtail I_1 \to \dots \to I_{n-2} \twoheadrightarrow D .$$

In fact, if

$$A \rightarrowtail J_1 \to \dots \to J_{n-2} \twoheadrightarrow D'$$

is another eact sequence of X-modules, with J_k , $1 \leqq k \leqq n-2$, injective X-modules, we have a commutative diagram

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, equal] & I_1 \arrow[r] \arrow[d, "h_1"] & \dots \arrow[r] & I_{n-2} \arrow[r, two heads] \arrow[d, "h_{n-2}"'] & D \arrow[d, "g"] \\
A \arrow[r, tail] & J_1 \arrow[r] & \dots \arrow[r] & J_{n-2} \arrow[r, two heads] & D'
\end{tikzcd}$$

where $h_1, \dots, h_{n-2}, g$ are morphisms of X-modules. The result follows from Proposition 3.1.2 and from the naturality of the connecting morphisms $\bar{\partial}$ and $\bar{\delta}$ in Propositions 1.1.6 and 1.2.2.

2.3.4. **Proposition.** *Suppose C has balanced cohomology. If* $f : A \to B$ *is a morphism of* X*-modules and* $(X_\bullet, s_\bullet, \chi_\bullet)$ *is a* $K(A \top X \to X, n)$*-torsor over* $1 : X \to X$ *relative to* U_X*, then*

$$f_n(W_A[(X_\bullet, s_\bullet, \chi_\bullet)]) = W_B(\bar{f}_n [(X_\bullet, s_\bullet, \chi_\bullet)]).$$

Proof. Let

$$A \rightarrowtail I_1 \to \dots \to I_{n-2} \twoheadrightarrow D$$

and

$$B \rightarrowtail J_1 \to \dots \to J_{n-2} \twoheadrightarrow D'$$

be two exact sequences of X-modules with I_k , J_k , $1 \leqq k \leqq n-2$, injective X-modules. There is a commutative diagram

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, "f"'] & I_1 \arrow[r] \arrow[d, "f_1"] & \dots \arrow[r] & I_{n-2} \arrow[r, two heads] \arrow[d, "f_{n-2}"'] & D \arrow[d, "g"'] \\
B \arrow[r, tail] & J_1 \arrow[r] & \dots \arrow[r] & J_{n-2} \arrow[r, two heads] & D'
\end{tikzcd}$$

where $f, f_1, \dots, f_{n-2}, g$ are morphisms of X-modules. The remainder of the proof is similar to that of Theorem 2.3.3.

3. Examples and applications.

3.1. *The long exact sequence*

$$\dots \to \mathrm{TORS}^n[X, \Pi] \to \mathrm{TORS}^n[X, \Pi''] \to \dots \qquad [15].$$

A $K(\Pi, n)$*-torsor over* X in an exact category *C* is a pair $(X_\bullet, \chi_\bullet)$, where $X_\bullet$ is an augmented simplicial object over X and $\chi_\bullet : X_\bullet \to K(\Pi, n)$ is a morphism of simplicial objects satisfying the rules (b) and (c) of Definition 1.1.3, together with the rule

(a') for any $m \geq 1$, if $p_i : K_m \to X_{m-1}$, $0 \leq i \leq m$, is the simplicial kernel of $d_i : X_{m-1} \to X_{m-2}$, $0 \leq i \leq m-1$, then the morphisms

$$\langle d_0, \dots, d_m \rangle : X_m \to K_m \quad \text{and} \quad d_{-1} : X_0 \to X$$

are coequalizers.

A *morphism* $f_\bullet : (X_\bullet, \chi_\bullet) \to (X'_\bullet, \chi'_\bullet)$ *of* $K(\Pi, n)$*-torsors over* X is an X-map $f_\bullet : X_\bullet \to X'_\bullet$ of augmented simplicial objects such that $\chi_\bullet' f_\bullet = \chi_\bullet$. The category of $K(\Pi, n)$-torsors over X and its class of connected components are denoted by $\mathrm{TORS}^n(X, \Pi)$ and $\mathrm{TORS}^n[X, \Pi]$ respectively.

In [15], Glenn defines an abelian group structure in $\mathrm{TORS}^n[X, \Pi]$ and, for any $f : \Pi' \to \Pi$, a homomorphism of abelian groups

$$\mathrm{TORS}^n[X, f] : \mathrm{TORS}^n[X, \Pi'] \to \mathrm{TORS}^n[X, \Pi],$$

and he obtains an exact sequence

$$0 \to C(X, \Pi') \to C(X, \Pi) \to C(X, \Pi'') \to \mathrm{TORS}^1[X, \Pi'] -$$
$$\xrightarrow{\mathrm{TORS}^1[X, f]} \mathrm{TORS}^1[X, \Pi] \xrightarrow{\mathrm{TORS}^1[X, g]} \mathrm{TORS}^2[X, \Pi''] \to \dots$$

associated with each short exact sequence

$$\Pi' \xrightarrow{f} \Pi \xrightarrow{g} \Pi''.$$

If C is an algebraic category and the underlying functor $U\colon C \to Set$ factors through the category of groups, then

$$\mathrm{TORS}^n(X, \Pi) = \mathrm{TORS}^n_U(X, \Pi),$$

and the morphisms $\mathrm{TORS}^n[X, f]$ and the additive structure of $\mathrm{TORS}^n[X, \Pi]$ coincide with those defined in Proposition 1.1.6 [15]. So, if C is an interest category with balanced cohomology, from Propositions 2.1.4, 2.1.5, 2.2.4, 2.2.5, 2.3.4 and Theorem 2.3.3 we see that Baer sum and the morphism f_n defined in 1.2.1 just give the addition in

$$\mathrm{TORS}^n[X \xrightarrow{1} X, \Pi]$$

and the morphism

$$\mathrm{TORS}^n[X \xrightarrow{1} X, f]$$

respectively. Furthermore, the exact sequence of Theorem 1.2.2 gives the long exact sequence of torsors over $1\colon X \to X$ obtained by Glenn. The details may be found in [32].

3.2. *An interpretation of the Eilenberg-MacLane cohomology.*

If G is a group, a *G-group* is a pair (A, α), where A is a group and $\alpha : G \to \mathrm{Aut}\, A$ is a homomorphism of groups. We put

$$x.a = \alpha(x)a.$$

A G-group (A, α) is said to be a *G-module* (i.e., ZG-module) if A is an abelian group. A *homomorphism of* G *-groups* $f : A \to B$ is a homomorphism of groups such that

$$f(x.a) = x.f(a).$$

The categories of G-groups and ZG-modules will be denoted by ${}_G Gr$ and ${}_G M$, respectively.

Barr [4] shows that there are natural isomorphisms

$$H^n(G \xrightarrow{f} H, A \mathrm{T} H \xrightarrow{\rho_H} H)_{G'_H} = \begin{cases} \mathrm{Der}(G, A)_f\,, & n = 0 \\ \text{E-M}^{n+1}(G, A), & n \geqq 1 \end{cases}$$

where E-M^n denotes the Eilenberg-MacLane cohomology and G'_H is the cotriple defined by the adjunction

$$\begin{tikzcd} (Set, H) \arrow[r, shift left, "F'_H"] & (Gr, H) \arrow[l, shift left, "U'_H"] \end{tikzcd}$$

with

$$F'_H(S \xrightarrow{g} H) = (F'S \xrightarrow{F'g} F'H \xrightarrow{\delta_H} H),$$

F' being the free functor and δ_H the counit of the adjunction (F', U') in H.

3.2.1. **Lemma.** *There are equivalences of categories*

(i) G *-Str* $\simeq$ $_{G}Gr$.

(ii) G *-Mod* $\simeq$ $_{G}M$.

Proof. (i) We define two functors

$$F_1 : G\text{-}Str \to {}_{G}Gr \quad \text{and} \quad F_2 : {}_{G}Gr \to G\text{-}Str$$

by

$$F_1 \left(A \rightarrowtail E \underset{s}{\overset{\twoheadrightarrow}{\longleftarrow}} G\right) = (A, \alpha)$$

with $\alpha : G \to \operatorname{Aut} A$ given by

$$\alpha(x)a = sx + a - sx,$$

and

$$F_2 (A, \alpha) = \left(A \rightarrowtail A \mathbin{\mathrm{T}} G \overset{\twoheadrightarrow}{\longleftarrow} G\right)$$

where A TG is the set A x G with the group structure

$$(a, x) + (a', x') = (a + x.a', x + x').$$

It is easy to see that

$$F_2 F_1 \simeq 1 \ , \qquad F_1 F_2 \simeq 1.$$

(ii) The restriction of F_1 and F_2 to G-*Mod* and $_{G}M$, respectively, gives the equivalence.

From these equivalences, the concepts of singular extension and *n*-fold special extension can be stated as follows :

A *singular extension* of G by the ZG-module A is a short exact sequence of groups

$$A \rightarrowtail E \xrightarrow{p} \twoheadrightarrow G,$$

where the ZG-module structure on A is given by any section of *p* (i.e.,

$$x.a = i^{-1} (sx + ia - sx), \quad x \in G, \quad a \in A).$$

An *n-fold special extension* of G by A is an exact sequence of groups

$$A \rightarrowtail A_1 \to \dots \to A_{n-2} \to E \xrightarrow{f} H \xrightarrow{p} \twoheadrightarrow G$$

where

$$A \rightarrowtail A_1 \longrightarrow \dots \quad A_{n-2} \twoheadrightarrow ker\ f$$

is an (*n*-2)-extension of ZG-modules and

$$ker\ f \rightarrowtail E \xrightarrow{f} H \xrightarrow{p} \twoheadrightarrow G$$

is an exact sequence of H-groups, where *ker f* is a ZH-module via *p*, H acts on itself by conjugation and such that

$$fe.e' = e + e' - e, \quad e, e' \in E \ .$$

Since the category of groups has balanced cohomology, we have the following propositions.

3.2.2. Proposition. *If* A *is a* ZG *-module, there are isomorphisms of abelian groups*

$$\mathrm{TORS}^n_{U_G}[G \xrightarrow{1} G, A \top G \to G] \simeq S^n(G, A) \quad \textit{for} \quad n \geq 1.$$

Proof. It follows from Propositions 2.1.5, 2.2.5 and Theorem 2.3.3.

3.2.3. Proposition. *If* A *is a* ZG *-module, there are isomorphisms*

$$E\text{-}M^{n+1}(G, A) \simeq S^n(G, A), \quad n \geq 1.$$

Proof. It follows from Theorem 1.1.5 and Proposition 3.2.2.

3.3. *An interpretation of the Chevalley-Eilenberg cohomology.*

Let $\boldsymbol{L}_K$ be the category of Lie algebras over a commutative unitary ring K. If $\boldsymbol{g}$ is a Lie algebra over K and U$\boldsymbol{g}$ its universal enveloping algebra, it is known that the categories of U$\boldsymbol{g}$-modules and $\boldsymbol{g}$-modules are equivalent [31].

The underlying functor $U : \boldsymbol{L}_K \to Set^{\bullet}$ factors through the category of K-modules L , and we have the following diagram of adjoint functors.

$$\begin{tikzcd}
Set^{\bullet} \arrow[d, shift right] \arrow[dd, bend right=90, "F"'] \\
L \arrow[u, shift right] \arrow[d, shift right, "F_L"'] \\
\boldsymbol{L}_K \arrow[u, shift right, "U_L"'] \arrow[uu, bend right=90, "U"']
\end{tikzcd}$$

where $F_L(M)$ is the quotient K-Lie algebra of the free non-associative algebra

$$A(M) = M + M \boxtimes M + (M \boxtimes M) \boxtimes M + M \boxtimes (M \boxtimes M) + \ldots$$

by the two sided ideal generated by elements of the form

$$m \boxtimes m, \qquad m_1 \boxtimes (m_2 \boxtimes m_3) + m_2 \boxtimes (m_3 \boxtimes m_1) + m_3 \boxtimes (m_1 \boxtimes m_2)$$

for m , $m_i \in$ M, and where

$$F\,(S, .) = F_L\ (K(S)/K(.)),$$

(K(S)/K(.) being the free K-module generated by the set S with identification . = 0). It is clear that the underlying functors U and U_L are tripleable [31].

In $\mathbf{L}_K$ the concepts of (L -)singular extension and (L -) -fold special extension can be state as follows :

A(n) (L -) *singular extension of* $\mathbf{g}$ *by a* U$\mathbf{g}$*-module* is a short exact sequence of Lie algebras

$$A \rightarrowtail^{i} \mathbf{n} \xrightarrow{p} \!\!\!\rightarrow \mathbf{g} \quad \text{with} \quad p \in E_{(L)},$$

where the U$\mathbf{g}$-module structure of A is given by any section s of p (i.e.

$$x.a = i^{-1}[\,sx,\ ia\,]\ ,\quad x \in \mathbf{g},\quad a \in A).$$

A(n) *(L-) 2-fold special extension of* $\mathbf{g}$ *by* A is an exact sequence in $\mathbf{L}_K$,

$$A \rightarrowtail^{\varphi_2} \mathbf{k} \xrightarrow{\varphi_1} \mathbf{n} \xrightarrow{\varphi_0} \!\!\!\rightarrow \mathbf{g},$$

with $\varphi_2^C, \varphi_0 \in E_{(L)}$, $\mathbf{k}$ a U$\mathbf{h}$-module, and such that

(1) $\qquad \varphi_2[\varphi_0 x,\ a\,] = x.\,\varphi_2(a),$

(2) $\qquad \varphi_1[x,\ k] = [\,x,\ \varphi_1 k]\ ,$

(3) $\qquad \varphi_1 k.k' = [\,k,\ k'\,]$

for each $x \in \mathbf{n}$, $a \in$ A, $k, k' \in \mathbf{k}$.

An *(L-)n-fold special extension of* $\mathbf{g}$ *by* A, $n > 2$, is an exact sequence

$$A \rightarrowtail^{\varphi_n} A_1 \rightarrow \ldots \longrightarrow A_{n-2} \xrightarrow{\varphi_2} \mathbf{k} \xrightarrow{\varphi_1} \mathbf{n} \xrightarrow{\varphi_0} \!\!\!\rightarrow \mathbf{g}$$

in $\mathbf{L}_K$ where

$$A \rightarrowtail^{\varphi_n} A_1 \rightarrow \ldots \quad \rightarrow A_{n-2} \xrightarrow{\varphi_3^C} \!\!\!\rightarrow ker\ \varphi_1$$

is an exact sequence of U$\mathbf{g}$-modules with $\varphi_k^C \in E_{(L)}$ for $k > 2$ and

$$ker\ \varphi_1 \rightarrowtail^{\varphi_1^k} k \xrightarrow{\varphi_1} h \xrightarrow{\varphi_0} \!\!\!\rightarrow \mathbf{g}$$

is a(n) (L -)2-fold special extension.

In [15], Shimada gives an interpretation of the group

$$H^2(\mathbf{g} \xrightarrow{1} \mathbf{g},\ A\,T\mathbf{g} \longrightarrow \mathbf{g})_{G_{(L)}\mathbf{g}}$$

as the set of equivalence classes of (L -)2-fold extensions of $\mathbf{g}$ by A, $G_{(L)}$ being the cotriple induced by the adjunction ($F_{(L)}$, $U_{(L)}$) in ($\mathbf{L}_K$, $\mathbf{g}$). In [2], is given an interpretation of

$$H^n(\mathbf{g} \xrightarrow{1} \mathbf{g},\ A\,T\,\mathbf{g} \longrightarrow \mathbf{g})_{G_{\mathbf{g}}}$$

for $n \geq 1$, in the case K being a field.

3.3.1. **Proposition.** *If* $U : \mathbf{L}_K \rightarrow Set^{\bullet}$ *and* $U_L : \mathbf{L}_K \rightarrow L$ *are the underlying* *functors, then*

$$\mathrm{TORS}^1_{U(L)\boldsymbol{g}}[\boldsymbol{g} \xrightarrow{1} \boldsymbol{g},\ A\top \boldsymbol{g} \longrightarrow \boldsymbol{g}]$$

and

$$\mathrm{TORS}^2_{U(L)\boldsymbol{g}}[\boldsymbol{g} \xrightarrow{1} \boldsymbol{g},\ A\top \boldsymbol{g} \longrightarrow \boldsymbol{g}]$$

classify (L-)singular extensions and (L-) 2 -fold special extensions, respectively.

Proof. It follows from Theorems 2.1.3 and 2.2.3.

3.3.2. **Proposition.** *If* $(X_\bullet, s_\bullet, \chi_\bullet)$ *is the standard* $K(A\top\, \boldsymbol{g} \rightarrow \boldsymbol{g}, 2)$*-torsor over* $1 : \boldsymbol{g} \rightarrow \boldsymbol{g}$ *defined by the 2-normalized cocycle*

$$f \in (L_K,\ \boldsymbol{g})(G^3_{(L)}\boldsymbol{g} \longrightarrow \boldsymbol{g},\ A\top\boldsymbol{g} \longrightarrow \boldsymbol{g})$$

then $MX_\bullet$ *coincides with the standard* [31] *(L-) 2-fold special extension defined by the cocycle* $-f$*.*

Proof. The standard (*L* -)2-fold special extension defined by the cocycle $-f$ is given by

$$\begin{tikzcd} A \arrow[r, tail, "\tau"] & N(\boldsymbol{g})\times_{-f}A \arrow[r, "\rho"] & G_{(L)}\boldsymbol{g} \arrow[r, two heads, "p"] & \boldsymbol{g} \end{tikzcd}$$

where p is the counit of the adjunction $(F(L), U(L))$ in $\boldsymbol{g}$ and where $N(\boldsymbol{g})\times_{-f} A$ is the set $Ker\, p \times A$ with the Lie algebra structures

$$(n_1, a_1) + (n_2, a_2) = (n_1+n_2,\ a_1+a_2 - p_A.f(\overline{\bar{n}_1+\bar{n}_2})),$$

$$k(n, a) = (kn,\ ka - p_A.f(\overline{k\bar{n}})),$$

$$[(n_1, a_1), (n_2, a_2)] = ([n_1, n_2],\ -p_A.f[\overline{\bar{n}_1, \bar{n}_2}]),$$

where we used the notation

$$\bar{x} = \eta_{UG_{(L)}\boldsymbol{g}}(x),$$

η being the unit of the adjunction. The maps τ and ρ are defined by

$$\tau(a) = (0, a), \quad \rho(n, a) = n.$$

Moreover, $G_{(L)}\boldsymbol{g}$ acts on $N(\boldsymbol{g})\times_{-f}A$ by

$$x.(n, a) = ([x, n],\ x.a - p\ .f[\overline{\bar{x}, \bar{n}}]).$$

This extension coincides with $MX_\bullet$, since there exists a commutative diagram

$$\begin{tikzcd} & A \arrow[r, tail] \arrow[d, equal] & N(\boldsymbol{g})\times_{-f}A \arrow[r] \arrow[d, "\beta"] & G_{(L)}\boldsymbol{g} \arrow[r, two heads] \arrow[d] & \boldsymbol{g} \arrow[d, equal] \\ MX_\bullet: & A \arrow[r, tail] & ker\ d_1 \arrow[r] & G_{(L)}\boldsymbol{g} \arrow[r, two heads] & \boldsymbol{g} \end{tikzcd}$$

where β is an homomorphism of $G_{(L)}\mathbf{g}$-modules which preserves the Lie algebra structure.

3.3.3. ***Proposition*** [27]. *If* K *is a field, then*

$$H^n(\mathbf{g} \xrightarrow{1} \mathbf{g}, A \Upsilon \mathbf{g} \to \mathbf{g})_{G_{L\mathbf{g}}} = H^n(\mathbf{g} \xrightarrow{1} \mathbf{g}, A \Upsilon \mathbf{g} \to \mathbf{g})_{G_{\mathbf{g}}}$$

$$= \text{Ch-E}^{n+1}(\mathbf{g}, A)$$

where Ch-E^n *denotes the Chevalley-Eilenberg cohomology.*

3.3.4. **Proposition.** *If* K *is a field, then there are isomorphisms of abelian groups*

$$\text{TORS}^n_{\mathcal{U}_{\mathbf{g}}}[\mathbf{g} \xrightarrow{1} \mathbf{g}, A \Upsilon \mathbf{g} \to \mathbf{g}] \simeq S^n(\mathbf{g}, A), \qquad n \geqq 1.$$

Proof. It follows from Propositions 1.1.5 and 2.2.5 and Theorem 2.3.3.

3.3.5. ***Corollary.*** *If* K *is a field, then*

$$\text{Ch-E}^{n+1}(\mathbf{g}, A) = S^n(\mathbf{g}, A), \qquad n \geqq 1.$$

Proof. It follows from Theorem 2.1.3 and Propositions 3.3.3 and 3.3.4.

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