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MARCO GRANDIS

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# ON DISTRIBUTIVE HOMOLOGICAL ALGEBRA, II. THEORIES AND MODELS

*by* Marco GRANDIS

**Résumé.** Cet article est le second, d'une série de trois, consacrée à l'étude des "théories homologiques distributives" (comme le complexe filtré ou le double complexe) et de leurs modèles canoniques. Ici nous prouvons que chaque "RE-théorie" a un modèle canonique ; les théories distributives et idempotentes sont étudiées plus particulièrement, et on donne des critères pour reconnaître leurs modèles canoniques.

## 0. Introduction.

**0.1.** The general frame and general purposes of this series are exposed in the Introduction of Part I [1]).

In Part I exact categories (in the sense of Puppe-Mitchell [15, 14]), or more precisely their categories of relations [2, 3, 1, 4] are generalized by *RE-categories,* i.e., involutive ordered categories satisfying certain conditions. RE-categories form a *strictly* complete 2-category **RE**, where *strict* universal problems can be solved ; a fact which simplifies our approach.

**0.2.** Here we introduce RE-theories, that is theories with values in RE-categories, and prove the existence of their canonical models via the completeness of **RE**.

This theorem being not constructive, the crucial (and generally heavy) task in studying a given theory will be to "devine" the good model and then to prove that it is canonical (Part III).

In order to simplify this task, we give here various criteria concerning transfer, distributive, idempotent theories. We also describe "universal" distributive or idempotent RE-categories.

The importance of the distributive case, both for having "concrete" representations and "good" algebraic properties, was already emphasized in I.0.1.

**0.3.** More particularly, the plan of Part II is the following.

([1]) Part I appeared in these "Cahiers", Vol. XXV-3 [9]. The references I.m or I.m.n or I.m.n.p concern Part I, and precisely § m or Section m.n or item (p) in Section m.n.

In § 1 graph morphisms $\Delta \to \underline{A}$ towards RE-categories are considered. Then, in § 2, a *RE-theory* T on the small graph $\Delta$ is semantically defined by assigning, for each RE-category $\underline{A}$, a set $T(\underline{A})$ of graph morphisms $\Delta \to \underline{A}$ (the models of T in $\underline{A}$), so that some coherence conditions (RT.1-3) are satisfied ; there is always a strict *canonical model* $t_0 : \Delta \to \underline{A}_0$, determined up to isomorphism and natural ; $\underline{A}_0$ is the *classifying RE-category* of T.

In § 3 various properties of RE-theories are considered : e.g., T is *transfer, distributive, boolean, idempotent, finite* whenever $\underline{A}_0$ is so (I.7 and 1.8) ; every idempotent theory is transfer and distributive, and every idempotent theory on a finite graph is finite. Moreover the RE-category *Mlr* of *modular lattices and modular relations* [6] is shown to be universal for transfer theories (3.7).

§ 4 introduces the *canonical transfer model* (c.t.m) $t_1 : \Delta \to Mlr$ of the theory T, and proves its existence, uniqueness and relations with the canonical model. The c.t.m. is generally easier to recognize than the latter and has some interest in itself as it "reveals" monics, epis, isomorphisms (4.8) ; chiefly, it determines the canonical model for transfer theories (4.5).

Then § 5 yields criteria for recognizing idempotent theories and their canonical models, derived from § 4 and the "running knot theorem" [7] ; the latter essentially says that a distributive theory on a "plane" order graph is idempotent, hence transfer.

In order to prepare the ground for the models of Part III, we recall in § 6 description and properties of the "standard" distributive RE-category Rel *(I)*, where *I* is the exact category of sets and partial bijections ; owing to a concreteness theorem for distributive exact categories [8] every distributive RE-theory has a classifying RE-category embedded in Rel *(I)* , which is universal for distributive theories. The pre-idempotent exact sub-category $I_0$ of sets and common parts plays a similar role for idempotent theories. The related categories $\mathcal{J}$ and $\mathcal{J}_0$ are also used, with some advantages.

Last, § 7 considers *EX-theories* T, i.e., theories with values in (categories of relations of) exact categories. The associated RE-theory $T^r$ provides an i-canonical model for T, determined up to equivalence (as, for example, it happens for theories with values in toposes [10, 13]). The i-classifying exact category of T is far richer in objects than the classifying RE-category of $T^r$ ; an encumbrance from a formal point of view, yet an advantage in applications : e.g., for the filtered complex or the double complex, the exact representation functor produces directly the terms of the associated spectral sequences.

**0.4.** We follow here the same conventions as in Part I. We only recall that a universe *U* is chosen once for all, and every RE-category (resp. exact category) is assumed to have objects and morphisms belonging to

$U$, and to be Prj-small (resp. Sub-small, i.e., well-powered) ; instead, we do not suppose it to have small Hom-sets.

$\Delta$ is always a small graph and $I(\Delta)$ the free involutive category generated by $\Delta$ ; for every morphism (of graphs) $t : \Delta \to \underline{A}$ with values in a RE-category, we write $\bar{t} : I(\Delta) \to \underline{A}$ the (unique) involution-preserving functor which extends $t$. Finally the notion of a *RE-transformation* $\tau : t \to t' : \Delta \dashrightarrow \underline{A}$ is obvious (see I.5.3, I.2.3).

## 1. Graphs and RE-graphs.

In order to study RE-theories, we examine here the morphisms of graphs with values in RE-categories, to which we extend the factorization of RE-functors (I.5.9-10). We also introduce RE-graphs, i.e., graphs with RE-conditions.

$\Delta$ is always a small graph, and $\underline{A}$, $\underline{B}$, $\underline{C}$ are RE-categories.

**1.1.** We call *q-morphism* a morphism of graphs $t_1 : \Delta \to \underline{C}$ such that :

a) $t_1$ is bijective on the objects,

b) $\underline{C}$ is RE-spanned by $t_1$ (i.e., by its subgraph $t_1(\Delta)$, according to I.5.8).

It is easy to see that any morphism $t : \Delta \to \underline{A}$ has an essentially unique *RE-factorization*

(1) $$\Delta \xrightarrow{t_1} \mathrm{RE}(t) \xrightarrow{F_2} \underline{A}, \qquad t = F_2 t_1,$$

where $t_1$ is a q-morphism and $F_2$ is a faithful RE-functor.

Indeed, first factorize $t$ as

(2) $$\Delta \xrightarrow{t'} \underline{A}' \xrightarrow{F'} \underline{A}, \qquad t = F't'$$

where $t'$ is bijective on the objects and $F'$ is faithful and full, by taking

$$\mathrm{Ob}\ \underline{A}' = \mathrm{Ob}\ \Delta \quad \text{and} \quad \underline{A}'(A_1, A_2) = \underline{A}(tA_1, tA_2).$$

Then let $\mathrm{RE}(t)$ be the RE-subcategory of $\underline{A}'$ spanned by $t'(\Delta)$, and $t_1$, $F_2$ be restrictions of $t'$, $F'$.

**1.2.** Extending I.5.11, the morphism $t : \Delta \to \underline{A}$ will be said *Rst-spanning* (or equivalently, *Prj-spanning*) if in the RE-factorization $t = F_2 t_1$ the faithful RE-functor $F_2$ is Rst-full (or equivalently, Prj-full).

Obviously, the composition of such a $t$ with a Rst-full RE-functor $\underline{A} \to \underline{B}$ is still Rst-spanning.

**1.3.** We also say that the morphism $t : \Delta \to \underline{A}$ is *transfer* (resp. *distributive, boolean, idempotent*) whenever the RE-category RE$(t)$ is so (Part I : 7.3, 7.4, 7.5, 8.1). •

**1.4. Lemma.** The following conditions on a morphism $t : \Delta \to \underline{A}$ are equivalent :

a) $t$ is distributive (resp. boolean, idempotent),

b) $t$ factorizes through a distributive (resp. boolean, idempotent) RE-category,

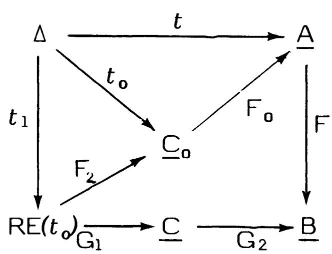
c) for every RE-functor $F : \underline{A} \to \underline{B}$, $Ft$ is distributive (resp. boolean, idempotent),

d) there exists a faithful RE-functor $F : \underline{A} \to \underline{B}$ such that $Ft$ is distributive (resp. boolean, idempotent).

Moreover in the condition d, for the distributive and boolean cases *only*, the functor F can be just Rst-faithful (e.g., $F = \mathrm{Rst}_{\underline{A}}$ is suitable).

**Proof.** a $\Rightarrow$ b : obvious. b $\Rightarrow$ c : let $t = F_0 t_0$ be a factorization of $t$ through the distributive (resp. boolean, idempotent) RE-category $\underline{C}_0$,

(1)



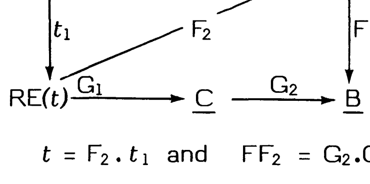

and consider the RE-factorizations

$$t_0 = F_2 . t_1 , \qquad F F_0 F_2 = G_2 . G_1 ;$$

then $F t = G_2.(G_1 t_1)$ is a RE-factorization of $Ft$ and $\underline{C} = \mathrm{RE}(Ft)$ is distributive by I.7.6 (resp. boolean by I.7.6 again ; idempotent by I.8.3) applied *first*, to the faithful RE-functor $F_2$ and *successively*, to the RE-quotient $G_1$. c $\Rightarrow$d : take $F = 1_{\underline{A}}$ . d $\Rightarrow$a : since F is faithful, RE$(t)$ = RE$(Ft)$.

Last, suppose that $F : \underline{A} \to \underline{B}$ is just *Rst-faithful*, and $Ft$ is distributive (resp. boolean) ; let :

(2)




$$t = F_2 . t_1 \ \text{ and } \ F F_2 = G_2 . G_1$$

be RE-factorizations : then $Ft = G_2.(G_1 t_1)$ is a RE-factorization and $\underline{C}$ is distributive (resp. boolean). Since $FF_2$ is Rst-faithful, so is $G_1$ ; from I.7.6 it follows that RE$(t)$ is distributive (resp. boolean).

**1.5.** It will be useful to consider a RE-structure on the graph $\Delta$ , much in the same way as a set S can be provided with group relations : these select those mappings from S to groups which preserve them, and define a universal problem whose solution is the group generated by S, with the given relations.

**1.6.** Similarly, a *RE-graph* will be a graph $\Delta$ provided with *RE-conditions,* that is a set of formulas of the following kinds :

a) $a = b,$

b) $a \leqq b,$

c) $a \in X ,$

where $a$ and $b$ are (parallel) morphisms of the free involutive category $I(\Delta)$ generated by $\Delta$, while X is any of the following *symbols*

(1) $\quad \mathrm{Prj}\ \Delta , \quad \mathrm{Prp}\ \Delta , \quad \mathrm{Nul}\ \Delta$

and the condition $a \in \mathrm{Prj}\ \Delta$ requires that $a$ is an endomorphism.

We remark that the symbols Prj $\Delta$ and Prp $\Delta$ could be easily spared (e.g., by writing

$$a = a\tilde{a} \text{ instead of } a \in \mathrm{Prj}\ \Delta ,$$

and

$$1 \leqq \tilde{a}a \ , \ a\tilde{a} \leqq 1 \quad \text{instead of} \quad a \in \mathrm{Prp}\ \Delta) \ ;$$

nevertheless we keep them, for the sake of evidence [2]).

**1.7.** A *RE-morphism* $t : \Delta \to \underline{A}$ from a RE-graph to a RE-category is a graph morphism $t$ whose involution-preserving extension $\bar{t} : I(\Delta) \to \underline{A}$ preserves the RE-conditions of $\Delta$ (in the obvious sense).

The RE-factorization $t = F_2 t_1$ of graph morphisms (1.1) extends trivially to RE-morphisms : the q-morphism $t_1$ is a RE-morphism too.

**1.8.** Last, we remark that, if $t: \Delta \to \underline{A}$ is a *graph morphism* from a RE-graph to a RE-category, and $F : \underline{A} \to \underline{B}$ a *faithful* RE-functor, then $t$ is a RE-morphism iff $F.t$ is so (I.5.5).

([2]) In the contrary, we avoid to use formulas with symbols $\underline{n}$, $\underline{d}$ for the sake of simplicity : otherwise the definition of RE-morphism (1.7) would require, in the place of $I(\Delta)$, the free RE-category on the underlying graph of $\Delta$ (whose existence will result from § 2).

## 2. RE-theories and canonical models.

RE-theories are semantically defined by assigning their models, so that some obvious coherence conditions be satisfied ; owing to the completeness of the 2-category **RE** (I.9) they have strict canonical models, determined up to isomorphism and natural.

$\Delta$ is always a small graph.

**2.1. Definition.** A *RE-theory* T is given by a small graph $\Delta$ and, for every RE-category $\underline{A}$, a set $T(\underline{A})$ of morphisms $t : \Delta \to \underline{A}$ satisfying the following *coherence conditions :*

(RT.1) if $F : \underline{A} \to \underline{B}$ is a RE-functor and $t \in T(\underline{A})$, then $Ft \in T(\underline{B})$,
(RT.2) if $F : \underline{A} \to \underline{B}$ is a faithful RE-functor, $t : \Delta \to \underline{A}$ is a morphism and $Ft \in T(\underline{B})$, then $t \in T(\underline{A})$,
(RT.3) if $t_i \in T(\underline{A}_i)$ for $i \in I$ (where I is a small set), then the morphism $(t_i) : \Delta \to \Pi \underline{A}_i$ belongs to $T(\Pi \underline{A}_i)$.

We also say that T is a RE-theory *on* $\Delta$, and that any $t \in T(\underline{A})$ is a *model of* T *in* $\underline{A}$. Notice that the constant morphism $\Delta \to \underline{1}$ is a model of T, by (RT.3) applied to the empty family.

**2.2.** A *canonical* (or *generic, universal*) model for T will be a model $t_0$: $\Delta \to \underline{A}_0$ such that for every model $t : \Delta \to \underline{A}$ there exists exactly one RE-functor F verifying :

$$(1) \qquad F : \underline{A}_0 \to \underline{A}, \quad t = Ft_0 .$$

Then $\underline{A}_0$ will be called a *classifying* RE-category for T ; the functor F in (1) will be the *representation* RE-functor of $t$.

The uniqueness of the canonical model (up to a uniquely determined isomorphism of RE-categories) is obvious ; its existence and naturality will now be proved.

**2.3. Main Theorem.** Let T be a RE-theory on $\Delta$. There exists a canonical model $t_0 : \Delta \to \underline{A}_0$ for T. The canonical model is natural in the following sense : for every RE-transformation $\tau : t_1 \to t_2 : \Delta \to \underline{A}$ between models of T (with values in the same RE-category $\underline{A}$) there exists a unique RE-transformation $\varphi : F_1 \to F_2 : \underline{A}_0 \to \underline{A}$ such that :

$$(1) \qquad \varphi t_0 = \tau \quad (\text{hence } F_i\, t_0 = t_i \quad \text{for } i = 1, 2).$$

Moreover $t_0$ is a q-morphism (1.1), $\underline{A}_0$ is small and :

$$(2) \qquad \text{card}(\text{Mor}\ \underline{A}_0) \leq \max(\text{card}(\text{Ob}\ \Delta), \text{card}(\text{Mor}\ \Delta), \aleph_0).$$

For every T-model $t$, $RE(t)$ is a quotient of $\underline{A}_0$.

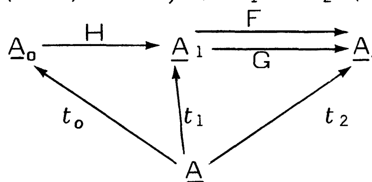
**Proof.** A) Consider the category $\hat{T}$ whose objects are all the models $t : \Delta \to \underline{A}$ of T, while a morphism $F : t_1 \hat{\to} t_2$ (where $t_i : \Delta \to \underline{A}_i$ ) is a RE-functor

$$F : \underline{A}_1 \to \underline{A}_2 \quad \text{such that} \quad F.t_1 = t_2.$$

Thus, a canonical model for T is just an initial object for $\hat{T}$ ; the existence of the latter will be proved by the Initial Object Theorem [12 ] [3]), in B and C.

B) First, $\hat{T}$ is small-complete. It has small products by axiom (RT.3). It has equalizers by (RT.2) : if F, $G : t_1 \to t_2$ ($t_i : \Delta \to \underline{A}_i$) are in $\hat{T}$ :

(3)




(4) $$F t_1 = t_2 = G t_1 ,$$

let $H : \underline{A}_0 \to \underline{A}_1$ be the equalizer of F and G in **RE** (complete) ; there exists exactly one RE-morphism

$$t_0 : \Delta \to \underline{A}_0 \quad \text{such that} \quad H t_0 = t_1 ;$$

since H is a faithful RE-functor and $t$ is a model of T, so is $t_0$. Now it is easy to check that $H : t_0 \hat{\to} t_1$ is an equalizer of F and G in $\hat{T}$.

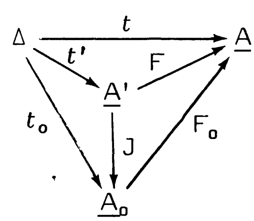
C) Second, $\hat{T}$ has a small *solution set* [12] : let

$$\alpha = \max(\text{card}(\text{Ob } \Delta), \text{card}(\text{Mor } \Delta), \aleph_0)$$

and assume that $\alpha$ and all the smaller cardinals belong to $U$ ($\Delta$ is small). Say S the small set of models $t_0 : \Delta \to \underline{A}_0$ where Ob $\underline{A}_0$ and Mor $\underline{A}_0$ are cardinals $\leq \alpha$. Their product $t_1 : \Delta \to \underline{A}$ has a small set of endomorphisms $\hat{T}(t_1, t_1)$, because $\underline{A}$ is small [3]).

Now, if $t : \Delta \to \underline{A}$ is any model of T, say $\underline{A}'$ the RE-subcategory of $\underline{A}$ spanned by the subgraph $t(\Delta)$. As card(Mor $\underline{A}'$) $\leq \alpha$ (I.5.8), there exists a RE-isomorphism $J : \underline{A}' \to \underline{A}_0$ where Ob $\underline{A}_0$ and Mor $\underline{A}_0$ are cardinals $\leq \alpha$. Thus we have a commutative diagram :

(3)

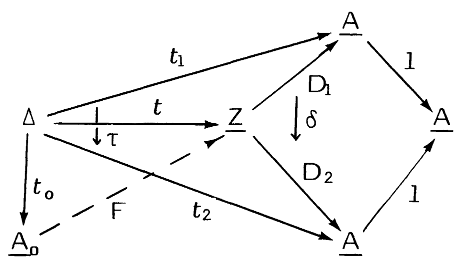



([3]) We actually use an easy extension of the Initial Object Theorem, where the category ($\hat{T}$ in our case) is not supposed to have small Hom-sets, but in the Solution Set condition the *product* of the given set of objects ($t_1$ in our case) is assumed to have a small set of endomorphisms. The proof is the same.

where $F_0$ is a faithful RE-functor, hence $t_0$ is a model of T and $F_0$ : $t_0 \hat{\to} t$ is a morphism of T, with $t_0 \in$ S.

D) Now, let $t_0 : \Delta \to \underline{A}_0$ be a canonical model. If $\tau : t_1 \to t_2 : \Delta \to \underline{A}$ is a RE-transformation of T-models :

(4)



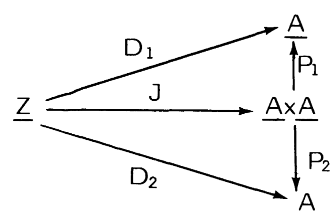

consider the comma square

$$\delta : 1_{\underline{A}} D_1 \to 1_{\underline{A}} D_2 : \underline{Z} \to \underline{A}$$

(I.9.5) : there exists exactly one RE-morphism $t : \Delta \to \underline{Z}$ such that

(5) $$\delta\, t = \tau \quad (D_1 t = t_1 \ , \ \ D_2 t = t_2 ).$$

Moreover, $t$ is a model of T : actually if J is the (faithful) RE-functor characterized by the commutative diagram :




(7) $$J(A_1, A_2, u : A_1 \to A_2) = (A_1, A_2), \qquad J(a_1, a_2) = (a_1, a_2)$$

then $P_i(Jt) = D_i t = t_i$ is a model ($i$ = 1, 2), therefore so is $J.t$ (RT.3) and so is $t$ (RT.2).

It follows that $t = F\ t_0$, for a unique RE-functor $F : \underline{A}_0 \to \underline{Z}$. By letting :

(8) $$F_i = D_i . F : \underline{A}_0 \to \underline{A} \ , \quad \varphi = \delta F : F_1 \to F_2 : \underline{A}_0 \to \underline{A}$$

the condition (1) is clearly satisfied.

E) As to the uniqueness of $\varphi$ , suppose that

$$\psi : F_1 \to F_2 : \underline{A}_0 \to \underline{A}$$

is also a RE-transformation verifying $\psi\, t_0 = \tau$. By the universal property of the comma-square in (4) there exists a unique RE-functor $F' : \underline{A}_0 \to \underline{Z}$ such that

(9) $$\delta F' = \psi \quad (D_i F' = F_i).$$

Write $t' = F' t_o$ ; then

$$\delta t' = \delta F' t_o = \psi t_o = \tau$$

hence $t' = t$ (by the uniqueness of $t$ satisfying (5)) that is $F' t_o = F t_o$ ; thus $F' = F$ ( $t_o$ is canonical) and

$$\psi = \delta F' = \delta F = \varphi .$$

F) Now, let $t_o = F t_1$ be the RE-factorization (1.1) of the canonical model $t_o$:

(10) $$\Delta \xrightarrow{t_1} \underline{A}_1 \underset{G}{\overset{F}{\rightleftarrows}} \underline{A}_0$$

Since F is faithful, $t_1$ is a model of T and there exists a unique RE-functor G such that $G t_o = t_1$ ; as

$$FG t_o = F t_1 = t_o \text{ and } GF t_1 = G t_o = t_1 ,$$

it follows that $FG = 1$ ($t_o$ is canonical) and $GF = 1$ ($t_1$ is a q-morphism). Thus also $t_o$ is a q-morphism, and $\underline{A}_0$ is RE-spanned by $t_o$: by I.5.8,

$$\mathrm{card}(\mathrm{Mor}\ \underline{A}_0) \leqq \alpha .$$

G) Last, let $t = F t_o$ be a model : the RE-factorization $F = F_2 . F_1$ of F yields a RE-factorization $t = F_2.(F_1 t_o)$, showing that $RE(t) = \mathrm{Cod}\ F_1$ is a RE-quotient of $\underline{A}_0 = \mathrm{Dom}\ F_1$ .

**2.4.** We shall need also the following property of naturality for the canonical model $t_o : \Delta \to \underline{A}_0$ of T :

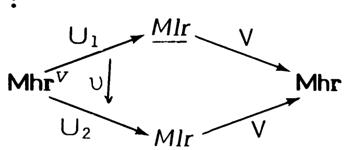
a) If $t_1, t_2 : \Delta \to \underline{Mlr}$ are T-models and

$$\vartheta : t_1 \to t_2 : \Delta \to \mathbf{Mhr}$$

(I.7.2) is a *horizontal* transformation of vertical graph morphisms, there is a unique horizontal transformation of vertical RE-functors

$$\varphi : F_1 \to F_2 : \underline{A}_0 \to \mathbf{Mhr} \qquad \text{such that} \qquad \vartheta = \varphi\ t_o \ (\text{and} \quad t_i = F_i t_o ).$$

The proof is similar to 2.3 D-E. First, observe that the vertical underlying category $\mathbf{Mhr}^V$ (squares with vertical composition) has an obvious RE-structure, and consider the horizontal comma square of vertical RE-functors :

(1)


where V is the (vertical) inclusion, $U_1$ and $U_2$ are the "horizontal domain and codomain" functors and $\upsilon$ is the horizontal transformation which turns the object $h: X_1 \to X_2$ into the horizontal morphism

$$h : U_1(h) \longrightarrow U_2(h)\ .$$

Since the functor

(2) $$J : \mathbf{Mhr}^V \longrightarrow \underline{Mlr} \times \underline{Mlr}$$

(3) $$(X_1 \xrightarrow{h} X_2) \mapsto (X_1, X_2), \qquad \begin{array}{ccc} X_1 & \xrightarrow{h} & X_2 \\ {\scriptstyle u_1}\downarrow & & \downarrow{\scriptstyle u_2} \\ Y_1 & \xrightarrow{k} & Y_2 \end{array} \mapsto (u_1, u_2)$$

is faithful, the proceeding used in 2.3 D-E can be adapted.

Last, we notice that the naturality properties in 2.3 and a have a common extension, which can be proved as above, and which we shall not use.

This extension concerns horizontal transformations of vertical models $\vartheta : t_1 \to t_2 : \Delta \to \mathbf{D}$, where $\mathbf{D}$ is a double category whose vertical underlying category $\mathbf{D}^V$ is provided with RE-structure, and whose cells are determined by "their arrows" (i.e., by their horizontal and vertical domains and codomains). For 2.3, just take $\mathbf{D}$ to be the double category of RO-squares of $\underline{A}$ (vertically RE).

**2.5.** The following terminology will be useful in stating some results of "universality" (2.6, 3.7, 6.10). Let T, T' be RE-theories on the same graph $\Delta$ ; we say that the *RE-statement* T $\Rightarrow$ T' *holds for the RE-category* $\underline{A}$ when $T(\underline{A}) \subset T'(\underline{A})$ ; we say that *it holds universally* whenever it holds for every RE-category $\underline{A}$.

Then, trivially :

**2.6. Lemma.** The RE-statement T $\Rightarrow$ T' holds universally iff the canonical model $t_0 : \Delta \to \underline{A}_0$ of T is a model of T'.

**2.7.** A small RE-graph $\Delta$ determines a theory $T_\Delta$ on its underlying graph : the models are *all* the RE-morphisms $\Delta \to \underline{A}$ with values in RE-categories.

The canonical model of $T_\Delta$ , existing and unique up to isomorphism, will be written $t_\Delta : \Delta \to RE(\Delta)$ and called the *free RE-category on* $\Delta$.

**2.8.** The model $t_\Delta$ is characterized by the following universal property : for each RE-morphism $t : \Delta \to \underline{A}$ there exists exactly one RE-functor F : $RE(\Delta) \to \underline{A}$ such that $t = F.t_\Delta$ ; moreover, the RE-factorization of the representation functor F :

(1) $$RE(\Delta) \xrightarrow{F_1} \underline{C} \xrightarrow{F_2} \underline{A}$$

has "central category" $\underline{C} \simeq \mathrm{RE}(t)$, since

$$t = F t_\Delta = F_2 .(F_1 t_\Delta)$$

is a RE-factorization of $t$ (2.3). It follows that RE$(t)$ is a RE-quotient of RE$(\Delta)$.

We also notice that, by 2.3, RE$(\Delta)$ is small and :

(2) $\qquad \mathrm{card}(\mathrm{Mor}\ \mathrm{RE}(\Delta)) \leqq \max(\mathrm{card}\ \mathrm{Ob}\Delta,\ \mathrm{card}\ \mathrm{Mor}\ \Delta,\ \aleph_0).$

**2.9.** Obviously, any small RE-category $\underline{A}_0$ classifies some RE-theory, for example $T_{\underline{A}_0}$ (where $\underline{A}_0$ is to be considered as a RE-graph, with conditions

$$a = 1_A, \quad a = cb, \quad a = \tilde{b}, \quad a \leqq b, \quad a \in \mathrm{Nul}\ \underline{A}_0$$

whenever this happens in the RE-category $\underline{A}_0$), whose canonical model is $1 : \underline{A}_0 \to \underline{A}_0$.

More generally a RE-morphism $t : \Delta \to \underline{A}_0$ is a canonical model for some RE-theory T on $\Delta$ iff it is a q-morphism : for the "if-part" of the proof take as models of T all the RE-morphisms which have a factorization $t = F t_0$, where F is any RE-functor (uniquely determined by $t$, because $t_0$ is a q-morphism) ; the "only-if-part" follows from 2.3.

## 3. Properties of RE-theories.

T is always a RE-theory on $\Delta$ with canonical model $t_0 : \Delta \to \underline{A}_0$.

**3.1.** Two RE-theories are said to be *equivalent* if their classifying RE-categories are RE-isomorphic ; they can be based on fairly different graphs.

**3.2.** A *RE-Theory T* will be a class of equivalent RE-theories, and the theory T $\in$ *T* will also be called a *presentation* of *T* (on the graph $\Delta$). By 2.3, 2.9, the RE-Theories are in biunivocal correspondence with the classes of isomorphic small RE-categories.

**3.3.** We are going to consider some properties of RE-theories which are equivalence-invariant, hence properties of RE-Theories. On the contrary the property "T is *proper*" (i.e., for every T-model $t : \Delta \to \underline{A}$, $t(\Delta) \subset \mathrm{Prp}\ \underline{A}$) is not so : for example the Theory of the (bi)filtered object can be presented via restrictions (as in Part III; non-proper theory), or via subobjects (proper theory).

**3.4.** We say that T is *finite* (resp. *Hom-finite, Rst-finite*) if its classifying category $\underline{A}_0$ is so.

Obviously T is finite iff it is Hom-finite and Ob$\Delta$ is finite.

**3.5.** We say that T is *transfer* (resp. *distributive, boolean, idempotent*) if its classifying category $\underline{A}_0$ (or equivalently its canonical model $t_0$) is so. Any idempotent theory is distributive and transfer (I.8.2).

By 1.4, the theory T is distributive (resp. boolean, idempotent) iff every model of T is so, iff every model of T factorizes through a distributive (resp. boolean, idempotent) RE-category. Instead, transfer (non-idempotent) theories can have some non-transfer model : see Part III § 7.

**3.6.** Every transfer theory which is Rst-finite is also Hom-finite, by I.7.3.

Every idempotent theory on a finite graph is finite (I.8.6). More generally, still by I.8.6, the idempotent theory T on the graph $\Delta$ is Hom-finite iff, for every model $t : \Delta \to \underline{A}$ and for every object A of $\Delta$, the set $X^o_A \subset \mathrm{Rst}_{\underline{A}}(tA)$ defined as in I.8.6.1 (by replacing $\Delta$ with $t(\Delta) \subset \underline{A}$) is finite.

**3.7. Theorem.** *(Universality for transfer theories)* If T and T' are RE-theories on $\Delta$ and T is transfer, the RE-statement T $\Rightarrow$ T' holds universally iff it holds for *Mlr* ([6], § 3). Shortly : the RE-category $\underline{Mlr}$ of modular lattices and modular relations *is universal for transfer theories.*

**Proof.** If T(*Mlr*) $\subset$ T'(*Mlr*), the T-model

$$t_1 = \mathrm{Rst}_{\underline{A}_0} . t_0 : \Delta \to Mlr$$

is a T'-model ; as $\mathrm{Rst}_{\underline{A}_0}$ is faithful, also $t_0$ is a T'-model, and the conclusion follows from 2.6.

**3.8.** We say that the theory T is *connected* (resp. *non* empty) if its classifying RE-category $\underline{A}_0$ is so ; this happens iff $\Delta$ itself is so.

Of course, it is always possible to consider the connected components of $\Delta$ , that is to restrict our attention to connected non empty theories.

## 4. Canonical transfer models.

We introduce here the notion of canonical transfer models (c.t.m.), weaker than that of canonical model but equivalent to the latter for transfer theories (4.5), and a fortiori for idempotent theories (§ 5). For a non-transfer theory the c.t.m. does not determine the canonical model, but can be of some use in itself (4.8).

T is always a RE-theory on a small graph $\Delta$.

**4.1. Definition.** A *canonical transfer model* (c.t.m.) for T will be a RE-morphism $t_1 : \Delta \to Mlr$ such that :

a) $t_1$ is a model of T,

b) for every model $t : \Delta \to Mlr$ there is exactly one horizontal transformation of vertical morphisms $\vartheta : t_1 \to t : \Delta \to \mathbf{Mhr}$.

Since the transfer functor of $Mlr$ is isomorphic to the identity (I.7.1.6), the condition b is equivalent to :

c) for every model $t : \Delta \to \underline{A}$ there is exactly one horizontal transformation $\vartheta : t_1 \to \mathrm{Rst}_{\underline{A}} . t : \Delta \to \mathbf{Mhr}$.

We now prove the existence and uniqueness of the c.t.m., together with its relations with the canonical model.

**4.2. Theorem.** If $t_o : \Delta \to \underline{A}_o$ is a canonical model of T, then

$$t_1 = \mathrm{Rst}_{\underline{A}_o} . t_o : \Delta \to Mlr$$

is a c.t.m. ; the c.t.m. of T is determined up to a unique isomorphism of RE-morphisms.

**Proof.** The RE-morphism $t_1$ is a model of T, by (RT.1). Moreover, if $t : \Delta \to Mlr$ is a model and $F : \underline{A}_o \to Mlr$ the RE-functor such that $t = Ft_o$, call

$$\rho : \mathrm{Rst}_{\underline{A}_o} \to F : \underline{A}_o \to \mathbf{Mhr}$$

the unique horizontal transformation (I.7.2) ; by 2.4 there is exactly one horizontal transformation $\vartheta : \mathrm{Rst}_{\underline{A}_o} . t_o \to Ft_o$. Last, if $t_1$ and $t_2$ are both c.t.m., there are unique horizontal transformations

$$\vartheta_1 : t_1 \to t_2 \quad \text{and} \quad \vartheta_2 : t_2 \to t_1 , \quad \text{and} \quad \vartheta_2\vartheta_1 = 1,\ \vartheta_1\vartheta_2 = 1 ;$$

thus

$$\vartheta_1 : t_1 \to t_2 : \Delta \to \mathbf{Mhr}$$

is a horizontal isomorphism, which is equivalent to saying that it is an isomorphism of RE-graphs $t_1 \to t_2 : \Delta \to Mlr$, since the isomorphisms of $Mlh$ and $Mlr$ coincide.

**4.3.** Thus, by 4.2 and 1.4, the theory T is distributive or boolean iff its c.t.m. is so.

**4.4. Proposition.** A model $t_1 : \Delta \to Mlr$ is a c.t.m. iff :

b') for every model $t : \Delta \to Mlr$ there is some horizontal transformation $\vartheta : t_1 \to t : \Delta \to \mathbf{Mhr}$.

b'') $t_1$ is Rst-spanning (1.2).

**Proof.** The c.t.m. $\mathrm{Rst}_{\underline{A}_0}.t_o$ satisfies b'' because $t_o$ is a q-morphism (2.3) and $\mathrm{Rst}_{\underline{A}_0}$ is Rst-spanning.

Conversely, suppose that the model $t_1 : \Delta \to Mlr$ satisfies b' and b'', and let $t_1 = Ft_o$, with $F : \underline{A}_0 \to Mlr$ RE-functor. If $F = F_2F_1$ is a RE-factorization, so is $t_1 = F_2(F_1 t_o)$, hence $F_2$ is Rst-full (by b'') and so is $F$ ; by (I.7.1-2) this proves that the unique horizontal transformation

$$\rho = (\iota F)\mathrm{Rst}_F : \mathrm{Rst}_{\underline{A}_0} \to F$$

is pointwise surjective. Now, if $\vartheta_i : t_1 \to t$ ($i = 1, 2$) are two horizontal transformations of model,

$$\bar{\vartheta}_i = \vartheta_i.(\rho\ t_o) : \mathrm{Rst}_{\underline{A}_0}.t_o \to t$$

are so and coincide by 4.2 :

$$\vartheta_1.(\rho\ t_o) = \vartheta_2.(\rho\ t_o).$$

Since $(\rho\ t_o)A = \rho(t_oA)$ is a surjective lattice-homomorphism for every $A \in \mathrm{Ob}\ \Delta$, it follows that $\vartheta_1 = \vartheta_2$ .

**4.5. Theorem.** Let $t_1 : \Delta \to Mlr$ be a c.t.m. of T and

$$(1) \qquad \Delta \xrightarrow{t_2} \underline{A}_2 \xrightarrow{G} Mlr$$

the RE-factorization of $t_1$. Then $t_2$ is a canonical model for T iff T is a transfer theory (3.5).

**Proof.** Let $t_o : \Delta \to \underline{A}_0$ be a canonical model for T, and

$$t_1 = Ft_o \qquad \text{with} \qquad F : \underline{A}_0 \to Mlr \ .$$

Now $t_1$ and $t_3 = \mathrm{Rst}_{\underline{A}_0}.t_o$ are both c.t.m. (4.2), hence there exists an isomorphism $\tau : t_1 \to t_3$ and an isomorphism

$$\varphi : F \to \mathrm{Rst}_{\underline{A}_0} : \underline{A}_0 \to Mlr \qquad \text{such that} \qquad \varphi\, t_o = \tau.$$

Therefore, if T is a transfer theory, $\mathrm{Rst}_{\underline{A}_0}$ is faithful and F too ; as $t_o$ is a q-morphism (2.3), this proves that $t_1 = Ft_o$ is a RE-factorization of $t_1$, like (1) : therefore $t_2 \simeq t_o$ is also a canonical model for T.

Conversely, if $t_2$ is a canonical model, take $t_o = t_2$ in the above argument : hence $\mathrm{Rst}_{\underline{A}_0} \simeq F = G$ is faithful (since (1) is a RE-factorization) and T is transfer.

**4.6.** We give now a criterion for recognizing c.t.m., which will be used in Part III. The hypotheses and constructions are so linked that we do not give it a "theorem-like" formulation.

Let $t_o : \Delta \to \underline{A}_0$ be a model,

$$t_1 = \mathrm{Rst}_{\underline{A}_0}.t_o : \Delta \longrightarrow Mlr$$

and let be given for every $i \in \mathrm{Ob}\,\Delta$ a subset $X_i^o$ of the modular lattice

$$X_i = t_1(i) = \mathrm{Rst}(t_o(i)),$$

provided with the induced order, so that

a) *$X_i$ is the free modular 0, 1-lattice generated by the ordered set $X_i^o$ ; moreover $X_i$ is distributive.*

Let us also have, for every $i$ and every restriction $e \in X_i^o$, a morphism $a_e$ in the free involutive category $I(\Delta)$ (0.4) and a "symbol"

$$\varepsilon_e \in \{1, \omega\}$$

so that :

b) *if* $e \in X_i^o$

(1) $$e = (\bar{t}_o a_e)_R(\varepsilon_e),$$

c) *if $e \leq f$ in $X_i$, for every model $t : \Delta \to \underline{A}$,*

$$(\bar{t} a_e)_R(\varepsilon_e) \leq (\bar{t} a_f)_R(\varepsilon_f).$$

Then, for every $i \in \mathrm{Ob}\,\Delta$ and every model $t : \Delta \to \underline{A}$, there is a unique homomorphism of 0, 1-lattices :

(3) $$\vartheta_i^t : X_i \to \mathrm{Rst}_{\underline{A}}(t(i))$$

such that, for every $e \in X_i^o$

(4) $$\vartheta_i^t(e) = (\bar{t} a_e)_R(\varepsilon_e).$$

Now, if

d) *for every $d \in \Delta(i, j)$, $e \in X_i^o$ , $f \in X_j^o$ and every model $t : \Delta \to \underline{A}$,*

(5) $$(td)_R(\vartheta_i^t e) = \vartheta_j^t((\bar{t}_o(d a_e))_R(\varepsilon_e)),$$

(6) $$(td)^R(\vartheta_j^t f) = \vartheta_i^t((\bar{t}_o(\tilde{d} a_f))_R(\varepsilon_f)),$$

$t_1$ is a c.t.m. for T, which is distributive.

Actually the family

$$\vartheta^t = (\vartheta_i^t) : t_1 \longrightarrow \mathrm{Rst}_{\underline{A}}.t : \Delta \longrightarrow \mathbf{Mhr}$$

is a horizontal transformation : if $d \in \Delta(i, j)$, in the square

(7) $$\begin{array}{ccc} X_i & \xrightarrow{\ \vartheta_i^t\ } & \mathrm{Rst}_{\underline{A}}(t_i) \\ \downarrow{\scriptstyle (t_o d)_R} & & \downarrow{\scriptstyle (td)_R} \\ X_j & \xrightarrow{\ \vartheta_j^t\ } & \mathrm{Rst}_{\underline{A}}(t_j) \end{array}$$

the horizontal arrows are lattice homomorphisms, and so are $(t_o d)_R$ and the restriction of $(td)_R$ to the lattice $\vartheta_i^t(X_i)$ because their domains are distributive ([6], § 6.3); therefore the commutativity of (7) can be checked on the generators $e \in X_i^o$ of $X_i$, which amounts to condition (5); analogously the commutativity of the "contravariant square" (with $(t_o d)^R$, $(td)^R$ for upward vertical arrows) comes from the distributivity of $X_j$ and (6).

Last, the uniqueness of the horizontal transformation

$$\vartheta^t : t_1 \to \mathrm{Rst}_{\underline{A}}.\, t$$

is trivial, since the "commutativity condition" for its extension

$$\bar{\vartheta}^t : \bar{t}_1 \to \mathrm{Rst}_{\underline{A}}.\bar{t} : \mathrm{I}(\Delta) \to \mathbf{Mhr}$$

gives, on $a_e \in \mathrm{I}(\Delta)(i_o, i)$ :

(8) $$\vartheta_i^t(e) = \vartheta_i^t((t_o a_e)_R(\varepsilon_e)) = (\bar{t}a_e)_R(\vartheta_{i_o}^t(\varepsilon_e)) = (ta_e)_R(\varepsilon_e).$$

**4.7.** With the same notations, it will be useful to remark that :

*if $h \in \Delta(i, i)$ is turned into a restriction by every model t, and moreover*

$$e_o = t_o(h) \in X_i^o, \quad a_{e_o} = h, \quad \varepsilon_{e_o} = 1$$

*then the checking of 4.6.5-6 for d = h can be spared.*

Indeed, for every $e \in X_i^o$ :

(1) $$(th)_R(\vartheta_i^t e) = (th)(\vartheta_i^t e)(th) = ((\bar{t}h)_R(1))(\vartheta_i^t e) =$$
$$((\bar{t}a_{e_o})_R(\varepsilon_{e_o})).(\vartheta_i^t e) = (\vartheta_i^t e_o).(\vartheta_i^t e) = \vartheta_i^t(e_o e) = \vartheta_i^t(e_{oR}(e)) =$$
$$= \vartheta_i^t((t_o h)_R(\bar{t}_o a_e)_R(\varepsilon_e)) = \vartheta_i^t(\bar{t}_o(ha_e)_R(\varepsilon_e)).$$

**4.8.** Notice that the c.t.m. $t_1 : \Delta \to Mlr$ of the theory T can be of some use in itself, in so far as it "detects" *monics, epis, isos, proper morphisms, null morphisms.*

More precisely, if $a \in \mathrm{I}(\Delta)$ (0.4) and $\bar{t}_1(a)$ is monic in $Mlr$ (or epi, ...) then, for every model $t : \Delta \to \underline{A}$, $\bar{t}(a)$ is monic in $\underline{A}$ (or epi,...).

Indeed, let $t_o : \Delta \to \underline{A}_o$ be the canonical model of T, and assume that $t_1 = \mathrm{Rst}_{\underline{A}_o}.\, t$ (4.2); let also $F : \underline{A}_o \to \underline{A}$ be the RE-functor such that $t = Ft_o$. Since :

(1) $$\bar{t}_1 = \mathrm{Rst}_{\underline{A}_o}.\bar{t}_o, \qquad \bar{t} = F\,\bar{t}_o$$

where $\mathrm{Rst}_{\underline{A}_o}$ reflects monics (and so on : I.7.1) and F preserves them, the conclusion follows.

Instead $t_1$ (or $\bar{t}_1$) may identify parallel morphisms which are "canonically" different, in so far as the theory (i.e., its classifying RE-category $\underline{A}_0$) is not transfer.

## 5. Criteria for idempotent theories.

We derive here, from § 4 and from the Running Knot Theorem [7], two criteria which will be used in Part III to prove that the given models are canonical.

$\underline{A}$ is always a RE-category. We recall that $\underline{A}$ is distributive iff it is orthodox (I.7.4), hence provided with a canonical order $a \Subset b$ *(domination)* on parallel morphisms, consistent with composition and involution :

$$a \Subset b \qquad \text{if} \qquad a = (a\tilde{a})b(\tilde{a}a) \,,$$

or equivalently if there exist idempotent endomorphisms e, $f$ such that $a = fbe$.

**5.1. Theorem.** Let $t_0 : \Delta \to \underline{A}_0$ be a model of T, and

$$t_1 = \mathrm{Rst}_{\underline{A}_0} . t_0 : \Delta \longrightarrow Mlr$$

then $t_0$ is a canonical model, $t_1$ is a c.t.m. and T is idempotent iff the following conditions hold :

a) $\underline{A}_0$ is distributive,
b) $t_0$ is a q-morphism,
c) for every model $t : \Delta \to \underline{A}$, if $\underline{A}$ is distributive and $t$ is a q-morphism, then $\underline{A}$ is idempotent,
d) for every model $t : \Delta \to \underline{A}$ there is some horizontal transformation

$$\vartheta : t_1 \to \mathrm{Rst}_{\underline{A}} . t : \Delta \to \mathbf{Mhr}.$$

**Proof.** The conditions a - d are trivially necessary.

Conversely, if they hold, $t_1$ is a c.t.m. by 4.4 (any q-morphism is Rst-spanning), obviously distributive. By 4.3 the theory T is ditributive ; by c it is also idempotent, hence it is transfer. By a – c, $t_1 = \mathrm{Rst}_{\underline{A}_0} . t_0$ is a RE-factorization and $t_0$ is a canonical model by 4.5.

**5.2. Lemma.** Let $t : \Delta \to \underline{A}$ be a morphism with values in the *distributive* RE-category $\underline{A}$ ; $t$ is a q-morphism iff the following conditions hold :

a) $t$ is bijective on the objects,
b) $t$ is Rst-spanning,
c) for each $a$ in $\underline{A}$ there is some $a_0$ in $I(\Delta)$ such that $a \Subset \bar{t}(a_0)$.

When these conditions are satisfied, $\underline{A}$ is idempotent iff :

d) $\bar{t}(I(\Delta))$, i.e. the involutive subcategory of $\underline{A}$ spanned by $t(\Delta)$, is idempotent.

**Proof.** Consider the RE-factorizations

$$t = F_2 t_1 \quad \text{and} \quad \bar{t} = F_2 \bar{t}_1$$

(1.1.1), and suppose that a, b, c hold. Then $F_2$ is faithful, bijective on the objects by a, and full : if $a$ belongs to $\underline{A}$ there exists some $a_0$ in $I(\Delta)$ such that

$$a \mathbin{\mathbb{C}} \bar{t}(a_0) = F_2(\bar{t}_1(a_0)) ;$$

hence

$$a = (a\tilde{a})F_2(\bar{t}_1(a_0))(\tilde{a}a),$$

where the projections $a\tilde{a}, \tilde{a}a$ are reached by $F_2$, because of b. Thus $F_2$ is an isomorphism and $t$ is a q-morphism.

Conversely, if $t$ is a q-morphism then $F_2$ is iso and a, b hold. Moreover, call $\underline{A}_0$ the subcategory of $\underline{A}$ having the same objects and those morphisms $a$ which are dominated by some $\bar{t}(a_0)$ with $a_0$ in $I(\Delta)$ : $\underline{A}_0$ is clearly a RE-subcategory of $\underline{A}$ containing $t(\Delta)$, hence it coincides with $\underline{A}$, and c follows.

The last remark is obvious.

**5.3. Criterion I for idempotent theories.** Let T be a RE-theory on $\Delta$, $t_0 : \Delta \to \underline{A}_0$ a model of T and

$$t_1 = \mathrm{Rst}_{\underline{A}_0} . t_0 : \Delta \to Mlr.$$

Assume also that $\Delta$ is the union [4]) of subgraphs $\Delta'$ and $\Delta''$ such that :

(C.1) all the morphisms of $\Delta'$ are endomorphisms, and they are turned by every model $t : \Delta \to \underline{A}$ into restrictions.
(C.2) $\Delta''$ is the graph underlying $\underline{\Gamma}$, the order category associated to a product IxJ where I and J are totally ordered sets ; every model $t : \Delta \to \underline{A}$ restricts to a functor on $\underline{\Gamma}$.

Then the following conditions are necessary and sufficient in order that $t_0$ be a canonical model for T and T be idempotent :

(C.3) $\underline{A}_0$ is distributive,
(C.4) $t_0$ is bijective on the objects,
(C.5) for every $a$ in $\underline{A}_0$ there is some $a_0$ in $I(\Delta)$ such that $a \mathbin{\mathbb{C}} t(a_0)$,
(C.6) $t$ is a c.t.m. for T.

Moreover for (C.5) it is sufficient to consider those morphisms $a$ which are not idempotent endomorphisms ; if $\Delta$ is connected, it is also sufficient to consider non-null morphisms $a$ .

([4]) In the applications $\Delta'$ and $\Delta''$ will be arrow-disjoint ; however this fact has no interest for the proof.

**Proof.** We apply Theorem 5.1. The condition 5.1 a coincides with (C.3), while the conjunction of b and d (in 5.1) is equivalent to (C.4, 5,6) by 4.4 and 5.2.

Thus we need only to prove that (C.1, 2) imply 5.1 c ; let $t : \Delta \to \underline{A}$ be a model of T, where $\underline{A}$ is distributive and $t$ is a q-morphism : the involutive subcategory of $\underline{A}$ spanned by $t(\Delta)$ is idempotent, by the Running Knot Theorem [7]; by 5.2 d this proves that $\underline{A}$ itself is idempotent.

The last remark is obvious (yet useful to spare trivial checkings!): an idempotent endomorphism is dominated by the parallel identity, while a null morphism is dominated by every parallel one.

**5.4. Criterion II for idempotent theories.** In the same general hypotheses, the same conclusion holds if the condition (C.2) is replaced by :

(C.2') I and J are intervals of **Z** ; $\Delta''$ has object-set I×J and the following morphisms (and only them) :

(1) $d'_{ij} : (i, j) \to (i+1, j)$ (for $i, i+1 \in I$, $j \in J$)

(2) $d''_{ij} : (i, j) \to (i, j+1)$ (for $i \in I$, $j, j+1 \in J$).

Moreover, for every model $t : \Delta \to \underline{A}$

(3) $$t(d''_{i+1,j}).t(d'_{ij}) = t(d'_{i,j+1}).t(d''_{ij}).$$

**Proof.** It follows immediately from 5.3 by extending $\Delta$ to $\Delta' \cup \Gamma$, where $\Gamma$ is the graph underlying the order category $\underline{\Gamma}$ associated to I×J.

## 6. Universal distributive and idempotent RE-categories.

Owing to the concreteness theorems of [8], here recalled in 6.8, the distributive RE-categories Rel($I$) and $L$ = Rel($J$) are universal for distributive theories, and their idempotent RE-subcategories Rel($I_0$) and $L_0$ = Rel($J_0$) are universal for idempotent theories (6.10). We recall here briefly from [5, 8] description and properties of these categories.

In Part III the canonical models of distributive (resp. idempotent) theories will be built in $L$ (resp. in $L_0$).

**6.1.** The category $I$ of *small sets and partial bijections* [11, 5, 8] has morphisms

$u = (H, K ; u_0) : S \to S'$ where $H \subset S$, $K \subset S'$ and $u_0 : H \to K$

is a bijective mapping ; the composition is obvious.

It is a boolean exact (hence inverse [6], Theorem 6.4) category, with :

(1) $$\ker u = (S\text{-}H,\ S\text{-}H\ ;\ 1) : S\text{-}H \to S,$$

(2) $$\mathrm{cok}\ u = (S'\text{-}K,\ S'\text{-}K\ ;\ 1) : S' \to S'\text{-}K,$$

(3) $$\mathrm{Sub}_I(S) \simeq P(S),$$

(4) $$u^{\rho} = (K, H\ ;\ u_0^{-1}) : S' \to S.$$

Notice that we write $u^{\rho}$ the generalized inverse of $u$ in $I$, while $\tilde{u}$ will denote the opposite of $u$ in Rel($I$) ; $u^{\rho}$ and $\tilde{u}$ coincide just when

$$u = (H, K\ ;\ u_0) : S \to S'$$

is an isomorphism, i.e., when H = S and K = S'.

An explicit construction of Rel($I$) is given in [5] (or can be derived from 6.3) ; besides, $\Theta(I) = I$, because $I$ is inverse.

**6.2.** The expansion

$$\mathcal{J} = \mathrm{Mdl}(I) = \mathrm{Dst}(I)$$

[6] can be described [5] as the category of *semitopological spaces and partial open-closed homeomorphisms* : the objects are the pairs S = ($S_0$, X) where $S_0$ is a small set and X a (distributive) sub-0,1-lattice of $S_0$ (containing the *closed* subsets of S) ; a morphism

$$u = (U, K\ ;\ u_0) : S \to S'$$

is given by a homeomorphism $u_0 : U \to K$ from an open subset U of S onto a closed subset K of S' ; the composition is obvious.

$\mathcal{J}$ is distributive exact, non boolean, as $\mathrm{Sub}_{\mathcal{J}}(S)$ is isomorphic to the lattice Cls(S) of closed subsets of S.

**6.3.** We shall need a direct description of the distributive RE-category $L$ = Rel( $\mathcal{J}$) [5] : a morphism

(1) $$a = (H_1, H_0\ ;\ K_1, K_0\ ;\ a_0) : S \to S'$$

is given by closed subspaces

$$H_0 \subset H_1 \subset S, \qquad K_0 \subset K_1 \subset S'$$

and a homeomorphism

$$a_0 : H_1 - H_0 \to K_1 - K_0\ ;$$

the composition, involution and order are easy to guess.

A $\mathcal{J}$-morphism $u = (U, K\ ;\ u_0)$ embeds in $L$ as

$$u = (S,\ S\text{-}U\ ;\ K,\ \emptyset\ ;\ u_0).$$

Moreover, for (1) :

(2) $$\underline{def}(a) = (H_1, \emptyset \; ; H_1, \emptyset \; ; 1_{H_1}) : S \to S,$$

(3) $$\underline{ann}(a) = (H_0, \emptyset \; ; H_0, \emptyset \; ; 1_{H_0}) : S \to S,$$

(4) $$(H_1, H_0 \; ; K_1, K_0 \; ; a_0) \subset\!\!\!\subset (H'_1, H'_0; K'_1, K'_0 \; ; a'_0)$$

iff $H_1 - H_0 \subset H'_1 - H'_0$, $K_1 - K_0 \subset K'_1 - K'_0$ and $a_0$ is a restriction of $a'_0$ .

It follows that a morphism of the inverse category $\Theta(\mathcal{J}) = L/\Phi$ can be described as :

(5) $$a = (H, K \; ; a_0) : S \to S'$$

where H and K are *locally closed* subsets (i.e., intersection of an open and a closed set) respectively in S and S', while $a_0 : H \to K$ is a homeomorphism.

**6.4.** We also need the exact, idempotent inverse subcategory $I_0$ of $I$ consisting of *small sets and partial identities* (or *common parts*), with morphisms

(1) $$\check{L} = (L, L \; ; 1_L) : S \to S'$$

where $L \subset S \cap S'$ ; the composition is the intersection.

**6.5.** Analogously we consider the exact pre-idempotent subcategory

$$\mathcal{J}_0 = \mathrm{Mdl}(I_0) = \mathrm{Dst}(I_0)$$

of $\mathcal{J}$ consisting of *small semitopological spaces and partial open-closed identities* (or *open-closed common parts)* :

(1) $$L : S \to S'$$

where L is a common *subspace* of S and S' (same induced semitopology), *open* in S and *closed* in S' ; the composition is the intersection.

The idempotent RE-category $L_0 = \mathrm{Rel}(\mathcal{J}_0)$ has morphisms

(2) $$a = (H, K \; ; L) : S \to S'$$

where H and K are closed subspaces of S and S', respectively, while L is a common subspace of H and K, open in both. Then :

(3) $$(H, K \; ; L) \subset\!\!\!\subset (H', K' \; ; L') \quad \text{iff} \quad L \subset L'.$$

The idempotent inverse category $\Theta(\mathcal{J}_0) = L_0/\Phi$ has morphisms of kind (1), where L is any locally closed subspace of S and S'.

**6.6.** We also use the full exact subcategories $I^f$, $\mathcal{J}^f$ of $I$ and $\mathcal{J}$ determined, respectively, by finite small sets and finite small semitopological spaces ; analogously we consider

(1) $$I_o^f = I^f \cap I_o\ , \quad \mathcal{J}_o^f = \mathcal{J}^f \cap \mathcal{J}_o$$
$$L^f = \mathrm{Rel}(\mathcal{J}^f)\ , \quad L_o^f = L^f \cap L_o.$$

**6.7.** Let R be a fixed, non trivial unitary ring ; $I$ has an exact embedding in the (abelian) category of left R-modules

(1) $$F : I \to \text{R-}Mod$$

where $F(S) = R^{(S)}$ is the free module on S, and for

$$u = (H, K\ ; u_o) \in I(S, S'),$$

$F(u) : R^{(S)} \to R^{(S')}$ is the unique R-homomorphism such that

(2) $$F(u)(x) = u_o(x)\ , \quad \text{for every } x \in H,$$

(3) $$F(u)(x) = 0, \quad \text{for every } x \in S\text{-}H.$$

Thus $I$ is isomorphic to its F-image $I_R$ , a (boolean) exact subcategory of R-*Mod*. We also call $I_{oR}, I^f_R, I^f_{oR}$ the F-images of $I_o$, $I^f$, $I^f_o$ in R-*Mod*.

**6.8.** *Embeddings.* From [8], § 4.9, 5.7, 4.10, we have that :

a) every small distributive exact category has an exact embedding in $I$ , and a Sub-full exact embedding in $\mathcal{J}$.

b) every pre-idempotent exact category has an exact embedding in $I_o$ , and a Sub-full exact embedding in $\mathcal{J}_o$ .

a') every small distributive RE-category has a RE-embedding in Rel($I$) and a Rst-full RE-embedding in Rel($\mathcal{J}$) $= L$.

b') every idempotent RE-category has a RE-embedding in Rel($\mathcal{J}_o$) and a Rst-full RE-embedding in Rel($\mathcal{J}_o$) $= L_o$.

Moreover $I$ can be replaced with $\mathcal{J}_R \subset$ R-*Mod* (6.7) and so on. In the Hom-finite case, $I$ can be replaced with $I^f$ and so on.

**6.9.** As a straightforward consequence of these embeddings, the following conditions on the RE-theory T, with classifying RE-category $\underline{A}_0$, are equivalent :

a) T is distributive (resp; idempotent),

b) $\underline{A}_0$ is (isomorphic to) a RE-subcategory of Rel($I$) (resp. of Rel($I_o$)).

c) $\underline{A}_0$ is (isomorphic to) a Rst-full RE-subcategory of $L$ (resp. of $L_o$).

d) $\underline{A}_0$ is (isomorphic to) a distributive (resp. idempotent) RE-subcategory of Rel(R-*Mod*).

In the Hom-finite case, $I$ can be replaced with $I^f$, and so on.

**6.10. Theorem** *(Universality for distributive and idempotent theories).* Let T and T' be RE-theories on the same graph. If T is distributive (resp. idempotent) the following conditions on the RE-statement T ⇒ T' are equivalent :

a) it holds universally,
b) it holds for Rel($I$) (resp. for Rel($I_0$)),
c) it holds for $L$ (resp. for $L_0$),
d) it holds for every distributive (resp. idempotent) RE-subcategory of Rel(R-*Mod*),
e) it holds for Rel(R- *Mod*).

When T is Hom-finite, $I$ can be replaced with $I^f$ and so on.

**Proof.** By 6.9 and 2.6.

## 7. EX-theories and representation functors.

EX-theories correspond 1-1 to RE-theories. Their i-canonical model, having *more* objects than the canonical model of the associated RE-theory, supplies richer representation functors, which will be useful in Part III. Δ is always a small graph.

**7.1. Definition.** An *EX-theory* T on Δ associates to each exact category $\underline{E}$ a set T($\underline{E}$) of graph morphisms $t : \Delta \to \mathrm{Rel}(\underline{E})$, the *models* of T in $\underline{E}$, so that :

(ET.1) if $F : \underline{E} \to \underline{E}'$ is an exact functor and $t \in T(\underline{E})$, then

$$(\mathrm{Rel}\ F).t \in T(\underline{E}'),$$

(ET.2) if $F : \underline{E} \to \underline{E}'$ is a faithful exact functor [5]), $t : \Delta \to \mathrm{Rel}(\underline{E})$ a morphism and $(\mathrm{Rel}\ F)t \in T(\underline{E}')$ then $t \in T(\underline{E})$,

(ET.3) if $t_i \in T(\underline{E}_i)$ for $i$ varying in the small set I, the morphism

$$(t_i) : \Delta \to \Pi\,\mathrm{Rel}(\underline{E}_i) = \mathrm{Rel}(\Pi\,\underline{E}_i)$$

belongs to $T(\Pi\,\underline{E}_i)$.

We say that T is *proper* whenever each model $t : \Delta \to \mathrm{Rel}(\underline{E})$ actually takes values in $\underline{E}$.

**7.2.** These theories do not have strict canonical models. An *i-canonical model* of T will be a model $t_0 : \Delta \to \mathrm{Rel}(\underline{E}_0)$ such that :

a) for eery model $t : \Delta \to \mathrm{Rel}(\underline{E})$ there exist an exact *representation* functor $F : \underline{E}_0 \to \underline{E}$ verifying $t = (\mathrm{Rel}\ F)t_0$ .

([5]) Notice that F is faithful iff Rel(F) is so ([4], Theorem 4.10).

b) for every RE-transformation

$$\tau : t_1 \to t_2 : \Delta \to \mathrm{Rel}(\underline{E})$$

of models there exists a natural transformation of exact functors

$$\varphi : F_1 \to F_2 : \underline{E}_0 \to \underline{E}$$

such that $\tau = (\mathrm{Rel}\ \varphi).t_o$ ; moreover $\varphi$ is uniquely determined by $\tau, F_1$ and $F_2$.

Notice that the condition a is superfluous. The exact functor F is determined up to a unique functorial iso $\varphi$ such that $(\mathrm{Rel}\ \varphi)t_o = t$. The exact category $\underline{E}_0$ is determined up to equivalence ; it will be called the *classifying exact category of* T.

The existence of the i-canonical model will be derived from the existence of the canonical model of the associated RE-theory.

**7.3.** Every RE-theory T defines an EX-theory $T^e$, on the same graph $\Delta$, just by setting $T^e(\underline{E}) = T(\mathrm{Rel}\ \underline{E})$ for every exact category $\underline{E}$.

Conversely the EX-theory T defines the RE-theory $T^r$ whose models in the RE-category $\underline{A}$ are the morphisms $t : \Delta \to \underline{A}$ verifying the following, trivially equivalent, conditions :

a) for every RE-functor $F : \underline{A} \to \mathrm{Rel}(\underline{E})$, where $\underline{E}$ is an exact category, $Ft \in T(\underline{E})$,

b) there exists a faithful RE-functor $F : \underline{A} \to \mathrm{Rel}(\underline{E})$, where $\underline{E}$ is an exact category and $Ft \in T(\underline{E})$,

c) $(\eta\underline{A})t \in T(\underline{E})$, where $\eta\underline{A} : \underline{A} \to \mathrm{Rel}(\underline{E})$ is the canonical RE-embedding of $\underline{A}$ in the category of relations on $\underline{E} = Z(\mathrm{Prp}(\mathrm{Fct}\ \underline{A}))$ (I.6.8).

It is easy to check that :

(1) $T^{er} = T$, for every RE-theory T,

(2) $T^{re} = T$, for every EX-theory T.

The properties of RE-theories considered in § 3 will also be referred to the associated EX-theory.

**7.4. Theorem.** Let T be an exact theory on $\Delta$, and $t_o : \Delta \to \underline{A}_0$ the canonical model of the associated RE-theory $T^r$ ; then the composed morphism $t_o^e$ (see I.6.8)

(1) $$\Delta \xrightarrow{t_o} \underline{A}_0 \xrightarrow{\eta\underline{A}_0} \mathrm{Rel}(\underline{E}_0)\ , \qquad \underline{E}_0 = Z(\mathrm{Prp}(\mathrm{Fct}\ \underline{A}_0))$$

is an i-canonical model for T.

**Proof.** Let

$$\tau : t_1 \to t_2 : \Delta \to \mathrm{Rel}(\underline{E})$$

be a RE-transformation of T-models. Since $t_1$ and $t_2$ are also $T^{r}$-models in Rel($\underline{E}$), by 2.3 there exists a unique RE-transformation

$$\gamma : G_1 \to G_2 : \underline{A}_0 \to \mathrm{Rel}(\underline{E}) \quad \text{such that} \quad \gamma\, t_0 = \tau.$$

By the i-universal property (I.0.6) of $\eta \underline{A}_0$ there exists a natural transformation of exact functors

$$\varphi : F_1 \to F_2 : \underline{E}_0 \to \underline{E} \quad \text{such that} \quad \mathrm{Rel}(\varphi).\,\eta A_0 = \gamma,$$

hence

$$\mathrm{Rel}(\varphi).\, t_0^e = \gamma\, t_0 = \tau .$$

Moreover $\varphi$ is determined by $F_1$, $F_2$ and $\gamma$ or, in other words, by $F_1$, $F_2$ and $\tau$.

**7.5.** For every RE-graph $\Delta$ call *i-free exact category generated by* $\Delta$ the i-classifying exact category $\underline{E}_0$ of the EX-theory $T_\Delta^e$, provided with the i-canonical morphism $t_0 : \Delta \to \mathrm{Rel}(\underline{E}_0)$.

In particular, every *graph* $\Delta$ (without RE-conditions) defines a RE-graph $\Delta'$ on itself, with RE-conditions

(1) $$a \in \mathrm{Prp}(\Delta') \quad \text{for every } \Delta\text{-morphism } a ,$$

and the *i-free exact category generated by the graph* $\Delta$ will be the one generated by the RE-graph $\Delta'$, according to the above definition.

**7.6.** Now, let T be a RE-theory, $T^e$ the associated EX-theory and $\underline{E}$ a fixed exact category : $T^e(\underline{E})$ can be made into a category, with morphisms the RE-transformations $\tau : t \to t'$ (for $t, t' : \Delta \to \mathrm{Rel}(\underline{E})$ models of T) and composition the vertical one.

For every model $t : \Delta \to \mathrm{Rel}(\underline{E})$ choose an exact *representation functor* $F_t : \underline{E}_0 \to \underline{E}$ (such that $t = \mathrm{Rel}(F_t).t_0$). For every RE-transformation $\tau : t \to t'$ in $T^e(\underline{E})$ take

$$F_\tau : F_t \to F_{t'} : \underline{E}_0 \to \underline{E}$$

the unique natural transformation such that $\tau = \mathrm{Rel}(F_\tau).\,t_0$. We have thus a functor

$$F : T^e(E) \to \mathbf{EX}(\underline{E}_0, \underline{E})$$

which gives a *global representation functor,* $\mathrm{Rpr} : \underline{E}_0 \times T^e(\underline{E}) \to \underline{E}$ :

(1) $$\underline{E}_0 \times T^e(\underline{E}) \xrightarrow{\ \underline{E}_0 \times F\ } \underline{E}_0 \times \mathbf{EX}(\underline{E}_0, \cdot \underline{E}) \xrightarrow{\ \mathrm{Evl}\ } \underline{E}$$

(2) $$\mathrm{Rpr}(E, t) = F_t(E),$$

(3) $$\mathrm{Rpr}(u, \tau) = F_\tau(E').F_t(u) = F_{t'}(u).F_\tau(E)$$

$$(4) \qquad \begin{array}{c} E \\ \downarrow u \\ E' \end{array} \qquad \begin{array}{ccc} F_t(E) & \xrightarrow{F_\tau(E)} & F_{t'}(E) \\ {\scriptstyle F_t(u)}\downarrow & & \downarrow{\scriptstyle F_{t'}(u)} \\ F_t(E') & \xrightarrow{F_\tau(E')} & F_{t'}(E') \end{array}$$

exact in the first variable, as $Rpr(-, t) = F_t$ . Shortly, we shall write :

$$(5) \qquad E(t) = Rpr(E, t) = F_t(E), \qquad u(\tau) = Rpr(u, \tau).$$

**7.7.** Fixing the second variable, i.e. a model $t$ in $T^e(E)$, the exact functor $F_t : \underline{E}_0 \to \underline{E}$ extends to the RE-functor

$$Rel(F_t) : Rel(\underline{E}_0) \to Rel(\underline{E}),$$

whose action on the relation $a : E \to E'$ will also be written :

$$(1) \qquad a(t) = Rel(F_t)(a) : E(t) \to E'(t) .$$

We recall that, for a RE-transformation $\tau : t \to t'$, the RE-transformation

$$Rel(F_\tau) : Rel(F_t) \to Rel(F_{t'})$$

is only *lax*-natural, that is the squares generalizing 7.6.4 are RO-squares

$$(2) \qquad \begin{array}{c} E \\ \downarrow a \\ E' \end{array} \qquad \begin{array}{ccc} E(t) & \xrightarrow{E(\tau)} & E(t') \\ {\scriptstyle a(t)}\downarrow & \leqq & \downarrow{\scriptstyle a(t')} \\ E'(t) & \xrightarrow{E'(\tau)} & E'(t') \end{array}$$

therefore Rpr does not extend to a functor on $Rel(\underline{E}_0) \times T^e(\underline{E})$. However, it will be useful to remark that *the square (2) commutes when $a(t)$ and $a(t')$ are both proper* (I.2.2).

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