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EXACT SEQUENCE INTERLOCKING AND FREE HOMOTOPY THEORY
by K. A. HARDIE and K. H. KAMPS

Résumé. On étudie des phénomènes d'exactitude entrelacés à l'extrémité non-abélienne d'un diagramme de Kervaire. On en déduit en particulier des critères pour l'exactitude des suites de Mayer-Vietoris associées. Ces suites sont étudiées dans le cas des diagrammes de Kervaire venant: (A) des suites d'homotopie d'un triple, (B) des suites d'homotopie d'un produit fibré d'homotopie, (C) des suites de paires d'homotopie pour une paire d'applications pointées; de nouvelles suites de Mayer-Vietoris sont indiquées et des applications sont données en théorie des paires d'homotopie et en théorie des crochets de Toda.

Dans une section finale une suite covariante d'ensembles d'homotopie libre, associée à une application libre est décrite. Cette suite a une structure enrichie à l'extrémité non-abélienne conduisant à un théorème de classification pour un nouvel ensemble d'homotopie. De plus les résultats du début permettent de déduire une suite de Mayer-Vietoris classifiant les applications dans un produit fibré d'homotopie libre.

0. INTRODUCTION.

In a wide variety of situations in algebra and topology invariants appear in the form of long exact sequences. Typically these sequences consist of homomorphisms of abelian groups becoming non-abelian at a certain stage and terminating with arrows between pointed sets. When several objects are interrelated the associated sequences of invariants tend to interlock. Interlocking phenomena for sequences of abelian groups have been studied by Wall [20] and results of the following kind obtained. The exactness of one of the sequences may be deduced from the exactness of the others, given certain information. Again from a (Kervaire) diagram of sequences interlocking in a certain way one can infer the existence of a sequence of Mayer-Vietoris type convenient for purposes of computation. No results of this kind have hitherto been available for the non-abelian parts of sequences. In Sections 1 and 2 we give an analysis of exactness properties, in the presence of weaker and weaker algebraic structure and prove certain results of the desired type. Section 3 is devoted to a study of the more immediate applications. Quite frequently we have found that when a particular Mayer-Vietoris sequence was known to exist and a proof could be constructed using the theory of Section 1, the methods also brought to light other sequences whose existence had not been suspected.

The systematic study of applications led to an advance of another sort. It has always been recognized that at the pointed set stage of an exact sequence the information about the objects expressed by the exactness of the sequence is severely limited. Whereas in the abelian group stage one may aspire to compute every third group (from information about the other two and provided one knows how to determine the group extensions), at the pointed set stage this has been impossible. An indication that it might be possible occasionally to enrich the structure of the objects of the sequence so as to perform computations was given by Rutter [16] when he discovered a sequence, in retrospect clearly related to the dual Puppe sequence of a map f , that could be used (given certain powerful assumptions) to classify maps into the homotopy fibre of f . Examination of the way Rutter's sequence could be placed in the framework of Sections 1 and 2 led to the discovery of an exact sequence for free homotopy sets that can specialize to Rutter's sequence in the pointed case but that even there is more general. Moreover the sequence throws light on the type of extra structure to be expected: one can hope to select arbitrary elements as base points, but one will need to contemplate parallel sets of arrows.

The covariant free homotopy sequence of a map is described in Section 4 together with the associated Kervaire diagram of such sequences resulting from the construction of the homotopy pullback of a pair of maps with common codomain. In an extended application of the theory of Sections 1 and 2 to this situation, a Mayer-Vietoris sequence is obtained by means of which it is possible to give a homotopy classification of the (free) maps into a (free) homotopy pullback. By specialization a classification is obtained also for the pointed case.

1. EXACT SEQUENCES OF GROUPS AND POINTED SETS.

We are dealing with pointed sets and pointed maps. The base point of a pointed set and the constant map to the base point of a pointed set will usually be denoted by $*$. In the case of a group the base point will be the neutral element.

A. Types of exact sequences.

For a pointed map $\alpha: A \rightarrow B$ let $\text{Ker}\alpha$ and $\text{Im}\alpha$ denote the pointed subsets $\alpha^{-1}(*)$ and $\alpha(A)$ of A and B , respectively.

For a sequence (α, β) of pointed maps,

$$(1.1) \quad A \xrightarrow{\alpha} B \xrightarrow{\beta} C,$$

the following types of exactness (at B) will occur.

(E0) (α, β) is differential: $\beta\alpha = *$.

(E1) (α, β) is exact: $\text{Im}\alpha = \text{Ker}\beta$.

(E2) A is a group operating on B from the left by an action $A \times B \rightarrow B$ such that

- (i) $\alpha(a) = a \cdot *$ for each $a \in A$;
- (ii) If $b, b' \in B$, then $\beta(b) = \beta(b')$ iff $b' = a \cdot b$ for some $a \in A$.

(E2*) (E2) with left operation replaced by right operation.

(E3) α is a group homomorphism such that for $b, b' \in B$

$$\beta(b) = \beta(b') \quad \text{iff} \quad b'b^{-1} \in \text{Im} \alpha.$$

(E3*) α is a group homomorphism such that for $b, b' \in B$

$$\beta(b) = \beta(b') \quad \text{iff} \quad b^{-1}b' \in \text{Im} \alpha.$$

(E4) α and β are group homomorphisms with $\text{Im} \alpha = \text{Ker} \beta$.

Clearly, type (E2) is a special case of type (E1), and type (E4) is a special case of both type (E3) and (E3*), the kernel of a group homomorphism being a normal subgroup. Type (E3) is a special case of type (E2), the operation of A on B being defined by $a \cdot b = \alpha(a)b$, where $\alpha(a)b$ is the product of $\alpha(a)$ and b in the group B . Similarly, type (E3*) is a special case of type (E2*).

B. Modification of exact sequences.

In the applications we will encounter the situation where a map in an exact sequence has to be modified using inversion in a group in order to turn a diagram which is anticommutative into a commutative one.

1.2. Notation. If $\alpha: A \rightarrow B$ is a pointed map and A is a group, let $\alpha^-: A \rightarrow B$ denote the pointed map defined by

$$\alpha^-(a) = \alpha(a^{-1}) \quad \text{for each } a \in A,$$

where a^{-1} is the inverse of a in the group A .

The following lemma will easily be verified.

1.3. Lemma. Let (α, β) be as in (1.1).

- (a) If (α, β) is exact and A is a group, then (α^-, β) is exact.
- (b) If (α, β) is exact of type (E2), then (α^-, β) is exact of type (E2*), and vice versa, the right operation $B \times A \rightarrow B$ being defined from the left operation $A \times B \rightarrow B$ by

$$b \cdot a = a^{-1} \cdot b \quad \text{for } a \in A, b \in B.$$

- (c) If (α, β) is exact and α is a group homomorphism, then (α, β^-) is exact.

(d) If (α, β) is exact of type (E3), then (α, β^-) is exact of type (E3*), and vice versa. \diamond

C. Examples.

We give some examples from algebra and topology.

1.4. An exact sequence of conjugacy classes. Let

$$1 \longrightarrow K \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 1$$

be a short exact sequence of groups and let $x \in K$. Then there is an exact sequence of groups and pointed sets

$$(1.5) \quad 1 \rightarrow C_K(x) \rightarrow C_G(\alpha x) \rightarrow H \rightarrow [K] \rightarrow [G] \rightarrow [H] \rightarrow 1,$$

where $C_K(x)$ is the centralizer of x in K , $[\]$ denotes the set of conjugacy classes and the base points in $[K]$, $[G]$ and $[H]$ are induced by x . This sequence is exact of type (E2) at $[K]$, exact of type (E3*) at H and exact of type (E4) at $C_G(\alpha x)$ and $C_K(x)$ ([17], Exact sequence 1).

1.6. The exact fibre sequence for groupoids. Let $p: G \rightarrow H$ be a fibration of groupoids (see [2], 2) and let x be an object of G . Then there is an exact sequence of groups and pointed sets

$$(1.7) \quad 1 \rightarrow F(x) \rightarrow G(x) \rightarrow H(px) \rightarrow \pi_0 F \rightarrow \pi_0 G \rightarrow \pi_0 H,$$

where $F = p^{-1}px$ is the fibre over px , $G(x)$ is the vertex group $G(x, x)$ at x , π_0 denotes the set of components, and the base points of $\pi_0 F$, $\pi_0 G$ and $\pi_0 H$ are induced by x . This sequence is exact of type (E2) at $\pi_0 F$, exact of type (E3*) at $H(px)$ and exact of type (E4) at $G(x)$ and $F(x)$ ([2], Theorem 4.3).

Note that (1.7) can be used to derive (1.5) and, as it has been shown in [9], more generally, exact orbit sequences for group operations on sets.

1.8. Exact sequences in homotopy theory. We use the notations of [8]. Let $f: X \rightarrow Y$ and $g: E \rightarrow B$ be pointed continuous maps between pointed topological spaces. Let

$$(1.9) \quad X \xrightarrow{f} Y \xrightarrow{Pf} C_f \xrightarrow{Qf} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma Pf} \Sigma C_f \xrightarrow{\Sigma Qf} \Sigma^2 X \rightarrow \dots$$

be the Puppe sequence of f , and let

$$(1.10) \quad \dots \rightarrow \Omega^2 B \xrightarrow{\Omega M g} \Omega F_g \xrightarrow{\Omega N g} \Omega E \xrightarrow{\Omega g} \Omega B \xrightarrow{M g} F_g \xrightarrow{N g} E \xrightarrow{g} B$$

be the dual Puppe sequence (Nomura sequence) of g . Then we have the following exact sequences of homotopy groups and homotopy sets due

to Puppe [15] and Nomura [12], respectively,

$$(1.11) \quad \dots \rightarrow \pi(C_f, V) \xrightarrow{\cdot \Sigma P f} \pi(\Sigma Y, V) \xrightarrow{\cdot \Sigma f} \pi(\Sigma X, V) \xrightarrow{\cdot Q f} \pi(C_f, V) \\ \xrightarrow{\cdot P f} \pi(Y, V) \xrightarrow{\cdot f} \pi(X, V),$$

$$(1.12) \quad \dots \rightarrow \pi(W, \Omega F_g) \xrightarrow{\Omega N g} \pi(W, \Omega E) \xrightarrow{\Omega g} \pi(W, \Omega B) \xrightarrow{M g} \pi(W, F_g) \\ \xrightarrow{N g} \pi(W, E) \xrightarrow{g} \pi(W, B),$$

where V, W are arbitrary pointed spaces and, for example, $\cdot f$ denotes precomposition with f , $g \cdot$ denotes postcomposition with g .

The sequence (1.11) will be referred to as the Puppe sequence of f at V , the sequence (1.12) as the Nomura sequence of g at W .

Similarly, we have the exact cofibre sequence of a pointed cofibration $f : X \rightarrow Y$, more generally a pointed h-cofibration (see [5], (2.2)), and the exact fibre sequence of a pointed fibration $g : E \rightarrow B$, more generally a pointed h-fibration (see [5], (6.4)).

As it has been discovered in [8] there are further two exact sequences involving sets of homotopy pair classes $\pi(f, g)$:

$$(1.13) \quad \dots \rightarrow \pi(\Sigma^2 X, B) \rightarrow \pi(\Sigma C_f, B) \rightarrow \pi(\Sigma f, g) \rightarrow \pi(\Sigma X, E) \\ \rightarrow \pi(C_f, B) \rightarrow \pi(f, g) \rightarrow \pi(X, E),$$

$$(1.14) \quad \dots \rightarrow \pi(Y, \Omega^2 B) \rightarrow \pi(X, \Omega F_g) \rightarrow \pi(f, \Omega g) \rightarrow \pi(Y, \Omega B) \rightarrow \pi(X, F_g) \\ \rightarrow \pi(f, g) \rightarrow \pi(Y, B).$$

The type of exactness at the bottom end of all those sequences is the same as in (1.7). We will come back to the subject in Section 3.

2. INTERLOCKING EXACT SEQUENCES AND MAYER-VIETORIS SEQUENCES.

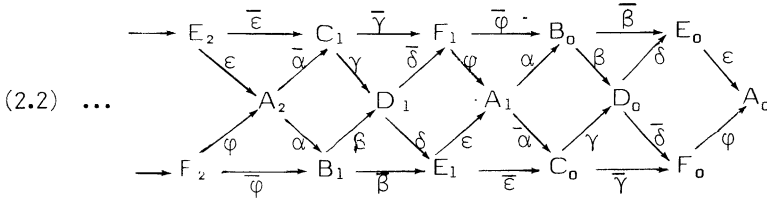
Wall's paper [20] is dealing with the exactness of interlocking sequences and Mayer-Vietoris sequences for interlocking exact sequences of abelian groups. In this section we investigate the essential ingredients of [20] in the non-abelian case.

As in the preceding section all arrows will be pointed maps between pointed sets.

A. Interlocking exact sequences.

Our first subject is the exactness of interlocking sequences in a so-called Kervaire diagram. The formulation of the following proposition is adapted to the applications and could partially be generalized.

2.1. Theorem. *Let*



be a commutative diagram of exact sequences (A, B, E) , (A, C, F) and (B, D, F) such that the arrows to the left of A_1 are group homomorphisms. Assume in addition that

- (1) at A_1 the sequence (A, B, E) is exact of type $(E3)$ and that (A, C, F) is exact of type $(E3^*)$,
- (2) (C, D, E) is differential at D_n for each $n \geq 0$.

Then the sequence (C, D, E) is exact except possibly at D_0 , and the type of exactness at E_1 is $(E3^*)$.

Proof. Exactness at C_0 : We have

$$\gamma \bar{\epsilon} = \gamma \bar{\alpha} \epsilon = \beta \alpha \epsilon = *.$$

Let $c_0 \in C_0$ and let $\gamma(c_0) = *$. Then $\bar{\gamma}(c_0) = \bar{\delta}\gamma(c_0) = *$, hence there is an element $a_1 \in A_1$ such that $c_0 = \bar{\alpha}(a_1)$. Since

we have

$$\beta \alpha(a_1) = \gamma \bar{\alpha}(a_1) = \gamma(c_0) = *,$$

$$\alpha(a_1) = \bar{\varphi}(f_1) = \alpha \varphi(f_1)$$

for a suitable element $f_1 \in F_1$. By (1) there exists

$$e_1 \in E_1 \quad \text{such that} \quad a_1 = \epsilon(e_1)\varphi(f_1).$$

Then by (1) we have

$$\bar{\epsilon}(e_1) = \bar{\alpha}\epsilon(e_1) = \bar{\alpha}(\epsilon(e_1)\varphi(f_1)) = \bar{\alpha}(a_1) = c_0.$$

Thus $c_0 \in \text{Im } \bar{\epsilon}$.

Exactness at E_1 : We prove exactness of type $(E3^*)$. Let $e_1 \in E_1$, $d_1 \in D_1$. Then by (1)

$$\begin{aligned} \bar{\epsilon}(e_1) \delta(d_1) &= \bar{\alpha}\epsilon(e_1) \delta(d_1) = \bar{\alpha}(\epsilon(e_1) \epsilon \delta(d_1)) = \\ &= \bar{\alpha}(\epsilon(e_1)\varphi\bar{\delta}(d_1)) = \bar{\alpha}\epsilon(e_1) = \bar{\epsilon}(e_1). \end{aligned}$$

Let $e_1, e_1' \in E_1$ and suppose $\bar{\epsilon}(e_1) = \bar{\epsilon}(e_1')$. Thus $\bar{\alpha}\epsilon(e_1) = \bar{\alpha}\epsilon(e_1')$, hence $\epsilon(e_1) = \epsilon(e_1')\varphi(f_1)$ for some $f_1 \in F_1$ by (1). Then

$$\bar{\varphi}(f_1) = \alpha\varphi(f_1) = \alpha(\epsilon(e_1)\varphi(f_1)) = \alpha\epsilon(e_1) = *$$

by (1). Thus $f_1 = \bar{\delta}(d_1)$ for some $d_1 \in D_1$. Since

$$\varepsilon(e_1') = \varepsilon(e_1)\varphi(f_1) = \varepsilon(e_1)\varphi\bar{\delta}(d_1) = \varepsilon(e_1)\varepsilon\delta(d_1) = \varepsilon(e_1\delta(d_1)),$$

there exists $b_1 \in B_1$ such that $e_1' = e_1\bar{\delta}(d_1)\beta(b_1)$. Thus $e_1' = e_1\delta(d_1\beta(b_1))$.

Exactness at D_1 : Since $\text{Im } \gamma \subset \text{Ker } \delta$ by assumption (2), we have to prove $\text{Ker } \delta \subset \text{Im } \gamma$. Let $d_1 \in D_1$ such that $\delta(d_1) = 1$. Then

$$\varphi\bar{\delta}(d_1) = \varepsilon\bar{\delta}(d_1) = 1 \quad \text{hence} \quad \bar{\delta}(d_1) = \bar{\gamma}(c_1) = \bar{\delta}\bar{\gamma}(c_1)$$

for some $c_1 \in C_1$. Then $d_1 = \gamma(c_1)\beta(b_1)$ for some $b_1 \in B_1$. Since $\text{Im } \gamma$ is included in $\text{Ker } \delta$, we have

$$\bar{\beta}(b_1) = \delta\beta(b_1) = \delta\gamma(c_1)\delta\beta(b_1) = \delta(\gamma(c_1)\beta(b_1)) = \delta(d_1) = 1.$$

Thus $b_1 = \alpha(a_2)$ for some $a_2 \in A_2$. Therefore

$$d_1 = \gamma(c_1)\beta(b_1) = \gamma(c_1)\beta\alpha(a_2) = \gamma(c_1)\gamma\bar{\alpha}(a_2) = \gamma(c_1\bar{\alpha}(a_2)) \in \text{Im } \gamma. \quad \diamond$$

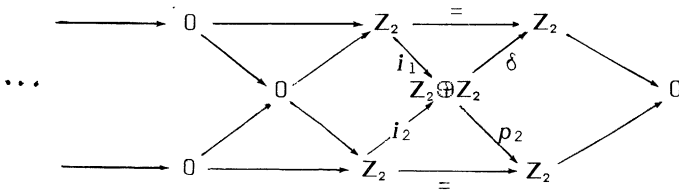
2.3. Definition. A diagram of interlocking sequences (A, B, E) , (A, C, F) , (B, D, F) and (C, D, E) as in (2.2) is called a *Kervaire diagram*.

Note that in the situation of Theorem 2.1 the exactness of (B, D, F) at F_1 in the simple form $\text{Im } \bar{\delta} = \text{Ker } \bar{\varphi}$ induces the strengthened form of exactness (E3*) for (C, D, E) at E_1 . Thus by a "dual" argument to the second part of the proof of 2.1 we have

2.4. Corollary. In the situation of Theorem 2.1 the strengthened form of exactness (E3) automatically holds for (B, D, F) at F_1 . \diamond

The following examples show that in the situation of Theorem 2.1 neither exactness of (C, D, E) at D_0 nor strengthened exactness properties at C_0 can be expected from general algebraic reasons even if (A, B, E) , (A, C, F) and (B, D, F) are exact sequences of homomorphisms of abelian groups.

2.5. Examples. 1. In the diagram

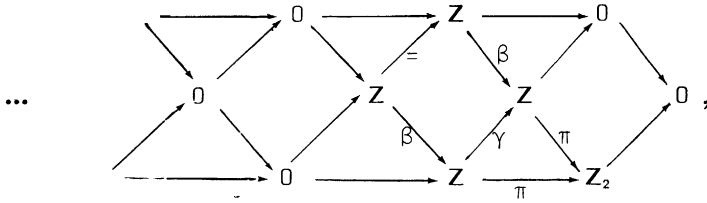


let Z_2 denote the integers modulo 2, let i_1 and i_2 be the injections into the first and second factor, respectively, let p_2 be the projection onto the second factor, and let δ be defined by

$$\delta(0, 0) = \delta(0, 1) = \delta(1, 1) = 0, \quad \delta(1, 0) = 1.$$

Then we are in the situation of Theorem 2.1, but (i_2, δ) is not exact.

2. Consider the diagram of interlocking exact sequences



where Z is the additive group of integers, β is multiplication by 2, π is the projection onto the integers modulo 2 and γ is defined by

$$\gamma(2a) = 2a, \quad \gamma(2a + 1) = \begin{cases} 2a + 1, & a \geq 0 \\ 2a + 3, & a < 0. \end{cases}$$

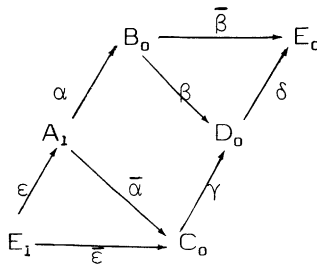
Then $\text{Ker } \gamma$ is trivial, but γ is not injective.

B. Mayer-Vietoris sequences.

Next we deduce two Mayer-Vietoris sequences for a Kervaire diagram (2.2) of interlocking exact sequences in the non-abelian case. In order to make things transparent the ingredients will be listed separately.

Note that similar algebraic investigations have been necessary for the construction of Mayer-Vietoris sequences for relative homotopy groups based on excision isomorphisms in [11].

2.6. Lemma. Let



be a commutative diagram of pointed maps. Let ε be a group homomorphism and let E_1 operate on C_0 from the left by an operation \cdot such that

$$(1) \quad \bar{\alpha}(\varepsilon(e_1)a_1) = e_1 \cdot \bar{\alpha}(a_1) \quad \text{for all } e_1 \in E_1, a_1 \in A_1.$$

Assume that

- (2) (γ, δ) is differential, $(\alpha, \bar{\beta})$ is exact,
- (3) (ε, α) is exact of type (E3), $(\bar{\varepsilon}, \gamma)$ is exact of type (E2).

Then



is a weak pullback of sets.

Proof. Diagram (2.7) is commutative by assumption. Let $b_0 \in B_0, c_0 \in C_0$ such that $\beta(b_0) = \gamma(c_0)$. We have to show the existence of an element $a_1 \in A_1$ such that $\alpha(a_1) = b_0, \bar{\alpha}(a_1) = c_0$. Since

$$\bar{\beta}(b_0) = \delta\beta(b_0) = \delta\gamma(c_0) = *$$

by (1), we have $b_0 = \alpha(a'_1)$ for a suitable $a'_1 \in A_1$. Since

$$\gamma\bar{\alpha}(a'_1) = \beta\alpha(a'_1) = \beta(b_0) = \gamma(c_0),$$

by (3) there exists $e_1 \in E_1$ such that $c_0 = e_1 \cdot \bar{\alpha}(a'_1)$. Then

$$\alpha(\varepsilon(e_1)a'_1) = \alpha(a'_1) = b_0$$

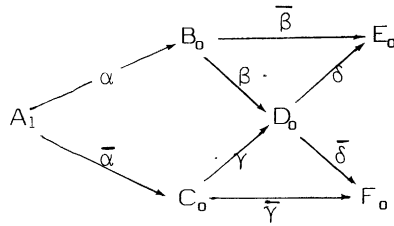
by (3), and

$$\bar{\alpha}(\varepsilon(e_1)a'_1) = e_1 \cdot \bar{\alpha}(a'_1) = c_0$$

by (1). Thus $a_1 = \varepsilon(e_1)a'_1$ is the desired element. \diamond

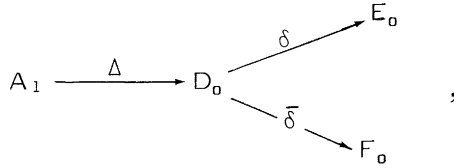
2.8. Remark. The typical situation in the applications is that an operation of A_1 on C_0 is given with $\bar{\alpha}(a_1) = a_1 \cdot *$ inducing the operation of E_1 on C_0 via the homomorphism $\varepsilon: e_1 \cdot c_0 = \varepsilon(e_1) \cdot c_0$. Then condition (1) of (2.6) is automatically fulfilled.

2.9. Lemma. Let



be a commutative diagram of pointed maps.

(a) If (γ, δ) , $(\beta, \bar{\delta})$ and $(\alpha, \bar{\beta})$ are exact, then



where $\Delta = \beta\alpha = \gamma\bar{\alpha}$, is exact in the following sense :

i.e.
 (2.10)
$$A_1 \xrightarrow{\Delta} D_0 \xrightarrow{(\delta, \bar{\delta})} E_0 \times F_0$$

is exact.

(b) If β and γ are group homomorphisms and if $(\alpha, \bar{\beta})$ is exact and (γ, δ) and $(\beta, \bar{\delta})$ are exact of type $(E3^*)$, then (2.10) is exact of type $(E3^*)$.

Proof. (a) Let $a_1 \in A_1$. Then

$$\delta\Delta(a_1) = \delta\gamma\bar{\alpha}(a_1) = * \quad \text{and} \quad \bar{\delta}\Delta(a_1) = \bar{\delta}\beta\alpha(a_1) = * .$$

Let $d_0 \in D_0$ such that $\delta(d_0) = *$ and $\bar{\delta}(d_0) = *$. Then $d_0 = \beta(b_0)$ for some $b_0 \in B_0$. Since

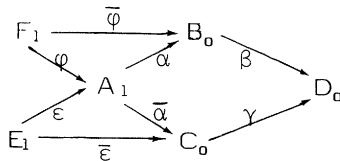
$$\bar{\beta}(b_0) = \bar{\delta}\beta(b_0) = \bar{\delta}(d_0) = * ,$$

we have $b_0 = \alpha(a_1)$ for some $a_1 \in A_1$. Then

$$\Delta(a_1) = \beta\alpha(a_1) = \beta(b_0) = d_0 .$$

(b) The proof is left to the reader. ◊

2.11. Lemma. Let

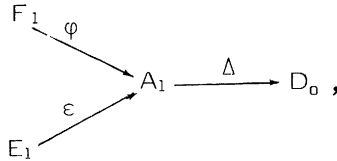


be a commutative diagram of pointed maps with group homomorphisms ε and φ . Let E_1 operate on C_0 from the left by an operation \cdot such that

$$(1) \quad \bar{\alpha}(\varepsilon(e_1)a_1) = e_1 \cdot \bar{\alpha}(a_1) \quad \text{for all } e_1 \in E_1, \quad a_1 \in A_1.$$

Assume that

(2) (ε, α) is exact of type (E3), $(\varphi, \bar{\alpha})$ is exact of type (E3*), and $(\bar{\varepsilon}, \gamma)$ is exact of type (E2). Then



where $\Delta = \beta\alpha = \gamma\bar{\alpha}$, is exact in the following sense :

If $a_1, a'_1 \in A_1$, then $\Delta(a_1) = \Delta(a'_1)$ iff there are $e_1 \in E_1, f_1 \in F_1$ such that $a'_1 = \varepsilon(e_1)a_1\varphi(f_1)$.

Proof. Let $a'_1 = \varepsilon(e_1)a_1\varphi(f_1)$. Then

$$\begin{aligned}
 \Delta(a'_1) &= \Delta(\varepsilon(e_1)a_1\varphi(f_1)) = \beta\alpha(\varepsilon(e_1)a_1\varphi(f_1)) = \\
 &= \beta\alpha(a_1\varphi(f_1)) = \gamma\bar{\alpha}(a_1\varphi(f_1)) = \gamma\bar{\alpha}(a_1) = \Delta(a_1)
 \end{aligned}$$

by (2). Conversely, let $\Delta(a_1) = \Delta(a'_1)$. Since $\gamma\bar{\alpha}(a_1) = \gamma\bar{\alpha}(a'_1)$, by (2) and (1) we have

$$\bar{\alpha}(a'_1) = e_1 \cdot \bar{\alpha}(a_1) = \bar{\alpha}(\varepsilon(e_1)a_1)$$

for some $e_1 \in E_1$. Thus $a'_1 = \varepsilon(e_1)a_1\varphi(f_1)$ for some $f_1 \in F_1$ by (2). ◊

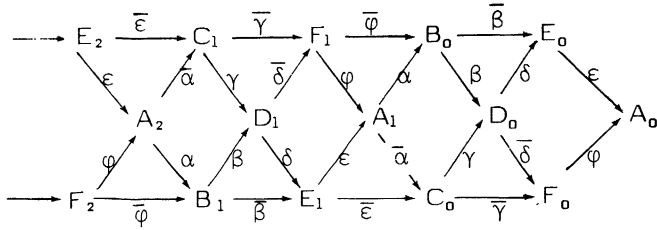
2.12. Remark. Note that in 2.11 the weaker form of exactness

$$\Delta(a_1) = * \quad \text{iff there are } e_1 \in E_1, f_1 \in F_1 \text{ such that } a_1 = \varepsilon(e_1)\varphi(f_1)$$

can be proved under weaker assumptions without the existence of an operation of E_1 on C_0 .

The following proposition giving the promised Mayer-Vietoris sequences for a Kervaire diagram is now an easy consequence of the preceding lemmas.

2.13. Theorem. Let be given the following commutative diagram of exact sequences $(A, B, E), (A, C, F), (B, D, F)$, and (C, D, E) such that the arrows to the left of A_1 are group homomorphisms.



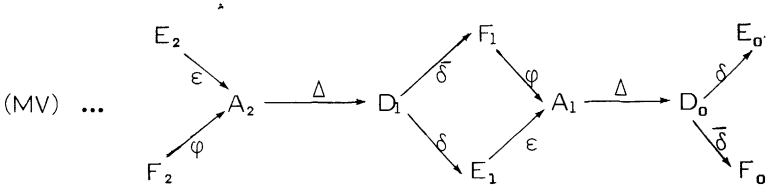
Let E_1 operate on C_0 from the left by an operation $\bar{\alpha}$ such that

$$(1) \quad \bar{\alpha}(\epsilon(e_1)a_1) = e_1.\bar{\alpha}(a_1) \quad \text{for all } e_1 \in E_1, a_1 \in A_1.$$

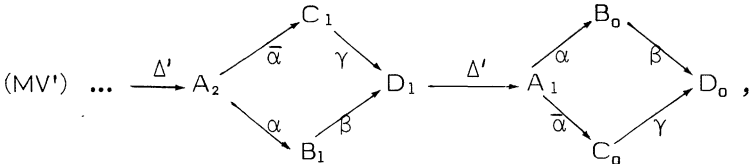
Assume that

(2) at A_1 the sequence (A, B, E) is exact of type (E3) and (A, C, F) is exact of type (E3*); (C, D, E) is exact of type (E2) at C_0 .

Then we have the following exact Mayer-Vietoris sequences



and



where $\Delta = \gamma\bar{\alpha} = \beta\alpha$ and $\Delta' = \epsilon\delta = \phi\bar{\delta}$. Exactness is understood in the sense of 2.6, 2.9, 2.11. \diamond

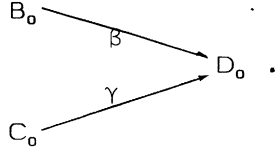
2.14. Remark. We have written the Mayer-Vietoris sequences in a form which seems to be closest to the given Kervaire diagram. This kind of description has already been used in [14, 13, 1, 3]. The Mayer-Vietoris sequences can of course be written in the following form which might be more familiar :

$$\begin{aligned} \dots \rightarrow A_2 \xrightarrow{\Delta} D_1 \xrightarrow{(\delta, \bar{\delta})} E_1 \times F_1 \xrightarrow{\epsilon^{-1}\phi} A_1 \xrightarrow{\Delta} D_0 \xrightarrow{(\delta, \bar{\delta})} E_0 \times F_0, \\ \dots \rightarrow A_2 \xrightarrow{(\alpha, \bar{\alpha})} B_1 \times C_1 \xrightarrow{\beta^{-1}\gamma} D_1 \xrightarrow{\Delta'} A_1 \xrightarrow{(\alpha, \bar{\alpha})} B_0 \amalg C_0 \rightarrow 1. \end{aligned}$$

Here, for example, $\epsilon^{-1}\phi$ is defined by

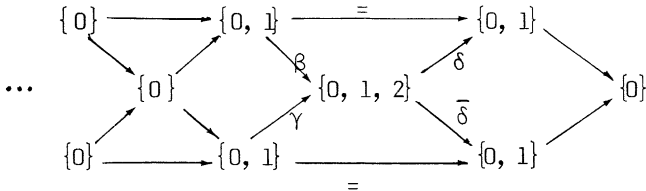
$$\epsilon^{-1}\varphi(e_1, f_1) = \epsilon(e_1)^{-1}\varphi(f_1) \quad \text{for each } e_1 \in E_1, f_1 \in F_1,$$

and $B_0 \amalg C_0$ denotes the pullback of



The following example shows that in the situation of Theorem 2.13 the right hand square need not be a weak pullback.

2.15. Example. In the following diagram of sets with base point 0



let β be the inclusion and let the pointed maps $\gamma, \delta, \bar{\delta}$ be determined by

$$\gamma(1) = 2, \quad \delta(1) = 1, \quad \delta(2) = 0, \quad \bar{\delta}(1) = 0, \quad \bar{\delta}(2) = 1.$$

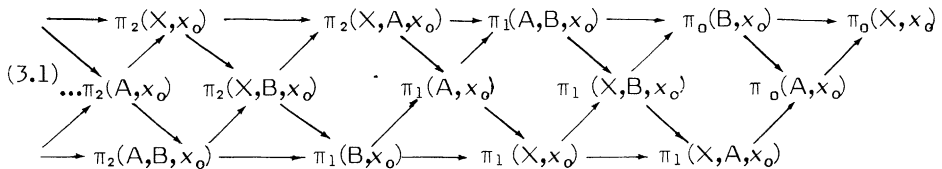
Then we are in the situation of 2.13, but the right hand square is not a weak pullback, since $\{0, 1, 2\} \rightarrow \{0, 1\} \times \{0, 1\}$ cannot be surjective.

3. APPLICATIONS.

A. The exact homotopy sequence of a triple.

The canonical application of Theorem 2.1 is to derive the exact homotopy sequence of a triple from the exact homotopy sequences of the pairs involved.

Let (X, A, B, x_0) be a triple of topological spaces $B \subset A \subset X$ with base point $x_0 \in B$. Then we have a commutative Kervaire diagram (see [18], 3.19)



composed by the exact homotopy sequences of the pairs (A, B, x_0) (X, A, x_0) , (X, B, x_0) and the homotopy sequence

$$(3.2) \quad \begin{aligned} \dots \rightarrow \pi_{n+1}(X, A, x_0) \rightarrow \pi_n(A, B, x_0) \rightarrow \pi_n(X, B, x_0) \rightarrow \pi_n(X, A, x_0) \rightarrow \dots \\ \dots \rightarrow \pi_2(X, A, x_0) \rightarrow \pi_1(A, B, x_0) \rightarrow \pi_1(X, B, x_0) \rightarrow \pi_1(X, A, x_0) \end{aligned}$$

of the triple (X, A, B, x_0) .

Then after verification that (3.2) is differential at $\pi_n(X, B, x_0)$ Theorem 2.1 gives the exactness of (3.2) except at $\pi_1(X, B, x_0)$ where exactness has to be proved separately.

B. Mayer-Vietoris sequences associated to a homotopy pullback.

Let

$$(3.3) \quad \begin{array}{ccc} Z & \xrightarrow{p_1} & X \\ \downarrow p_2 & & \downarrow f \\ Y & \xrightarrow{g} & B \end{array}$$

be a homotopy pullback of pointed topological spaces such that Z is the double mapping track

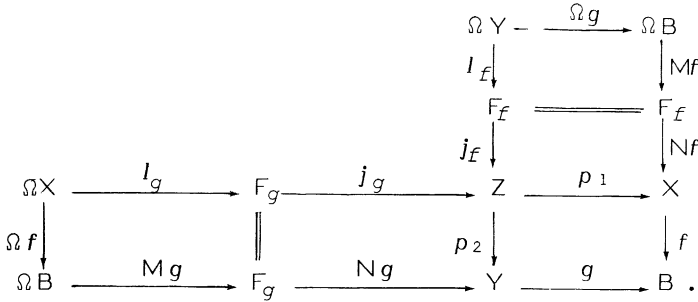
$$Z = \{ (x, y, \lambda) \in X \times Y \times B^I \mid \lambda(0) = f(x), \lambda(1) = g(y) \},$$

B^I being the space of paths into B , and p_1, p_2 are the obvious projections. Note that p_1 and p_2 are pointed fibrations ([13], Proposition 1.9).

Then, for any pointed space A , we have a Kervaire diagram of homotopy sets and homotopy groups which (up to isomorphism) contains the Nomura sequences of f and g and the fibre sequences of p_1 and p_2 :

$$(3.4) \quad \begin{array}{ccccccc} \pi(\Sigma A, F_g) & \longrightarrow & \pi(\Sigma A, Y) & \xrightarrow{\overline{T}_f} & \pi(A, F_f) & \xrightarrow{Nf} & \pi(A, X) \\ & \searrow & \nearrow p_2 & \searrow g & \nearrow \overline{M}f & \searrow j_f & \nearrow p_1 \\ & & \pi(\Sigma A, Z) & & \pi(\Sigma A, B) & & \pi(A, Z) \\ \dots & & \nearrow p_1 & & \nearrow f & & \nearrow p_2 \\ \pi(\Sigma A, F_f) & \longrightarrow & \pi(\Sigma A, X) & \xrightarrow{\overline{T}_g} & \pi(A, F_g) & \xrightarrow{Ng} & \pi(A, Y) \\ & & & & \nearrow j_g & & \nearrow p_2 \\ & & & & \searrow \overline{M}g & & \searrow g \\ & & & & & & \pi(A, B) \end{array}$$

Here, for example, $f.$ denotes postcomposition with f , and $\overline{}$ marks the application of the adjunction between the suspension functor Σ and the loop functor Ω . The maps inducing (3.4) appear in the following diagram of spaces with commutative squares except for (3.3) which is homotopy commutative :



Here, $F_{\mathcal{F}}$ is the homotopy fibre of f ,

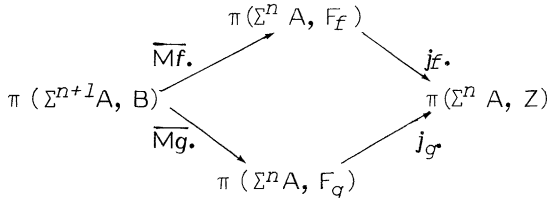
$$F_{\mathcal{F}} = \{ (x, \lambda) \in X \times B^I \mid \lambda(0) = *, \lambda(1) = f(x) \},$$

where $*$ denotes the base point, F_g is the homotopy fibre of g , correspondingly, Nf and Ng are the obvious projections, Mf and Mg are the obvious injections, and we have

$$\begin{aligned}
 I_g(v) &= (*, f \circ v), & I_{\mathcal{F}}(\mu) &= (*, g \circ \mu), \\
 j_g(y, \lambda) &= (*, y, \lambda), & j_{\mathcal{F}}(x, \lambda) &= (x, *, -\lambda),
 \end{aligned}$$

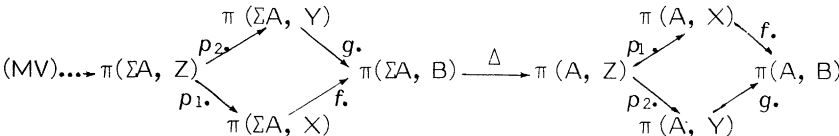
$-\lambda$ being the inverse path.

We observe that diagram (3.4) is commutative except for the squares



which are anticommutative. We replace \overline{Mf} by \overline{Mf}^- and $\overline{I_{\mathcal{F}}}$ by $\overline{I_{\mathcal{F}}}^-$ (see 1.2). The resulting Kervaire diagram is commutative, the modified sequence (f, \overline{Mf}^-) is exact of type (E3) whereas the sequence (g, \overline{Mg}) is exact of type (E3*). Thus we are now in a position to apply Theorem 2.13. We obtain

3.5. Proposition. *There are two Mayer-Vietoris sequences induced by the homotopy pullback (3.3)*



and

$$\begin{array}{c}
 \Sigma A, F_g \xrightarrow{j_g} \Sigma A, Z \\
 \Sigma A, F_f \xrightarrow{j_f} \Sigma A, Z \\
 \text{---} \xrightarrow{\Delta'} \pi(\Sigma A, B)
 \end{array}
 \xrightarrow{\Delta'}
 \begin{array}{c}
 \pi(A, F_f) \xrightarrow{j_f} \pi(A, Z) \\
 \pi(A, F_g) \xrightarrow{j_g} \pi(A, Z) \\
 \pi(\Sigma A, B) \xrightarrow{\overline{Mf}} \pi(A, F_f) \\
 \pi(\Sigma A, B) \xrightarrow{\overline{Mg}} \pi(A, F_g)
 \end{array}$$

where

$$\Delta = j_g \cdot \overline{Mg} = j_f \cdot \overline{Mf} \quad \text{and} \quad \Delta' = g \cdot p_2 = f \cdot p_1$$

Note that the fact that

$$\begin{array}{ccc}
 & \pi(A, X) & \\
 p_1 \cdot & \nearrow & f \cdot \\
 \pi(A, Z) & & \pi(A, B) \\
 p_2 \cdot & \searrow & g \cdot \\
 & \pi(A, Y) &
 \end{array}$$

is a weak pullback of sets does not follow from 2.13 but has to be proved separately. \diamond

3.6. Remark. The first Mayer-Vietoris sequence (MV) of (3.5) is up to $\Sigma \Omega$ adjunction the exact sequence of [13], Theorem 1.7. It induces [13], Theorem 1.8 in case g is a pointed fibration, or more generally, a pointed h-fibration. By specializing A to be the 0-sphere S^0 we obtain [3], 4.2 and [6], Theorem.

The second Mayer-Vietoris sequence (MV') of (3.5) can be prolonged as follows. Let the map $\Gamma : F_f \times F_g \rightarrow Z$ to be defined by

$$\Gamma((x, \lambda), (y, \mu)) = (x, y, -\lambda + \mu),$$

where $-$ denotes the addition of paths. For any pointed space A we have an induced map

$$\Gamma : \pi(A, F_f) \times \pi(A, F_g) = \pi(A, F_f \times F_g) \rightarrow \pi(A, Z).$$

3.7. Proposition. The sequence (3.8)

$$\pi(\Sigma A, Z) \xrightarrow{\Gamma'} \pi(\Sigma A, B) \xrightarrow{(\overline{Mf}, \overline{Mg})} \pi(A, F_f) \times \pi(A, F_g) \xrightarrow{\Gamma} \pi(A, Z) \xrightarrow{\Delta'} \pi(A, B)$$

is exact, more precisely, it is exact of type (E3*) at $\pi(\Sigma A, B)$, and exact of type (E2) at $\pi(A, F_f) \times \pi(A, F_g)$.

Note that (3.8) involves the unmodified map \overline{Mf} . The exactness at $\pi(\Sigma A, B)$ follows from 2.9 (b). The remaining parts have to be proved directly. This is left to the reader. \diamond

3.9. Remark. (i) The second Mayer-Vietoris sequence (MV') of (3.5) and its prolongation (3.8) do not seem to have been previously noted.

(ii) Since $\pi(A, F_{\mathcal{L}})$ can be interpreted as the relative homotopy set $[i, f]$ where $i : A \rightarrow CA$ is the inclusion of A into the (reduced) cone over A (see [5], (13.3)), one has the possibility of writing the sequences (MV') of (3.5) and (3.8) in terms of relative homotopy sets. In the more classical case of $A = S^0$ and in which f and g are inclusions, one obtains an exact sequence

$$\dots \rightarrow \pi_2(B, X) \times \pi_2(B, Y) \rightarrow \pi_1(Z) \rightarrow \pi_1(B) \rightarrow \pi_1(B, X) \times \pi_1(B, Y) \rightarrow \pi_0(Z) \rightarrow \pi_0(B)$$

involving the relative homotopy groups (sets) $\pi_n(B, X), \pi_n(B, Y)$.

C. Mayer-Vietoris sequences for a pair of maps.

Let $f : X \rightarrow Y$ and $g : E \rightarrow B$ be pointed continuous maps between pointed topological spaces.

Then we can associate a Kervaire diagram of homotopy sets and homotopy groups which, up to $\Sigma\Omega$ adjunction, contains the Puppe sequence (1.11) of f at B , the Nomura sequence (1.12) of g at X and the exact sequences (1.13), (1.14) in homotopy pair theory :

$$(3.10) \dots \begin{array}{ccccccc} \pi(\Sigma C_{\mathcal{L}}, B) & \xrightarrow{\cdot\Sigma Pf} & \pi(\Sigma Y, B) & \xrightarrow{\beta'} & \pi(X, F_g) & \xrightarrow{Ng} & \pi(X, E) \\ & \searrow \delta & \nearrow c & \searrow \cdot\Sigma f & \nearrow \overline{Mg} & \searrow \delta' & \nearrow d \\ & & \pi(\Sigma f, g) & & \pi(\Sigma X, B) & & \pi(f, g) & & \pi(X, B) \\ & & \nearrow \delta' & \searrow d & \nearrow g & \searrow \cdot Qf & \nearrow \delta & \searrow c & \nearrow \cdot f \\ \pi(\Sigma X, F_g) & \xrightarrow{Ng} & \pi(\Sigma X, E) & \xrightarrow{\beta} & \pi(C_{\mathcal{L}}, B) & \xrightarrow{\cdot Pf} & \pi(Y, B) \end{array}$$

We will not give the complete details of the construction which can essentially be found in [8] but restrict ourselves to point out the general method of applying the results of Section 2.

The construction is such that the triangles and rectangles of (3.10) commute except for the rectangles

$$\begin{array}{ccc} & \pi(\Sigma^n X, F_g) & \\ \overline{Mg} \nearrow & & \searrow \delta' \\ \pi(\Sigma^{n+1} X, B) & & \pi(\Sigma^n f, g) \\ \cdot \Sigma^n Qf \searrow & & \nearrow \delta \\ & \pi(\Sigma^n C_{\mathcal{L}}, B) & \end{array}$$

in case n is even, $n \geq 0$, which are anticommutative. Thus, for n even, we replace \overline{Mg} by \overline{Mg}^- and β' by β'^- (see (1.2)). Then we are in a position to apply the results of Section 2. We can use Theorem 2.1

to derive the essential parts of exactness of (1.14) from the corresponding properties of (1.13) though, of course, there is the possibility of obtaining (1.14) from (1.13) by duality. Theorem 2.13 provides us with two Mayer-Vietoris sequences.

3.11. Proposition. For a pair of maps $f : X \rightarrow Y, g : E \rightarrow B$ we have the following exact Mayer-Vietoris sequences

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & \pi(\Sigma Y, B) & \\
 c \nearrow & & \searrow \cdot \Sigma f \\
 \pi(\Sigma f, g) & & \pi(\Sigma X, B) \\
 d \searrow & & \nearrow g \cdot \\
 & \pi(\Sigma X, E) &
 \end{array}
 & \xrightarrow{\Delta = \delta(\cdot Qf)} &
 \begin{array}{ccc}
 & \pi(X, E) & \\
 d \nearrow & & \searrow g \cdot \\
 \pi(f, g) & & \pi(X, B) \\
 c \searrow & & \nearrow \cdot f \\
 & \pi(Y, B) &
 \end{array}
 \end{array}$$

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \pi(\Sigma C_f, B) & & \\
 \delta \searrow & & \nearrow \\
 & \pi(\Sigma f, g) & \\
 \delta' \nearrow & & \searrow \\
 \pi(\Sigma X, F_g) & &
 \end{array}
 & \xrightarrow{\Delta' = g \cdot d} &
 \begin{array}{ccc}
 & \pi(X, F_g) & \\
 \overline{Mg} \cdot \nearrow & & \searrow \delta' \\
 \pi(\Sigma X, B) & & \pi(f, g) \\
 \cdot Qf \searrow & & \nearrow \delta \\
 & \pi(C_f, B) &
 \end{array}
 \end{array}
 \quad \diamond$$

The special topological situation enables one to prolongate the second Mayer-Vietoris sequence (MV') of (3.11) in a similar way as in the preceding example.

3.12. Proposition. There is a map

$$\Gamma : \pi(X, F_g) \times \pi(C_f, B) \rightarrow \pi(f, g)$$

such that the sequence (3.13)

$$\pi(\Sigma f, g) \xrightarrow{\Delta'} \pi(\Sigma X, B) \xrightarrow{(\overline{Mg} \cdot \cdot Qf)} \pi(X, F_g) \times \pi(C_f, B) \xrightarrow{\Gamma} \pi(f, g) \xrightarrow{\Delta'} \pi(X, B)$$

is exact. More precisely, (3.13) is exact of type (E3*) at $\pi(\Sigma X, B)$ and exact of type (E2) at $\pi(X, F_g) \times \pi(C_f, B)$.

The proof will be given elsewhere. \(\diamond\)

The connecting map $\Delta : \pi(\Sigma X, B) \rightarrow \pi(f, g)$ of Proposition 3.11 is intimately connected with the theory of the Toda bracket or secondary composition [19]. Let $h : Y \rightarrow E$ be a map such that $hf \simeq *$ and $gh \simeq *$. Then the Toda bracket

$$\{g, h, f\} \subset \pi(\Sigma X, B)$$

is a certain double coset of the subgroups $(\cdot \Sigma f) \pi(\Sigma Y, B)$ and $g \cdot \pi(\Sigma X, E)$, in that order. As an application we give a short proof of the following proposition.

3.14. Proposition. The bracket $\{g, h, f\}$ contains the neutral element of $\pi(\Sigma X, B)$ iff the element θ of $\pi(f, g)$ represented by the commutative square

$$\begin{array}{ccc} X & \xrightarrow{hf} & E \\ f \downarrow & & \downarrow g \\ Y & \xrightarrow{gh} & B \end{array}$$

is trivial.

Remark. Proposition 3.14 is equivalent to [8], Theorem 8.6, but the proof in [8] depended on the rather complicated theory of diagonal factorization of a homotopy pair class.

Proof of 3.14. Let $\tau \in \{g, h, f\}$ and let τ^{-1} denote its inverse in $\pi(\Sigma X, B)$. In [8], Theorem 5.3 it is proved that $\Delta\tau^{-1} = \theta$. Hence τ^{-1} belongs to the kernel of Δ iff $\theta = 0$. But τ^{-1} is an element of a double coset of the subgroups $g.\pi(\Sigma X, E)$ and $(.\Sigma f)\pi(\Sigma Y, B)$ in that order and by Proposition 3.11 the trivial double coset of these subgroups is precisely the kernel of Δ . \diamond

3.15. Remark. There is a third exact sequence involving the homotopy pair sets $\pi(\Sigma^n f, g)$ ($n \geq 0$) described in [8]:

$$(3.16) \dots \rightarrow \pi(\Sigma^{n+1}f, g) \xrightarrow{\Gamma} \pi(\Sigma^n C_f, F_g) \xrightarrow{\beta''} \pi(\Sigma^n Y, E) \xrightarrow{\Delta} \pi(\Sigma^n f, g) \rightarrow \dots$$

The exactness of 3.16 is established via the theory of diagonal factorization in [8]. We note that it can also be obtained by applying a variant of Theorem 2.1 to the Kervaire diagram

$$\begin{array}{ccccccc} \rightarrow \pi(\Sigma Y, E) & \xrightarrow{.\Sigma f} & \pi(\Sigma X, E) & \xrightarrow{\beta} & \pi(C_f, B) & & \\ & \Delta \searrow & \nearrow d & & \searrow \delta & & \\ \dots & & \pi(\Sigma f, g) & & \pi(C_f, E) & & \pi(f, g) \\ & \nearrow \delta & \Gamma \searrow & & \nearrow .Pf & \Delta & \searrow d \\ \rightarrow \pi(\Sigma C_f, B) & \xrightarrow{\overline{M}g.} & \pi(C_f, F_g) & \xrightarrow{\beta''} & \pi(Y, E) & \xrightarrow{.f} & \pi(X, E) \end{array}$$

D. Mayer-Vietoris sequences for groupoids.

The exact Mayer-Vietoris sequence of [4], Theorem 3.3 associated to a homotopy pullback (3.3) of groupoids can be obtained in much the same way as sequence (MV) of (3.5) above. Though in this special situation this is certainly not the simplest way this approach leads to the discovery of a new exact sequence in groupoid theory involving the groupoid homotopy fibres of the morphisms f and g . This sequence

is the prototype of the bottom end of the prolonged Mayer-Vietoris sequences (MV') of (3.5) and (3.11) in 3.7 and 3.12.

This sort of topic will be dealt with elsewhere.

4. EXACT SEQUENCES FOR FREE HOMOTOPY SETS.

The constructions of this section can be carried out both in the free and pointed category of topological spaces or, more generally, in a suitable abstract homotopy category (see for example [10]).

Let W and B be topological spaces. We recall the definition of the track groupoid $\pi_1^W(B)$ ([2], see also [16]).

Let w and w' be maps from W to B and suppose that h_t and h'_t are homotopies such that

$$h_{\cdot 0} = h'_{\cdot 0} = w \quad \text{and} \quad h_{\cdot 1} = h'_{\cdot 1} = w'.$$

Then h and h' are defined to be *equivalent (relatively homotopic)*, denoted $h_t \cong h'_t$, if there exists a homotopy of homotopies $H_{t,s}$ that satisfies

$$H_{t,0} = h_t, \quad H_{t,1} = h'_t, \quad H_{0,s} = w, \quad H_{1,s} = w'$$

($s, t \in I$). Let $\{h_t\}$ denote the equivalence class of h_t . It is often convenient to denote this class diagrammatically by the homotopy square

$$\begin{array}{ccc} W & \xrightarrow{w'} & B \\ \parallel & \nearrow \{h_t\} & \parallel \\ W & \xrightarrow{w} & B \end{array}$$

or simply

$$\begin{array}{ccc} W & \xrightarrow{w'} & B \\ \parallel & \nearrow h_t & \parallel \\ W & \xrightarrow{w} & B. \end{array}$$

Then we have the (w, w') -track set

$$\pi_1^W(B; w, w') = \{ \{h_t\} \mid h_0 = w, h_1 = w' \}.$$

If $w = w'$ then this becomes the w -based track group $\pi_1^W(B; w)$ with operation

$$\{h_t\} \times \{k_t\} = \{h_t + k_t\} = \begin{array}{ccc} W & \xrightarrow{w} & B \\ \parallel & \nearrow k_t & \parallel \\ W & \xrightarrow{w} & B \\ \parallel & \nearrow h_t & \parallel \\ W & \xrightarrow{w} & B. \end{array}$$

Note. (i) The track addition defines a right action of $\pi_1^W(B; w')$, resp. a left action of $\pi_1^W(B; w)$ on $\pi_1^W(B; w, w')$.

(ii) $\pi_1^W(B; w)$ is an invariant of the homotopy class of w , for if h_t has $h_0 = w$ and $h_1 = w'$ then

$$\begin{array}{ccc}
 \begin{array}{ccc}
 W & \xrightarrow{w} & B \\
 \parallel & \nearrow k_t & \parallel \\
 W & \xrightarrow{w} & B
 \end{array} & \longmapsto & \begin{array}{ccc}
 W & \xrightarrow{w'} & B \\
 \parallel & \nearrow h_t & \parallel \\
 W & \xrightarrow{w} & B \\
 \parallel & \nearrow k_t & \parallel \\
 W & \xrightarrow{w} & B \\
 \parallel & \nearrow h_{1-t} & \parallel \\
 W & \xrightarrow{w'} & B
 \end{array}
 \end{array}$$

defines an isomorphism $\{h_t\}_\# : \pi_1^W(B; w) \rightarrow \pi_1^W(B; w')$.

A. A new homotopy set.

Let $w : W \rightarrow B$ and $f : X \rightarrow B$ be maps. A map $u : W \rightarrow X$ and an element $\{\mu_t\} \in \pi_1^W(B; w, fu)$ determine a homotopy commutative square

$$(4.1) \quad \begin{array}{ccc}
 W & \xrightarrow{u} & X \\
 w \downarrow & \nearrow \mu_t & \downarrow f \\
 B & \xlongequal{\quad} & B,
 \end{array}$$

denoted by (μ_t, u) . Let $\pi(w, f/B)$ denote the set of those squares factored out by the relation \sim where if we are given a homotopy $h_t : W \rightarrow X$ we have

$$\begin{array}{ccc}
 \begin{array}{ccc}
 W & \xrightarrow{h_0} & X \\
 w \downarrow & \nearrow \mu_t & \downarrow f \\
 B & \xlongequal{\quad} & B
 \end{array} & \sim & \begin{array}{ccc}
 W & \xrightarrow{h_1} & X \\
 \parallel & \nearrow h_t & \parallel \\
 W & \xrightarrow{h_0} & X \\
 w \downarrow & \nearrow \mu_t & \downarrow f \\
 B & \xlongequal{\quad} & B.
 \end{array}
 \end{array}$$

The equivalence class of (μ_t, u) in $\pi(w, f/B)$ will be denoted by $\{\mu_t, u\}$. The notation $\pi(w, f/B)$ is intended to suggest "classes of maps from w to f over B ".

4.2. Remark. There is a canonical map from the set $[w, f]_B$ of homotopy classes over B (see [5], (0.22)) to $\pi(w, f/B)$. This map is a bijection if f is a fibration (or, more generally, an h-fibration) (see [10], 3.7). If we are in the pointed category and if we let $w = * : W \rightarrow B$ (i.e. the constant map) then there is an obvious bijection of $\pi(w, f/B)$ with the

homotopy pair set $\tau(W^*, f)$ and hence with $\pi(W, F_f)$, see [8]. Hence $\pi(w, f/B)$ can be regarded as a generalization of $\pi(W, F_f)$. We pursue this line of thought by studying what may be regarded as the covariant free homotopy sequence of a free map $f : X \rightarrow B$.

First note that $d\{\mu_t, u\} = \{u\}$ defines a *domain restriction operator* $d : \pi(w, f/B) \rightarrow \pi(W, X)$.

Let $u : W \rightarrow X$ be a map such that $fu \simeq w$. Choose (the class of) a homotopy μ_t from w to fu . This determines the homotopy commutative square $U = (\mu_t, u)$ (see (4.1)) and its class

$$\tilde{U} = \{\mu_t, u\} \in \pi(w, f/B).$$

Then an operator $m_U : \pi_1^W(B; fu) \rightarrow \pi(w, f/B)$, that depends on the choice of U is defined by the rule

$$\begin{array}{ccc} W & \xrightarrow{fu} & B \\ \parallel & \nearrow h_t & \parallel \\ W & \xrightarrow{fu} & B \end{array} \xrightarrow{m_U} \begin{array}{ccc} W & \xrightarrow{u} & X \\ \parallel & \nearrow h_t & \parallel \\ W & \xrightarrow{\mu_t} & B \end{array} \begin{array}{ccc} & & X \\ & & \downarrow f \\ & & B \end{array}$$

4.3. Theorem. *The following sequence of pointed sets (with base points $\bigcup_{\tilde{U}} u, w$ as indicated) is exact. Moreover it is exact of type (E3*) at $\pi_1(B; fu)$.*

$$\pi_1^W(X; u) \xrightarrow{f_*} \pi_1^W(B; fu) \xrightarrow{m_U} \pi(w, f/B) \xrightarrow{d} \pi(W, X)_U \xrightarrow{f_*} \pi(W, B)_w.$$

Proof. The exactness at $\pi(W, X)$ is obvious and clearly $dm_U\{h_t\} = \{u\}$. Let $\{\sigma_t, v\} \in \pi(w, f/B) \tilde{U}$ and suppose that $d\{\sigma_t, v\} = \{u\}$. Then there is a homotopy ψ_t with $\psi_0 = v$ and $\psi_1 = u$. Then m_U sends the element

$$\begin{array}{ccc} W & \xrightarrow{u} & X & \xrightarrow{f} & B \\ \parallel & \nearrow \psi_t & \parallel & & \parallel \\ W & \xrightarrow{v} & X & \xrightarrow{f} & B \\ \parallel & \nearrow \sigma_t & \parallel & & \parallel \\ W & \xrightarrow{w} & B & & B \\ \parallel & \nearrow \mu_{1-t} & \parallel & & \parallel \\ W & \xrightarrow{fu} & B & & B \end{array}$$

to

$$\begin{array}{ccc} W & \xrightarrow{u} & X \\ \parallel & \nearrow \psi_t & \parallel \\ W & \xrightarrow{v} & X \\ \parallel & \nearrow \sigma_t & \parallel \\ B & \xrightarrow{w} & B \end{array} \sim \begin{array}{ccc} W & \xrightarrow{v} & X \\ \parallel & \nearrow \sigma_t & \parallel \\ B & \xrightarrow{w} & B \end{array} \begin{array}{ccc} & & X \\ & & \downarrow f \\ & & B \end{array}$$

Exactness at $\pi_1^W(B ; fu)$: First observe that

$$\begin{array}{ccc}
 \begin{array}{ccc}
 W & \xrightarrow{u} & X \\
 \parallel & \nearrow n_t & \parallel \\
 W & \xrightarrow{u} & X
 \end{array} & \xrightarrow{m_U f.} & \begin{array}{ccc}
 W & \xrightarrow{u} & X \\
 \parallel & \nearrow n_t & \parallel \\
 W & \xrightarrow{u} & X \\
 \downarrow w & \nearrow \mu_t & \downarrow f \\
 B & \xrightarrow{\quad} & B
 \end{array} \sim U.
 \end{array}$$

Suppose that $m_U\{h_t\} = \tilde{U}$. Then for some $\{n_t\} \in \pi_1^W(X ; u)$ we have

$$\mu_t + h_t + fn_t \cong \mu_t.$$

Thus $fn_{1-t} \cong h_t$ and hence $f \cdot \{n_{1-t}\} = \{h_t\}$. Finally suppose that $\{h_t\}$ and $\{h'_t\}$ are sent to the same element of $\pi(w, f/B)$ by m_U . Then

$$\mu_t + h_t + f\psi_t \cong \mu_t + h'_t \quad \text{for some } \{\psi_t\} \in \pi_1^W(X ; u).$$

Hence $f\psi_t \cong h_{1-t} h'_t$, so that $\{h_t\}^{-1} \times \{h'_t\} \in \text{Im } f$, as required. \diamond

4.4. Remark. (i) Alternatively, Theorem 4.3 can be deduced using groupoid methods from the exact fibre sequence for groupoids (1.7) by replacing the induced morphism of track groupoids $f. : \pi_1^W(X) \rightarrow \pi_1^W(B)$ by its associated mapping track fibration. This implies that we have exactness of type (E2) at $\pi(W, X)_u$. Here we have given preference to the simpler direct approach.

If W is locally compact or if we are in a convenient category of topological spaces, then the track groupoid $\pi_1^W(X)$ is isomorphic to the fundamental groupoid $\pi_1(X^W)$ of the function space X^W and the homotopy set $\pi(w, f/B)$ can be identified with the component set $\pi_0(F_{f^W})$ of the homotopy fibre of the induced map of function spaces $f^W : X^W \rightarrow B^W$.

(ii) The sequence in 4.3 can be continued to the left. The ramifications will be considered elsewhere.

4.5. Remark. If we are in a pointed category and $w = *$ then the sequence in 4.3 is equivalent to the sequence studied by Rutter [16] and used there to classify the maps from any space into the homotopy fibre of f . Theorem 4.3 leads to a similar classification of $\pi(w, f/B)$. Let $K(u, w)$ denote the set of left cosets of $f. \pi_1^W(X ; u)$ in $\pi_1^W(B ; fu)$, observe that $K(u, w)$ depends only on the homotopy classes $\{u\}$ and $\{w\}$.

4.6. Corollary. *There is a bijection*

$$\{u\} \in (f.)^{-1}\{w\} \cup K(u, w) \longleftrightarrow \pi(w, f/B).$$

B. Classifying the maps into a homotopy pullback.

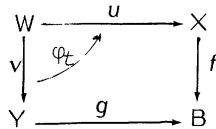
Let $f : X \rightarrow B$, $g : Y \rightarrow B$ and $w : W \rightarrow B$ be fixed maps and let $u : W \rightarrow X$ and $v : W \rightarrow Y$ be maps such that $fu \simeq w$ and $gv \simeq w$. Then homotopies μ_t and ν_t with

$$\mu_0 = \nu_0 = w, \quad \mu_1 = fu \quad \text{and} \quad \nu_1 = gv$$

determine homotopy commutative squares $U = (\mu_t, u)$, $V = (\nu_t, v)$ and elements

$$\tilde{U} = \{ \mu_t, u \} \in \pi(w, f/B), \quad \tilde{V} = \{ \nu_t, v \} \in \pi(w, g/B),$$

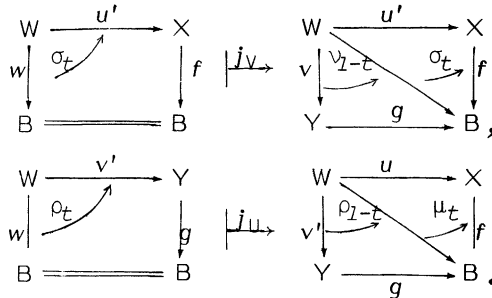
respectively. If $\varphi_t = \nu_{1-t} + \mu_t$ then the square



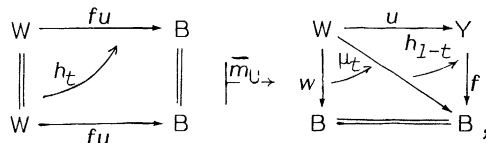
represents an element $\{\varphi\} \in \pi(W, Z)$, where Z is the homotopy pullback as in (3.3), and every element can be represented in this way. It is a consequence of the theory of homotopy pullbacks that $\{\varphi\}$ depends only on the homotopy classes of u and v and on the equivalence class of φ_t . Thus we will use the notation $\{\varphi\} = \{v, \varphi_t, u\}$. Now consider operators

$$j_V : \pi(w, f/B) \rightarrow \pi(W, Z) \quad \text{and} \quad j_U : \pi(w, g/B) \rightarrow \pi(W, Z)$$

given by the rules

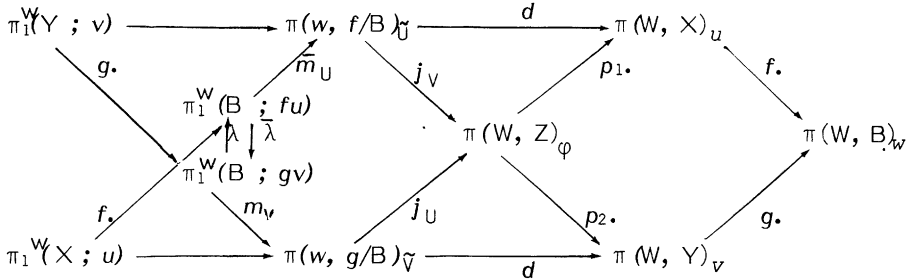


Finally let $\bar{m}_U : \pi_1^W(B ; fu) \rightarrow \pi(w, f/B)$ be given by



noting the distinction between the definitions of m_U and \bar{m}_U . Then we have the following theorem.

4.7. Theorem. *The following diagram, in which $\lambda = \{\varphi_t\}_\#$ and $\bar{\lambda} = \lambda^{-1}$ is commutative and its sequences of pointed sets, with base points $\{w\}, \{u\}, \{v\}, \{\varphi\}, \bar{U}, \bar{V}$ as indicated, are exact.*



Moreover

$$\cdot \xrightarrow{f \cdot} \cdot \xrightarrow{\bar{m}_U} \cdot \quad (\text{resp.} \quad \cdot \xrightarrow{g \cdot} \cdot \xrightarrow{m_V} \cdot)$$

is exact of type (E3) at $\pi_1^W(B; fu)$ (resp. of type (E3*) at $\pi_1^W(B; gv)$). There is an operation of $\pi_1^W(X; u)$ on $\pi(w, g/B)$ given by

$$\{\psi_t\} \cdot \{\rho_t, v'\} = \{\mu_t + f\psi_t + \mu_{-t} + \rho_t, v'\}$$

that satisfies

$$(4.8) \quad \{\psi_t\} \cdot m_V \{k_t\} = m_V(\bar{\lambda}(f \cdot) \{\psi_t\} \times \{k_t\}).$$

Further

$$\cdot \xrightarrow{n_V} \cdot \xrightarrow{j_U} \cdot$$

is exact of type (E2) at $\pi(w, g/B)$, where $n_V = m_V \bar{\lambda} f$.

Proof. The commutativity of the diagram can readily be checked. The exactness of

$$\cdot \xrightarrow{g \cdot} \cdot \xrightarrow{m_V} \cdot \xrightarrow{d} \cdot \xrightarrow{g \cdot} \cdot$$

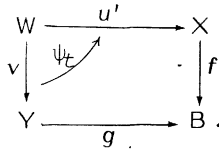
is a consequence of Theorem 4.3, in particular exactness of type (E3*) at $\pi_1^W(B; gv)$. Next consider

$$\pi_1^W(Y; v) \rightarrow \pi(w, f/B) \rightarrow \pi(W, Z) \rightarrow \pi(W, Y).$$

Certainly

$$p_2 \cdot j_V \{\sigma_t, u'\} = \{v\}.$$

Suppose that $p_2 \cdot \{\psi\} = \{v\}$. Then $\{\psi\}$ is represented by a square of form



Then

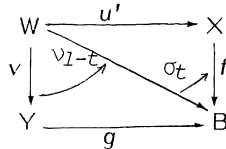
$$j_V \{v_t + \psi_t, u'\} = \{v, v_{I-t} + v_t + \psi_t, u'\} = \{v, \psi_t, u'\} = \{\psi\}.$$

Exactness at $\pi(w, f/B)$: Let $n_U = \bar{m}_U \lambda g$, then if $\{h_t\} \in \pi_1^W(Y; v)$, we have

$$j_V n_U \{h_t\} = \{v, v_{I-t} + \mu_t + \varphi_{I-t} + gh_{I-t} + \varphi_t, u\} = \{v, gh_{I-t} + \varphi_t, u\}$$

$$(\text{since } v_{I-t} + \mu_t = \varphi_t) = \{v, \varphi_t, u\} = \{\varphi\}.$$

Now if $\{\sigma_t, u'\} \in \pi(w, f/B)$ then $j_V \{\sigma_t, u'\}$ is represented by a square



and if this square represents $\{\varphi\} \in \pi(W, Z)$ then there exist

$$\{w_t\} \in \pi_1^W(X; u', u) \quad \text{and} \quad \{w'_t\} \in \pi_1^W(Y; v) \quad \text{with} \quad gw'_t + v_{I-t} + \sigma_t + fw_t \approx \varphi_t.$$

Then

$$\begin{aligned}
 n_U \{w'_t\} &= \{\mu_t + \varphi_{I-t} + gw'_{I-t} + \varphi_t, u\} = \{\mu_t + \mu_{I-t} + v_t + gw'_{I-t} + \varphi_t, u\} = \\
 &= \{v_t + gw'_{I-t} + \varphi_t, u\} = \{v_t + gw'_{I-t} + gw'_t + v_{I-t} + \sigma_t + fw_t, u\} = \\
 &= \{\sigma_t + fw_t, u\} = \{\sigma_t, u'\},
 \end{aligned}$$

as required. The exactness of

$$\pi_1^W(X; u) \rightarrow \pi_1^W(B; fu) \rightarrow \pi(w, f/B) \rightarrow \pi(W, X)$$

is only affected by the switch from m_U to \bar{m}_U in that type (E3) is produced at $\pi_1^W(B; fu)$. To check the equality (4.8):

$$\begin{aligned}
 m_V(\bar{\lambda}(f.)\{\psi_t\} \times \{k_t\}) &= m_V \{\varphi_t + f\psi_t + \varphi_{I-t} + k_t\} = \\
 &= \{v_t + v_{I-t} + \mu_t + f\psi_t + \mu_{I-t} + v_t + k_t, v\} = \{\psi_t\} \cdot m_V \{k_t\}.
 \end{aligned}$$

It remains to check that

$$. \xrightarrow{n_V} . \xrightarrow{j_U} .$$

is exact of type (E2) at $\pi(w, g/B)$. Regarding the action of $\pi_1^W(X; u)$ on $\pi(w, g/B)$, we have to check

$$(i) \quad n_V\{\psi_t\} = \{\psi_t\} \cdot \tilde{V}$$

and we have

$$n_V\{\psi_t\} = m_V \{ \varphi_{t+} f \psi_{t+} \varphi_{1-t} \} = \{ v_t + \varphi_{t+} f \psi_{t+} \varphi_{1-t}, v \} = \{\psi_t\} \cdot \tilde{V}.$$

(ii) Suppose that

$$\{\psi_t\} \cdot \{\rho_t, v'\} = \{\rho'_t, v''\}.$$

Then

$$\{\rho'_t, v''\} = \{ \mu_t + f \psi_{t+} \mu_{1-t} + \rho_t, v' \}$$

so that there exists a homotopy k_t , from v' to v'' with

$$\rho'_t \cong \mu_t + f \psi_{t+} \mu_{1-t} + \rho_t + gk_t.$$

Then

$$\begin{aligned} j_U\{\rho'_t, v''\} &= \{ v'', \rho'_{1-t} + \mu_t, u \} = \\ &= \{ v'', gk_{1-t} + \rho_{1-t} + \mu_t + f \psi_{1-t+} \mu_{1-t} + \mu_t, u \} = \\ &= \{ v', \rho_{1-t} + \mu_t, u \} = j_U\{\rho_t, v'\}. \end{aligned}$$

Conversely, if $j_U\{\rho_t, v'\} = j_U\{\rho'_t, v''\}$ then there exist

$$\{h_t\} \in \pi_1^W(X; u) \quad \text{and} \quad \{k_t\} \in \pi_1^W(Y; v', v'')$$

so that

$$gk_t + \rho'_{1-t} + \mu_t + fh_t \cong \rho_{1-t} + \mu_t.$$

Hence

$$\begin{aligned} \{\rho_t, v'\} &= \{ \mu_t + fh_{1-t} + \mu_{1-t} + \rho'_t + gk_{1-t}, v' \} = \\ &= \{ \mu_t + fh_{1-t} + \mu_{1-t} + \rho'_t, v'' \} = \{h_{1-t}\} \cdot \{\rho'_t, v''\} \end{aligned}$$

as required, completing the proof of Theorem 4.7. \diamond

Applying Theorem 2.13 and Lemma 2.11 we have the following corollary.

4.9. Corollary. *The sequence*

$$\pi_1^W(X; u) \times \pi_1^W(Y; v) \xrightarrow{\Theta} \pi_1^W(B; fu) \xrightarrow{\Delta} \pi(W, Z) \xrightarrow[\varphi]{(\rho_1, \rho_2)} \pi(W, X)_U \times \pi(W, Y)_V$$

is exact, where

$$\Delta = j_V \bar{m}_U \quad \text{and} \quad \Theta(\{h_t\}, \{k_t\}) = f \cdot \{h_{1-t}\} \times \lambda g \cdot \{k_t\}.$$

Moreover the images of two elements under Δ coincide iff they belong to the same double coset of the subgroups $f \cdot \pi_1^W(X; u)$ and $\lambda g \cdot \pi_1^W(Y; v)$. \diamond

Let $K(w, u, v)$ denote the set of double cosets in $\pi_1^W(B; fu)$ of the subgroups $f \cdot \pi_1^W(X; u)$ and $\lambda g \cdot \pi_1^W(Y; v)$. Then the following is an immediate consequence of Corollary 4.9.

4.10. Classification Theorem. *There is a bijection*

$$\pi_0(W, Z) \longleftrightarrow \bigcup_{\{w\} \in \pi(W, B)} (\{u\}, \{v\}) \bigcup_{\{w\} \in f^{-1}\{w\} \times g^{-1}\{w\}} K(w, u, v). \quad \diamond$$

4.11. Remark. The classification 4.10 is also valid in the pointed category. If X^W and Y^W are H_0 -spaces in the sense of Rutter [16] then f and g are homomorphisms of abelian groups and can sometimes be computed as discussed in [16].

4.12. Remark. The Kervaire diagram 3.4 can be recovered through specialization of the diagram in Theorem 4.7. A point free treatment of homotopy pair theory can be given enabling the Kervaire diagram 3.10 to be obtained by specialization. Details will be given elsewhere.

4.13. Remark. Proposition 3.14 can be regarded as an item of information about the secondary structure of the relevant Mayer-Vietoris sequence. A more systematic study of such secondary structure is in preparation.

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