CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

KIMMO I. ROSENTHAL

Sheaves and local equivalence relations

Cahiers de topologie et géométrie différentielle catégoriques, tome 25, nº 2 (1984), p. 179-206

<http://www.numdam.org/item?id=CTGDC_1984__25_2_179_0>

CAHIERS DE TOPOLOGIE
ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

Vol. XXV-2 (1984)

SHEAVES AND LOCAL EQUIVALENCE RELATIONS

by Kimmo I. ROSENTHAL

Résumé. Une relation d'équivalence locale r sur un espace topologique X est une section globale du faisceau des germes de relations d'équivalence sur X. On construit une catégorie $sh(X\ ;r)$ de r-faisceaux, qui sous certaines conditions sur r est un topos, et même une étendue. D'après un théorème de Grothendieck caractérisant les étendues, il s'ensuit que ce topos est équivalent à un topos de faisceaux muni d'une action d'un groupoïde topologique étale. En utilisant une construction de Pradines pour le groupoïde d'holonomie d'un morceau différentiable de groupoïde, on décrit explicitement ce groupoïde pour une grande classe d'équivalences r.

Introduction.

Much of topology deals with analyzing local information about a space and trying to relate it to global properties of the space. Sheaf theory plays a central role in this process and $sh(X)$, the category of sheaves on a space X, is the fundamental example of a topos. Often a space X comes equipped with some additional data such as a group action or an equivalence relation and it is natural to consider sheaves which have a compatible such structure. These categories will again be topoi. Given the local nature of topology, it may be that X locally has some structure in addition to its topology. The significance and need of systematically understanding this point of view was first emphasized by C. Ehresmann [4]. In this paper, we consider an example of local structure on a space, namely that of a local equivalence relation. We then construct a category of sheaves relative to this local equivalence relation and investigate its properties.

A local equivalence relation on a topological space X is a global section of the sheaf of germs of equivalence relations on X. This notion was introduced by A. Grothendieck in SGA4 [11], p. 485, in a series of exercises, presented essentially as open problems, for the purpose of constructing a certain kind of étendue.

Constructions similar to local equivalences appear earlier in Differential Geometry, namely Reeb's "système dynamique généralisé" [14] and Ehresmann's "partition locale" [6]. Grothendieck wished to construct an étendue of sheaves relative to a given local equivalence relation. An étendue is a topos E, which has a covering U of the terminal object such that E/U is generated by subobjects of 1. If this locale has enough points, we call E a topological étendue. (A study of local equivalence relations appears in [17], while étendues are investigated in [16]). A characterization theorem of Grothendieck states that every topological étendue is equivalent to a category $sh(X, R)$ of R-sheaves, which are sheaves equipped with an action from an étale topological groupoid R with space of objects X. In [16], it was shown that étendues can have more general presentations, for example by topological categories with monic maps. Another means of obtaining étendues is from strictly open local equivalence relations. The details of this appear in [15], however there the presentation and proofs are rather lengthy and cumbersome and there is no description of the associated étale groupoid as provided by Grothendieck's Theorem.

In this paper, by considering a certain class of strictly open local equivalence relations r, we present the étendue of r-sheaves by explicitly exhibiting its étale groupoid. A motivating example from Reeb and Ehresmann and also Grothendieck was that of the local equivalence relation defined by a foliation on a manifold. Ehresmann defined holonomy groupoids for foliations [6] and our construction is based on a general notion of holonomy groupoid as introduced by Pradines [13] and described by Brown [2]. We hope this will provide some insight into how the various properties of local equivalence relations relate to sheaves and étendues.

In § 1, we present the relevant definitions and results about local equivalence relations, describing the various coherence assumptions and the functors *loc( )* and *glob( )* relating local and global equivalence relations. Most of these results appear in [17] and proofs in general will be omitted.

In § 2, we similarly give definitions and results concerning étendues. Most of this section appears in [16], to which the reader is referred for details.

In §3, the category $sh(X\ ;\ r)$ of r-sheaves is presented, where r is a local equivalence relation on X. We investigate some of its properties, in particular for globally coherent r.

In §4, strictly regular r are defined and for such a local equivalence relation we construct an étale topological groupoid Γ^r such that

$$sh(X\ ;\ r) \simeq sh(X\ ;\ \Gamma^r).$$

If r is the local equivalence relation of a foliation on a manifold, we can take Γ^r to be the holonomy groupoid of the foliation. In this context, we can think of r-sheaves as sheaves which locally have a connection relative to the foliation. Consideration of this kind of structure goes back to the work of Ehresmann [5]. Hopefully, the topos theorists find this an interesting example of a topos and it provides another perspective on the holonomy groupoid construction.

I am very grateful to Professor R. Brown for sending me his preprint [2], which proved invaluable, and for his general interest in this work. I would also like to thank Professor A. Ehresmann for making me aware of the work of C. Ehresmann and G. Reeb, which served

as a motivation for the constructions presented in this paper. This not only places things in a proper historical perspective, but it also provides increased insight and understanding.

1. Local equivalence relations.

A local equivalence relation on a topological space X is a global section of the sheaf of germs of equivalence relations on X. In this section, we present some of the definitions and results we shall use later on. Most of this material appears in [17], and proofs will only be provided for results which do not appear there.

Let X be a topological space. If U is open in X, let

$$Equiv\,(U) = \{\text{equivalence relations on } U\} .$$

This defines a presheaf on X, which however is not a sheaf (see p. 168 in [17]). Let E_X denote the associated sheaf.

Definition 1.1. A global section r of E_X is called a *local equivalence relation* on X.

A local equivalence relation r is given by the following local data : an open cover

$$\{U_x\}_{x \in X} \qquad \text{with } x \in U_x \qquad \text{and} \qquad R_x \in Equiv(U_x),$$

such that if $z \in U_x \cap U_y$, there is an open neighborhood W of z with

$$W \subset U_x \cap U_y \quad \text{and} \quad R_x | W = R_y | W.$$

A motivating example is that of the local equivalence relation r of a foliation on a manifold X. (See [17], p. 169.) If the foliation is defined by submersions

$$f_i : U_i \to \mathbf{R}^q \qquad (i \in I, \text{ an index set}),$$

then r is defined by

$$R_i \in Equiv\,(U_i) \qquad \text{where} \qquad \langle x, y\rangle \in R_i \quad \text{iff} \quad f_i(x) = f_i(y) \quad .$$

If r_x, s_x are germs at $x \in X$ in the presheaf $Equiv$ (X) locally defined by

$$R \in Equiv\,(U) \qquad \text{and} \qquad S \in Equiv\,(V)$$

respectively (where U and V are open neighborhoods of x), then

$r_x \leqq s_x$ iff there is an open neighborhood W of x with $W \subset U \cap V$ and $R|W \subset S|W$.

Definition 1.2. Let r and s be local equivalence relations on X with

$$r = (r_x)_{x \in X} \quad \text{and} \quad s = (s_x)_{x \in X} .$$

Then $\underline{r \leqq s}$ iff $r_x \leqq s_x$ for all $x \in X$.

Using this ordering we can try to study the relationship between local and global equivalence relations via the following definitions.

Definition 1.3. (a) Let $R \in$ *Equiv* (X). Then *loc* (R) denotes the local equivalence relation it defines.

(b) Let $r \in E_X$. Let

$$glob(r) = \cap\{ R \in Equiv(X) \mid r \leqq loc(R)\} .$$

Remarks. *glob(r)* represents an attempt to "approximate" r locally by a single global relation. Let $\underline{U} = \{U_x\}_{x \in X}$ denote the open cover, where r is locally defined by $R_x \in Equiv(U_x)$. If $\underline{V} = \{V_x\}_{x \in X}$ is an open cover, we say

$$\underline{V} \leqq \underline{U} \quad \text{iff} \quad x \in V_x \subset U_x \quad \text{for all} \quad x \in \underline{X}.$$

It can be shown that

$$glob(r) = \cap\{ R_{\underline{V}} \mid \underline{V} \leqq \underline{U}\}$$

where $R_{\underline{V}}$ is the equivalence relation generated by $\{R_x | V_x\}_{x \in X}$. One should also note that if $R \in$ *Equiv* (X), then *glob(loc* (R)) $\subset$ R always holds. However, $r \leqq loc(glob(r))$ need not always be true. To have an adjoint relationship between *loc* and *glob* , we need the notion of coherence.

Definition 1.4. Let r be a local equivalence relation on X.

(i) r is *coherent* iff $r \leqq loc(glob(r))$. .

(ii) r is *totally coherent* if $r|U$ is coherent for every open set U in X.

(iii) r is *globally coherent* iff $r = loc(glob(r))$.

Definition 1.5. Let R be an equivalence relation on X.

(i) R is *locally coherent* iff *loc*(R) is coherent.

(ii) R is *coherent* iff R = *glob(loc*(R)).

When we restrict our attention to coherent local equivalence relations and locally coherent global equivalence relations, *glob* becomes the left adjoint of *loc* . The following proposition indicates some of the relationships between the above definitions.

Proposition 1.6. *Let r be a local equivalence relation on X locally defined by $R_x \in$ Equiv (U_x), where U_x is an open neighborhood of x in X.*

(a) Suppose r is globally and totally coherent. Then, if U is open in X, $r|U$ is globally coherent.

(b) Suppose r is totally coherent. Then, $r|U_x$ is globally coherent.

(c) If there is an open cover

$$\{V_x\}_{x\in X} \quad \text{with} \quad x \in V_x \subset U_x$$

such that $r|V_x$ is globally coherent for each $x \in X$, then r is totally coherent.

(d) If r is globally coherent and $R = glob(r)$, then for any open cover $\{V_x\}_{x\in X}$, $R = R_{\underline{V}}$, the equivalence relation generated by

$$\{R|V_x\}_{x\in X}.$$

Proof. For (a) and (c), see Proposition 2.8 in [17]. (d) is immediate, since $R_{\underline{V}} \subset R$ for all covers $\underline{V}$, yet by the remarks following Definition 1.3, $R \subset R_{\underline{V}}$.

To prove (b), since r is totally coherent, $r|U_x$ is coherent, hence

$$r|U_x \leqq loc(glob(r|U_x)).$$

We also have $r|U_x = loc(R_x)$, since the other equivalence relations R_y are locally compatible with R_x;

$$glob(loc(R_x)) \subset R_x$$

is true, i.e. $glob(r|U_x) \subset R_x$. Applying the functor loc, we obtain

$$loc(glob(r|U_x)) \leqq loc(R_x) = r|U_x. \qquad \Diamond$$

The above notions of coherence are closely related to connectedness of equivalence classes (for details, see [17]). The next couple of results indicate some of these relationships and will be crucial in the sequel.

Proposition 1.7. *Let R be an equivalence relation on X. If R has connected equivalence classes, then $R = glob(loc(R))$, i.e. R is coherent. If R is coherent and has closed equivalence classes, then it has connected equivalence classes.*

Proof. See Theorem 2.7 in [17]. ◊

Proposition 1.8. *Let r be coherent. Then, $glob(r)$ has connected equivalence classes.*

Proof. This follows immediately from Proposition 3.6 of [17]. ◊

Corollary 1.9. *If r is totally coherent, then r is locally definable by equivalence relations with connected equivalence classes.*

Proof. By Proposition 1.6 (b),

$$r|U_X = loc(glob(r|U_X)).$$

Since total coherence implies that $r|U$ is coherent, by Proposition 1.8, *glob* $(r|U_X)$ has connected equivalence classes. ◊

We shall need one further definition concerning equivalence relations. Recall that an equivalence relation R on X is open iff the canonical quotient mapping $X \to X/R$ is an open mapping.

Definition 1.10. (i) A local equivalence relation *r* on X is *open* iff it is locally definable by open equivalence relations.

(ii) *r* is *strictly open* iff it is totally coherent and open.

These definitions and results will be used in §3 and §4 to analyze the category of *r*-sheaves, and the important role played by the various coherence assumptions will become clear.

As concluding remarks to provide some historical perspective on local equivalence relations, they can be traced back to the "systèmes dynamiques généralisés" introduced by G. Reeb in [14]. (Many of the ideas of this section and [17] can be found in Reeb's paper.) The "systèmes dynamiques généralisés" were somewhat generalized by C. Ehresmann in [6] under the name "partition locale". We get a correspondence to the strictly open local equivalence relations of Definition 1.10 (ii) and of course the local equivalence relation of a foliation is strictly open.

2. Étendues.

Étendues are a class of topoi introduced by Grothendieck in SGA 4 [11], page 479. An étendue is a topos E with an object U covering the terminal object 1, such that $E/U \simeq sh(\underline{H})$, where $\underline{H}$ is a locale ; i.e. E/U is generated by subobjects of 1 (see [12], Chapter 5). If $\underline{H}$ has enough points, $E/U \simeq sh(X)$, where X is a topological space, and we call such an étendue "topological". A theorem of Grothendieck characterizes a topological étendue as equivalent to a category of sheaves equipped with an "action" from an étale topological groupoid. More generally, étendues can be presented by categories with monic maps [16] or, as we shall see in §4, by certain local equivalence relations. Finally, we should note that every Grothendieck topos F is equivalent to one of the form E_G, a topos a G-coalgebras, where G is a certain kind of left exact cotriple on an étendue E, [18]. In this section, we state the main theorems about étendues, and discuss the examples which will be relevant to our discussion in §4.

Let

$$C = \Big(X_2 \;\begin{array}{c}\xrightarrow{\pi_1}\\ \xrightarrow{\pi_2}\\ \xrightarrow[m]{}\end{array}\; X_1 \;\begin{array}{c}\xrightarrow{d_1}\\ \xrightarrow[d_0]{}\end{array}\; X\Big)$$

be a topological category. (We can think of X as the space of objects, X_1 as the space of morphisms and X_2 as the space of composable pairs of morphisms. m represents composition, d_0 and d_1 are the domain and codomain maps, which have a common section $i : X \to X_1$.)

Definition 2.1. (i) C is an *étale topological category* iff d_0 and d_1 are local homeomorphisms.

(ii) C has monic maps iff

$$m < g_1, f > = m < g_2, f > \quad \text{implies} \quad g_1 = g_2$$

(note we are writing composition algebraically).

Definition 2.2. Let

$$C = \Big(X_2 \;\begin{array}{c}\xrightarrow{\pi_1}\\ \xrightarrow{\pi_2}\\ \xrightarrow[m]{}\end{array}\; X_1 \;\begin{array}{c}\xrightarrow{d_1}\\ \xrightarrow[d_0]{}\end{array}\; X\Big)$$

be a topological category. A *C-sheaf* is a sheaf $p : Y \to X$ together with a map $\alpha : X_1 \underset{d_1}{\times} Y \to Y$ such that

$$\begin{array}{ccc} X_1 \underset{d_1}{\times} Y & \xrightarrow{\alpha} & Y \\ \downarrow & & \downarrow p \\ X_1 & \xrightarrow{d_0} & X \end{array}$$

commutes (where $X_1 \underset{d_1}{\times} Y$ denotes the pullback along $d_1 : X_1 \to X$) and furthermore

$$Y \xrightarrow{i \times 1} X_1 \underset{d_1}{\times} Y \xrightarrow{\alpha} Y \;=\; Y \xrightarrow{id} Y$$

and

$$X_2 \underset{d_1}{\times} Y \xrightarrow{m \times 1} X_1 \underset{d_1}{\times} Y \xrightarrow{\alpha} Y \;=\; X_2 \underset{d_1}{\times} Y \xrightarrow{1 \times \alpha} X_1 \underset{d_1}{\times} Y \xrightarrow{\alpha} Y$$

(i.e. α is an "action" of C on Y).

A C-sheaf morphism is a sheaf map which preserves the actions and so we get a category $sh(X\ ;\ C)$ of C-sheaves.

Theorem 2.3. *Let*

$$C = \left(X \;\begin{array}{c}\xrightarrow{\pi_1}\\ \xrightarrow{\pi_2}\\ \xrightarrow[m]{}\end{array}\; X \;\begin{array}{c}\xrightarrow{d_1}\\ \xrightarrow[d_0]{}\end{array}\; X\right)$$

be an étale topological category with monic maps. Then, $sh(X\ ;\ C)$ *is an étendue.*

Proof. See Theorem 3.13 in [16]. ◊

Examples. 1. If X is discrete, C is just a category in *Sets* and

$$sh(X\ ;\ C) \simeq Sets^{C^{op}}$$

In fact, $Sets^{C^{op}}$ is an étendue iff maps in C are monic (see Theorem 1.5 [16]).

2. Let

$$R = \left(R \underset{X}{\times} R \longrightarrow R \;\begin{array}{c}\xrightarrow{\pi_1}\\ \xrightarrow[\pi_2]{}\end{array}\; X\right)$$

be an étale topological groupoid, i.e. R is étale and every map in R is invertible. (For more on topological groupoids, see [3] and the references therein.) As special cases of this, we have equivalence relations on X, i.e.

$$R \xrightarrow{\langle \pi_1,\ \pi_2 \rangle} X \times X$$

is monic, and discrete group actions on X, where $R = G \times X$ and π_1 is $\mu : G \times X \longrightarrow X$, the action of the group G on X. In this case, we obtain the topos of G-sheaves.

If R is an étale groupoid on X, then $\pi_1 : R \to X$ becomes an R-sheaf with the action given by the composition in R. Using this, we have the following

Theorem 2.4. *If R is an étale topological groupoid on X, then $sh(X; R)$ is an étendue and*

$$sh(X\ ;\ R)/R \simeq sh(X).$$

Proof. See Theorem 4.2 [16]. ◊

For the more general topological categories of Theorem 2.3, we need not be able to recover the space X. However, the following characterization Theorem of Grothendieck outlines the central role played by étale groupoids.

Theorem 2.5. *(Grothendieck) If E is a topological étendue, then there is an étale topological groupoid R with space of objects X such that*

$$E \simeq sh(X \; ; R).$$

Proof. See Theorem 4.5 [16]. ◊

It should be noted that different groupoids can give rise to equivalent étendues. For some examples, such as the Jonsson-Tarski topos [16], pp. 198-200, an explicit description of the associated étale groupoid is not known.

C. Ehresmann initiated the study of topological categories and emphasized their significance. The idea of a C-sheaf goes back to his paper [7], where the notion of a topological category C acting on a space is presented under the name "espèce de structures" over C. These objects are studied in a differentiable setting and the important role of actions by topological groupoids is outlined.

In this paper, we are, primarily, interested in the case of equivalence relations. If

$$R \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X$$

is an equivalence relation on X and R denotes the topological groupoid it defines, then even when π_1 and π_2 are not local homeomorphisms, the R-sheaves, $sh\,(X \; ; R)$, form a topos. Since R is a groupoid, it can be shown that an R-sheaf can be described as a sheaf $p : Y \to X$ with an equivalence relation S on Y such that if

$$\langle x_1, x_2 \rangle \in R \qquad \text{and} \qquad y_1 \in p^{-1}(x_1), \;,$$

there is a unique

$$y_2 \in p^{-1}(x_2) \qquad \text{with} \qquad \langle y_1, y_2 \rangle \in S.$$

Of course

$$\langle y, y' \rangle \in S \quad \text{implies} \quad \langle p(y), p(y') \rangle \in R.$$

(This follows from Proposition 4.1 in [16].) We shall be interested in when we can recover $sh\,(X/R)$, the spatial topos of sheaves on the quotient space. We make the following definition.

Definition 2.6. A continuous map $p : X \to Y$ of topological spaces *locally admits sections* if for every $x \in X$, there is an open neighborhood V of $p(x)$ and a continuous section $\sigma : V \to X$ with $\sigma(p(x)) = x$.

(If X and Y are manifolds and p is smooth, then if σ can be chosen smooth, this is equivalent to saying that p is a submersion [1].)

Theorem 2.7. *(Grothendieck) Suppose*

$$R \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X$$

is an equivalence relation on X *and* $q : X \to X/R$ *(the canonical quotient map) locally admits sections. Then,* $sh(X\ ; R) \simeq sh(X/R)$.

Proof. It suffices to show that $sh(X\ ; R)$ is generated by subobjects of 1 ([12], Chapter 5), i.e., by sub-R-sheaves of X. An open $U \subset X$ is a sub-R-sheaf iff it is saturated under R and thus of the form $q^{-1}(V)$ for V open in X/R. Suppose

$$p : Y \longrightarrow X, \qquad p' : Y' \longrightarrow X$$

are R-sheaves and

$$Y \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y'$$

are R-sheaf morphisms with $f_1 \neq f_2$. Let S denote the equivalence relation in Y making it into an R-sheaf. Since $f_1 \neq f_2$ there is

$$y \in Y \qquad \text{with} \qquad f_1(y) \neq f_2(y).$$

Let $p(y) = x$ and choose an open neighborhood W of y such that $p(W)$ is open in X and homeomorphic to W. Since $q : X \to X/R$ locally admits sections, choose an open neighborhood $\hat{U}$ of $[x]$ in X/R and a section

$$\sigma : \hat{U} \longrightarrow X \qquad \text{with} \qquad \sigma([x]) = x.$$

$\sigma^{-1}(p(W)) = \hat{W}$ is open in X/R and $[x] \in \hat{W}$. Let $U = q^{-1}(\hat{W})$. Then, U is a sub-R-sheaf of X.

Define $s : U \to Y$ as follows. If $x_1 \in U$, then

$$q(x_1) = [x_1] \in \hat{W}.$$

Since $\hat{W} = \sigma^{-1}(p(W))$, there is

$$x_2 \in p(W) \qquad \text{with } \sigma([w_1]) = x_2 .$$

Thus, $\langle x_1, x_2 \rangle \in R$. Since $p(W)$ is homeomorphic to W, choose the unique

$$y_2 \in W \qquad \text{with} \qquad p(y_2) = x_2 .$$

Since Y is an R-sheaf, there is a unique

$$y_1 \in p^{-1}(x_1) \qquad \text{with} \quad \langle y_1, y_2\rangle \in S.$$

Let $s(x_1) = y_1$. By construction s is an R-sheaf morphism and separates f_1 and f_2 since $s(x) = y$. Thus, subobjects of 1 generate $sh(X ; R)$. ◊

This result will prove useful in §4, when we consider local equivalence relations locally defined by equivalence relations satisfying the above hypothesis.

3. r-sheaves.

Given a local equivalence relation r, we shall define a category of r-sheaves, sh (X ; r). It can be shown, [15], that if r is strictly open, then $sh(X ; r)$ is an étendue. In §4, we shall prove this for a slightly less general class of r, the strictly regular local equivalence relations. In this section, after defining an r-sheaf, we shall discuss some properties of $sh(X ; r)$, in particular for globally coherent r, that will be needed in what follows.

Let $p : Y \to X$ be a sheaf over X, and let U be open in X. Let Q(U, Y) consist of pairs (R_U, S_U) where R_U is an equivalence relation on U, S_U is an equivalence relation on $Y|U$, such that $p: Y|U \to U$ is compatible with R_U and S_U (i.e.,

$$\langle y_1, y_2\rangle \in S_U \Rightarrow \langle p(y_1), p(y_2)\rangle \in R_U)$$

and we have the following pullback

$$\begin{array}{ccc} Y|U & \longrightarrow & Y|U/S_U \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ U & \longrightarrow & U/R_U \end{array}$$

with q a local homeomorphism.

This implies that if $p(y) = x$ and

$$x' \in [x] \text{ , the equivalence class of } x \text{ mod } R_U,$$

there is a unique

$$y' \in p^{-1}(x') \quad \text{with} \qquad \langle y', y\rangle \in S_U.$$

Note that distinct elements in the same fiber in Y cannot be equivalent.

We shall make use of the following categorical lemma.

Lemma 3.1. *Let* $\underline{A}$ *be a category. Suppose we have the following diagram*

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h''} \\ D & \xrightarrow{f'} & E & \xrightarrow{g'} & F \end{array}$$

such that the whole square is a pullback, the right hand square commutes and g is monic. Then, the left hand square is a pullback.

Proof. It is a straightforward category theoretical argument and will be omitted. ◊

Lemma 3.2. *Let* Y *be a sheaf on* X. *Let* O(X) *denote the open sets of* X. *Then,*

$$Q(-, Y) : O(X)^{op} \longrightarrow \mathit{Sets}$$

is a presheaf.

Proof. If $V \subset U$ in O(X), let $Q(U, Y) \to Q(V, Y)$ be given by restricting the equivalence relations on U and $Y|U$ to V and $Y|V$ respectively. If $(R_U, S_U) \in Q(U, Y)$, let

$$R_V = R_U | V \quad \text{and} \quad S_V = S_U|V.$$

The following square is a pullback, since the two smaller squares are.

$$\begin{array}{ccccc} Y|V & \longrightarrow & Y|U & \longrightarrow & Y|U/S_U \\ \downarrow & & \downarrow & & \downarrow \\ V & \longrightarrow & U & \longrightarrow & U/R_U \end{array}$$

The maps $Y|V \to Y|U/S_U$ and $V \to U/R_U$ factor as follows :

$$\begin{array}{ccccc} Y|V & \longrightarrow & Y|V/R_V & \longrightarrow & Y|U/S_U \\ \downarrow & & \downarrow & & \downarrow \\ V & \longrightarrow & V/R_V & \longrightarrow & U/R_U \end{array}$$

Thus

$$\begin{array}{ccccc} Y|V & \longrightarrow & Y|V/S_V & \xrightarrow[g]{} & Y|U/S_U \\ \downarrow & & \downarrow & & \downarrow \\ V & \longrightarrow & V/R_V & \longrightarrow & U/R_U \end{array}$$

is a pullback. One can see that the right hand square is a pullback, and hence $Y|V/S_V \to V/R_V$ is a local homeomorphism. Since g is monic, apply

Lemma 3.1 to conclude that $(R_V, S_V) \in Q(V, Y)$. ◊

Q will not be a sheaf however, and let Q_Y denote the associated sheaf. We have a forgetful functor of sheaves $Q_Y \to E_X$, the sheaf of local equivalence relations on X.

Let r be a local equivalence relation on X.

Definition 3.3. An *r-structure* on a sheaf Y is a local equivalence relation s on Y such that (r, s) is a global section of Q_Y.

Definition 3.4. An *r-sheaf on* X is a pair (Y, s), where Y is a sheaf on X and s is an r-structure on Y.

If (Y, s) and (Z, t) are r-sheaves, an r-sheaf morphism is a sheaf map $Y \to Z$, which locally preserves the r-structures. Thus, we have a category $sh(X\ ;\ r)$ of r-sheaves.

If r is locally defined by $R_x \in Equiv(U_x)$, where U_x is an open neighborhood of x in X, then (Y, s) being an r-sheaf means that s is locally defined by equivalence relations $(S_x)_{x \in X}$ defined over open neighborhoods of x in X such that on some $V_x \subset U_x$, we have

$$\begin{array}{ccc} Y|V_x & \longrightarrow & Y|V_x/S_x \\ \downarrow & & \downarrow \\ V_x & \longrightarrow & V_x/R_x \end{array}$$

is a pullback and the right hand map a local homeomorphism. (Thus, locally we have parallel translations along equivalence classes.)

The following lemma will prove useful.

Lemma 3.5. *If* $q : (Z, t) \to (Y, s)$ *is a morphism of r-sheaves, then t is an s-structure, i.e., we can view* (Z, t) *as an s-sheaf over* Y.

Proof. Suppose t is locally defined by equivalence relations $(T_x)_{x \in X}$ and s by $(S_x)_{x \in X}$. Since (Z, t) and (Y, s) are r-sheaves and q is an r-sheaf map, we can find open neighborhoods W_x of x in X such that

$$\begin{array}{ccc} Z|W_x & \longrightarrow & Z|W_x/T_x \\ \downarrow & & \downarrow \\ W_x & \longrightarrow & W_x/R_x \end{array} \qquad \begin{array}{ccc} Y|W_x & \longrightarrow & Y|W_x/S_x \\ \downarrow & & \downarrow \\ W_x & \longrightarrow & W_x/R_x \end{array}$$

are pullbacks with the right hand maps local homeomorphisms and over W_x, if

$$\langle z_1, z_2\rangle \in T_x \quad \text{then} \quad \langle q(z_1), q(z_2)\rangle \in S_x \,..$$

Thus we obtain

$$\begin{array}{ccc} Z|W_x & \longrightarrow & Z|W_x/T_x \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ Y|W_x & \longrightarrow & Y|W_x/S_x \\ \downarrow & & \downarrow \\ W_x & \longrightarrow & W_x/R_x \end{array}$$

It follows that q' is a local homeomorphism and since the whole square is a pullback and the bottom one is, then so is the top square. Thus t is an s-structure on Z over Y. ◊

We shall now look more closely at globally coherent local equivalence relations. If r is globally coherent and $R = glob(r)$, then we can think of r as being locally defined by R at every point. We obtain a functor

$$q^* : sh(X/R) \longrightarrow sh(X; r)$$

by pulling back along the canonical quotient map $q : X \to X/R$. If $q : Z \to X/R$ is a sheaf, define $q^*(Z)$ by

$$\begin{array}{ccc} q^*(Z) & \longrightarrow & Z \\ \downarrow & \text{pullback} & \downarrow{\scriptstyle p} \\ X & \xrightarrow{q} & X/R \end{array}$$

On $q^*(Z)$ define an equivalence relation S by

$$\langle x_1, z_1\rangle \sim \langle x_2, z_2\rangle \mod S \quad \text{iff} \quad z_1 = z_2 \text{ and } \langle x_1, x_2\rangle \in R.$$

With this relation $q^*(Z)$ becomes an r-sheaf. If R were open, it could be shown that $q^*(Z)/S \simeq Z$ in $sh(X/R)$.

Let us now look at an r-sheaf for globally coherent r. Let $p : Y \to X$ be an r-sheaf with r-structure s. We can find open neighborhoods V of x in X such that s is locally defined by equivalence relations S_x on $Y|V_x$ and we have the following pullback conditions for each $x \in X$,

(*)
$$\begin{array}{ccc} Y|V_x & \longrightarrow & Y|V_x/S_x \\ \downarrow & & \downarrow \\ V_x & \longrightarrow & V_x/R \end{array}$$

with the right hand map a local homeomorphism.

If R is an open equivalence relation with quotient map $q: X \to X/R$, we can identify V_x/R with the open set $q(V_x)$ in X/R and thus view $Y|V_x/S_x$ as a sheaf over X/R. Let us denote this sheaf by Z_x for $x \in X$.

The following technical lemma gives some insight into the nature of an r-sheaf in this special context.

Lemma 3.6. *Let r be globally coherent with $R = glob(r)$ and R an open equivalence relation with quotient map $q: X \to X/R$. Let $p: Y \to X$ be an r-sheaf with r-structure s locally defined by equivalence relations S_x over open neighborhoods V_x of x in X such that the pullback conditions (*) hold for each x in X. Then, if $x, x' \in X$ have non-empty fibers in Y and $\langle x, x'\rangle \in R$ there is an open neighborhood W of $[x]$ in X/R such that over W, $Z_x = Z_{x'}$.*

Proof. If $\langle x, x'\rangle \in R$, then since r is globally coherent (by Proposition 1.6 (d)), $R = R_{\underline{V}}$, the equivalence relation generated by $\{R|V_x\}_{x\in X}$ and therefore we can find $x_1, \ldots, x_{n-1}, a_1, \ldots, a_n$ in X such that

$$\langle x, x_1\rangle \in R|V_{a_1}, \qquad \langle x_1, x_2\rangle \in R|V_{a_2}, \quad \ldots, \quad \langle x_{n-1}, x'\rangle \in R|V_{a_n}.$$

Let $y \in Y$ with $p(y) = x$. Since $y \in Y|V_{a_1}$, applying the pullback condition (*), there is a unique

$$y_1 \in Y|V_{a_1} \quad \text{with} \quad p(y_1) = x_1 \quad \text{and} \quad \langle y, y_1\rangle \in S_{a_1}.$$

Similarly, using the pullback conditions (*) we get a unique

$$y_2 \quad \text{with} \quad p(y_2) = x_2 \quad \text{and} \quad \langle y_1, y_2\rangle \in S_{a_2},$$

$$\cdots\cdots$$

$$y_{n-1} \quad \text{with} \quad p(y_{n-1}) = x_{n-1} \quad \text{and} \quad \langle y_{n-2}, y_{n-1}\rangle \in S_{a_{n-1}}$$

and finally

$$y' \quad \text{with} \quad p(y') = x' \quad \text{and} \quad \langle y_{n-1}, y'\rangle \in S_{a_n}.$$

Since

$$y \in Y|V_x \cap V_{a_1}, \quad y_1 \in Y|V_{a_1} \cap V_{a_2}, \quad \ldots, \quad y' \in Y|V_{a_n} \cap V_{x'},$$

by the local compatibility of the equivalence relations defining s, we can find open sets $W_0, \ldots, W_n$ in Y with

$$y \in W_0, \quad y_1 \in W_1, \quad \ldots, \quad y_{n-1} \in W_{n-1}, \quad y' \in W_n$$

such that

$$S_x = S_{a_1} \text{ on } W_0, \quad S_{a_1} = S_{a_2} \text{ on } W_1, \quad \ldots, \quad S_{a_n} = S_{x'} \text{ on } W_n.$$

Since p is an open map and so is q, $q(p(W_i))$ is open in X/R for all $i = 0, \ldots, n$ and

$$[x_i] \in \bigcap_{i=0}^{n} q(p(W_i)). \qquad \text{Let} \quad W = \bigcap_{i=0}^{n} q(p(W_i)).$$

W is open in X/R, $[x] \in$ W and over W,

$$Z_x = Z_{a_1} = Z_{a_2} = \dots = Z_{a_n} = Z_{x'} \;.$$

Note that for the element y' obtained above with $p(y') = x'$, we have that over W, $Z_x = Z_{x'}$ and the equivalence classes $[y] = [y']$. The pullback conditions (*) guarantee that y' is the unique element in the fiber of x' with this property. ◊

We record this last observation as a Corollary.

Corollary 3.7. *Let r be globally coherent with $R = glob(r)$ and R is an open equivalence relation. Let $p : Y \to X$ be an r-sheaf with r-structure s locally defined by equivalence relations, S_x over open neighborhoods V_x of $x \in X$ with the pullback conditions (*) holding for each $x \in X$. Then, if*

$$\langle x, x'\rangle \in R \quad \text{and} \quad y \in Y \quad \text{with} \quad p(y) = x,$$

there is a unique $y' \in Y$ with $p(y') = x'$ such that over some open neighborhood W of $[x]$ in X/R we have

$$Z_x = Z_{x'} \qquad \text{and} \qquad [y] = [y'] \;.$$

We now obtain a theorem about globally coherent r, which will be very useful in §4. More general results about local equivalence relations relating them to spatial topoi and étendues can be proved [15], but the following will be sufficient for our purposes.

Theorem 3.8. *Let r be globally coherent and let $R = glob(r)$ be such that the canonical quotient map $q : X \longrightarrow X/R$ locally admits sections. Then, we have that the following categories are equivalent :*

$$sh(X\;;\;r) \simeq sh(X\;;\;R) \simeq sh(X/R).$$

Proof. We know

$$sh(X;\,R) \simeq sh\,(X/R)$$

by Theorem 2.6. From the discussion preceding Lemma 3.6, it is clear that because of the above equivalence, every R-sheaf can be viewed as an r-sheaf. It remains to show that every r-sheaf can be made into an R-sheaf. If $p : Y \to X$ is an r-sheaf with an r-structure s, we must produce an R-action $\alpha : R \times_{\pi_2} Y \to Y$ making the following diagram commute

$$\begin{array}{ccc} R \times_{\pi_2} Y & \xrightarrow{\ \alpha\ } & Y \\ \downarrow & & \downarrow p \\ R & \longrightarrow & X \end{array}$$

(see definition 2.2 and the ensuing discussion). Since q locally admits sections, it is an open mapping. If

$$(\langle x', x\rangle, y) \in R \times Y, \quad \text{i.e.,} \quad \langle x', x\rangle \in R \text{ and } p(y) = x,$$

let $y' \in Y$ be the unique element with $p(y') = x'$ obtained in Corollary 3.7. Let

$$\alpha(\langle x', x\rangle, y) = y'.$$

From the proof of Lemma 3.6 it follows that this will be an R-action on Y and after a moment's reflection one can see that this will yield our desired equivalence $sh(X\,;\,r) \simeq sh(X\,;\,R)$. ◊

We do wish to state here without proof a result from [15]. The proof is quite lengthy and cumbersome and is best omitted, as it would not shed any light on the topics discussed in §4.

Theorem 3.9. *Let r be a strictly open local equivalence relation on X. Then, $sh(X\,;\,r)$ is a topos.*

4. Strictly regular r and the groupoid Γ^r.

In this section, after defining the class of strictly regular local equivalence relations, we shall show that for these r, $sh(X\,;\,r)$ is an étendue and we shall explicitly describe an étale topological groupoid Γ^r (as guaranteed by Theorem 2.5), such that

$$sh(X\,;\,r) \simeq sh(X\,;\,\Gamma^r).$$

The construction of Γ^r is essentially that of the holonomy groupoid of a differentiable piece of a groupoid as defined by Pradines [13] and described by Brown [2]. In the case where r is the local equivalence relation of a foliated manifold, the Pradines construction gives the holonomy groupoid of the foliation.

We shall first describe Pradines'Theorem and sketch its proof following Brown. This will motivate our definition of strictly regular r and gives us our étale topological groupoid Γ^r. We then show that Γ^r has an r-structure s making it an r-sheaf and that

$$sh(X\,;\,r)/(\Gamma^r\,;\,s) \simeq sh(\Gamma^r\,;\,s) \simeq sh(X)$$

from which we draw our desired conclusions.

First, we need some notation : let

$$G \underset{\beta}{\overset{\alpha}{\rightrightarrows}} X$$

be a topological groupoid. $\varepsilon : X \to G$ denotes the common section of α and β. $G \underset{\alpha}{\times} G$ denotes the pullback of α along itself and $\delta : G \underset{\alpha}{\times} G \to G$ is the map

$$\delta(<f, g>) = f^{-1}g$$

(note, we are writing composition algebraically). For the following, let us suppose X is a manifold.

Definition 4.1. A *differentiable piece of a groupoid* is a pair (G, W) such that :

(i) W contains $\varepsilon(X)$ and generated G algebraically.

(ii) W has a manifold structure and $\varepsilon : X \to W$ gives X as a regularly embedded submanifold.

(iii) $\alpha | W : W \to X$ is a submersion (i.e., locally admits sections ; see Definition 2.6 and [1]).

(iv) $V = (W \underset{\alpha}{\times} W) \cap \delta^{-1}(W)$ is open in $W \times W$ and $\delta : V \to W$ is differentiable.

(G, W) is *α-connected* if $\alpha^{-1}(x) \cap W$ is connected for all $x \in X$.

Theorem 4.2. *(Pradines' Holonomy Theorem [13,2]) Let* (G, W) *be an α-connected piece of a differentiable groupoid. Then, there is an α-connected differentiable groupoid* Q(G, W), *and a groupoid morphism*

$$\varphi : Q(G, W) \to G$$

such that :

(1) φ is the identity on objects and $\varphi : \varphi^{-1}(W) \to W$ *is differentiable.*

(2) If A *is an α-connected differnetiable groupoid and* $\xi : A \to G$ *is a groupoid morphism, which is the identity on objects and with* $\xi : \xi^{-1}(W) \to W$ *differentiable, there is a unique morphism*

$$\xi' : A \to Q(G, W) \qquad \textit{with} \qquad \varphi\xi' = \xi.$$

This groupoid has a sheaf topology over X and hence is an étale topological groupoid. We shall be interested in general topological spaces and shall sketch the construction of the holonomy groupoid (as it appears in Brown [2]), however we shall omit the differentiability assumptions. Let

$$G \overset{\alpha}{\underset{\beta}{\rightrightarrows}} X$$

be a topological groupoid, where X is a topological space. Let W be as in Definition 4.1, deleting (ii) and replacing "differentiable" in (iv) by continuous.

Definition 4.3. Let U be open in X. $\sigma : U \to G$ is an *invertible local sec-*

tion of α iff $\alpha \circ \sigma = id_U$ and $\beta(\sigma(U)) = U'$ is open in X and homeomorphic to U.

If $\sigma : U \to G$ is as above and $\tau : U' \to G$ is another local invertible section, define

$$\sigma\tau : U \to G \quad \text{by} \quad (\sigma\tau)(x) = \sigma(x)\tau(\beta(\sigma(x))).$$

Let Γ be the sheaf of germs of local invertible sections of α. The above composition makes Γ into an étale topological groupoid over X. Let $\Gamma(W)$ denote the subsheaf of germs of sections $\sigma : U \to G$ with $\sigma(U) \subset W$. Let Γ' be the subgroupoid of Γ generated by $\Gamma(W)$. This will be a subsheaf and there is a map

$$\xi : \Gamma \to G \quad \text{given by} \quad \xi(\sigma_x) = \sigma_x(x).$$

If we let

$$\Gamma_0 = \Gamma(W) \cap \ker \xi ,$$

then Γ_0 will be a normal subgroupoid of Γ'. Define

$$Q(G, W) = \Gamma'/\Gamma_0.$$

It is a topological groupoid and still has a sheaf topology over X. One should observe that

$$\sigma_x \sim \tau_x \mod \Gamma_0 \text{ iff } \sigma_x(x) = \tau_x(x) \text{ and } \sigma\tau^{-1} \text{ locally has values in W.}$$

This type of construction for topological groupoids appears in a more general form for topological categories in [8]. In that paper, Ehresmann constructs categories of local sections of a topological category and then develops the idea of prolongations of topological categories. These ideas are utilized in some of his subsequent work [9, 10]. Recalling that the motivating example is that of a foliation on a manifold, these constructions, together with the holonomy groupoids of general foliations, which Ehresmann defined in [6], serve as a backdrop for Pradines' Holonomy Theorem and the special case of topological groupoids and invertible local sections.

Motivated by this construction, we now turn our attention to local equivalence relations and make the appropriate definitions necessary to be able to apply the preceding.

Definition 4.4 Let r be a local equivalence relation. A family $(R_x, W_x)_{x \in X}$ is called *r-adaptable* iff

(1) $R_x \in Equiv(W_x)$, with W_x an open neighborhood of x in X, and these locally define r.

(2) R_x had connected equivalence classes.

(3) $glob(r)$ = equivalence relation generated by $\{R_x\}_{x \in X}$

Proposition 4.5. *Let r be totally coherent. Then, r admits an r-adaptable family.*

Proof. Suppose r is locally defined by $S_x \in Equiv(U_x)$. By coherence, there is an open cover

$$\{W_x\}_{x \in X} \quad \text{with} \quad x \in W_x \subset U_x$$

such that

$$glob(r) = S_{\underline{W}} = \text{the equivalence relation generated by } \{S_x | W_x\}_{x \in X} .$$

By Proposition 1.6 (b), $r|W_x$ is globally coherent, hence

$$r|W_x = loc(glob(r|W_x)).$$

As in corollary 1.9, we can conclude that $glob(r|W_x)$ has connected equivalence classes. Let $R_x = glob(r|W_x)$. Since $glob(r) = S_{\underline{W}}$ and in some neighborhood of x, we have $S_x = R_x$, it follows that (3) above will also be satisfied for (R_x, W_x). ◊

Definition 4.6. (i) A local equivalence relation r is *regular* iff it is totally coherent and has an r-adaptable family $(R_x, W_x)_{x \in X}$ such that $\pi_1 : R_x \to W_x$ (the first projection) locally admits invertible sections for each $x \in X$.
(Call such an r-adaptable family *regular.*)

(ii) r is *strictly regular* iff it is regular and there is a regular r-adaptable family $(R_x, W_x)_{x \in X}$ such that if

$$W = \bigcup_{x \in X} R_x \quad \text{and} \quad V = (W \underset{\alpha}{\times} W) \cap \delta^{-1}(W),$$

then V is open in W x W. Note that

$$V = \{(\langle x_1, x_2\rangle, \langle x_1, x_3\rangle) \in W \times W \mid \langle x_2, x_3\rangle \in W\} .$$

If r is regular, it is open and hence strictly open. Furthermore, if $(R_x, W_x)_{x \in X}$ is a regular r-adaptable family, then by Theorem 2.6 the topos of R_x-sheaves, $sh(W_x ; R_x)$, is spatial and equivalent to $sh(W_x/R_x)$.

Suppose r is strictly regular and $(R_x, W_x)_{x \in X}$ is a regular r-adaptable family as in Definition 4.5 (2). Let

$$R = glob(r) \quad \text{and} \quad W = \bigcup_{x \in X} R_x .$$

Then, (R, W) will be α-connected, since if

$$R \underset{\beta}{\overset{\alpha}{\rightrightarrows}} X$$

denote the projections onto X, then $\alpha^{-1}(x) \cap W$ is the union of all the equivalence classes of x in the various relations R_y . Since these are connected and have x in common, it follows that their union is connected. We can now apply the holonomy groupoid construction to (R, W). We shall denote Q(R, W) by Γ^r and $\rho : \Gamma' \to \Gamma^r$ will denote the quotient map. We shall call Γ^r the ***holonomy groupoid of r.***

We must now describe various properties of Γ' and Γ^r for r strictly regular. As notation, since elements of Γ^r are equivalence classes of germs of invertible sections, we shall denote them by $[\sigma_x]$ where this is the class of σ_x, i.e., $\sigma_x \in \rho^{-1}([\sigma_x])$. For (R, W), recall that $\Gamma(W)$ is the sheaf of germs of invertible sections, which locally are defined by sections having their values lie in W.

Lemma 4.7. *Let r be strictly regular and let* $(R_x, W_x)_{x \in X}$ *be a regular r-adaptable family satisfying Definition 4.5 (2). Let* $W = \bigcup_{x \in X} R_x$. *Suppose* σ_x *and* τ_y *are composable germs of local invertible sections with*

$$\sigma_x(x) = \langle x, y\rangle \qquad \textit{and} \qquad \tau_x(y) = \langle y, z\rangle .$$

If

$$\sigma_x \in \Gamma(W) \qquad \textit{and} \qquad \tau_y \in \Gamma(W) \qquad \textit{and} \qquad \langle x, z\rangle \in W ,$$

then $(\sigma\tau)_x \in \Gamma(W)$.

Proof. Choose U an open neighborhood of x so that $\sigma : U \to W$ defines σ_x and $\beta\sigma(U) = U'$ is homeomorphic to U and $\tau : U' \to W$ locally defines τ_y . Let

$$V = \{(\langle x_1, x_2\rangle, \langle x_1, x_3\rangle) \in W \times W \mid \langle x_2, x_3\rangle \in W\} .$$

It is open in $W \underset{\alpha}{\times} W$ and if we take the following pullback

$$\begin{array}{ccc} V' & \longrightarrow & V \\ \downarrow & & \downarrow \\ U' \times U' & \xrightarrow{(\sigma^{-1}, \tau)} & W \times W \end{array}$$

then V' is open in U' x U' and it is non-empty since

$$(\sigma^{-1}(y), \tau(y)) \in V$$

because $\langle x, y\rangle \in W$. Choose an open neighborhood S of x with $S \subset U$ such that

$$\beta\sigma(S) \times \beta\sigma(S) \subset V'.$$

Define a section $\gamma: S \to W$ to be the composite

$$S \xrightarrow{\Delta} S \times S \xrightarrow{(\beta\sigma,\ \beta\sigma)} \beta\sigma(S) \times \beta\sigma(S) \xrightarrow{(\sigma^{-1},\ \tau)} V \xrightarrow{\delta} W.$$

if $z \in S$,
$$\gamma(z) = \sigma(z).\tau(\beta\sigma(z)),$$

hence $\gamma = \sigma\tau$ on S. Thus, $\sigma\tau$ locally has values in W and $(\sigma\tau)_x \in \Gamma(W)$. ◊

Define $\Gamma_X \subset \Gamma(W)$ to be the subsheaf of germs of invertible sections, which are locally defined by sections having values in R_X, i.e., σ_z is in Γ_X iff it is defined by

$$\sigma : U \to R \quad \text{where} \quad U \subset W_X \quad \text{and } \sigma(U) \subset R_X .$$

Definition 4.8 (1) Let $A \subset \Gamma'|W_X$. A is Γ_X *-saturated* iff given $\sigma_z \in A$ and $\tau_y \in \Gamma_X$ with τ_y and σ_z composable (i.e.

$$\tau_y(y) = \langle y, z\rangle),$$

then $(\tau\sigma)_y \in A$.

(2) Let $A \subset \Gamma'|W_X$. The Γ_X-saturation of A,

$$\tilde{A} = \{ \gamma_y \in \Gamma' \mid \gamma_y = (\tau\sigma)_y \text{ where } \tau_y \in \Gamma_X \ , \ \tau_y(y) = \langle y, z\rangle \quad \text{and } \sigma_z \in A \}.$$

(By slight abuse of notation, we shall use the letter α and β for the domain and codomain maps of our various groupoids. It will always be clear from the context as to which groupoid we are referring.)

Proposition 4.9. *Let* A *be open in* Γ', *with* $A \subset \Gamma'|W_X$. *Then,* $\tilde{A}$, *the* Γ_X *-saturation of* A, *is open and* Γ_X*-saturated.*

Proof. We have the following pullback diagram

$$\begin{array}{ccc} \Gamma_X \times_{W_X} \Gamma'|W_X & \xrightarrow{\ m\ } & \Gamma'|W_X \\ \downarrow & & \downarrow \\ \Gamma_X & \xrightarrow{\ \alpha\ } & W_X \end{array}$$

where m denotes the multiplication in the groupoid Γ'. Since Γ_X is an étale groupoid over W_X, it follows that $\Gamma'|W_X$ is a Γ_X-sheaf. Also, m is a local homeomorphism and the proof can proceed similarly to Proposition 3.12 in [16] with suitable changes in notation. ◊

We shall now show that Γ^r has an r-structure s making it into an r-sheaf. We shall then prove that s is globally coherent on Γ^r and combining these facts with some results obtained in §3, we shall be able to conclude that *sh* (X ; r) is an étendue with Γ^r as its associated étale topological groupoid.

Theorem 4.10. *Suppose* r *is strictly regular on* X *and* Γ^r *is its holonomy groupoid with domain and codomain maps*

$$\Gamma^r \overset{\alpha}{\underset{\beta}{\rightrightarrows}} X.$$

Then $\alpha : \Gamma^r \to X$ *has an r-structure s so that* $(\Gamma^r\ ;\ s)$ *is an r-sheaf.*

Proof. Let $(R_x, W_x)_{x \in X}$ be a regular r -adaptable family satisfying the conditions of Definition 4.5 (2). As remarked earlier, existence of local sections gives us that

$$sh(W_x\ ;\ R_x) \simeq sh(W_x/R_x).$$

Thus, if we can show that $\Gamma^r|W_x$ is an R_x-sheaf, this will be sufficient to guarantee the pullback conditions (see §3) necessary for having an r-structure. Let $R_x \underset{\pi_2}{\times} \Gamma^r|W_x$ denote the pullback of

$$\alpha : \Gamma^r|W_x \to W_x \text{ along } \pi_2 : R_x \to W_x$$

(the second projection). We need to define an action

$$R_x \underset{\pi_2}{\times} \Gamma^r|W_x \xrightarrow{\psi} \Gamma^r|W_x$$

such that the following diagram commutes

$$\begin{array}{ccc} R_x \underset{\pi_2}{\times} \Gamma^r|W_x & \xrightarrow{\quad\psi\quad} & \Gamma^r|W_x \\ \downarrow & & \downarrow \alpha \\ R_x & \xrightarrow{\quad\pi_1\quad} & W_x \end{array}$$

Suppose

$$<x_1, x_2> \in R_x \quad \text{and} \quad [\sigma_{x_2}] \in \Gamma^r|W_x\ .$$

Let σ_{x_2} be a representative of $[\sigma_{x_2}]$. Choose a germ τ_{x_1} of a local invertible section $\tau : U \to R_x$ with $x_1 \in U$ open in W_x and $\tau_{x_1}(x_1) = <x_1, x_2>$. Define

$$\psi\,(<x_1, x_2>,\ [\sigma_{x_2}]) = [(\tau\sigma)_{x_1}]\ .$$

We must show that this action is independent of the choices of sections made. Suppose

$$\tau_{x_1} \sim \mu_{x_1} \quad \text{and} \quad \gamma_{x_2} \in [\sigma_{x_2}] \quad \text{i.e.,} \quad \gamma_{x_2} \sim \sigma_{x_2}\ .$$

Then

$$\mu_{x_1}(x_1) = <x_1, x_2>$$

and $\tau\mu^{-1}$ locally has values in W. Similarly,

$$\gamma_{x_2}(x_2) = \sigma_{x_2}(x_2)$$

and $\sigma\gamma^{-1}$ locally has values in W. We must show that $(\tau\sigma)_{x_1} \sim (\mu\gamma)_{x_1}$.

We have that

$$(\tau\sigma\gamma^{-1}\mu^{-1})_{x_1}(x_1) = \langle x_1, x_1 \rangle$$

and thus is in ker ξ. To see that it is locally defined by a section having values in W, we shall apply Lemma 4.7 twice. $\sigma\gamma^{-1}$ and τ both locally have values in W and

$$(\tau\sigma\gamma^{-1})_{x_1} = \langle x_1, x_2 \rangle \in W,$$

hence by Lemma 4.7, $\tau\sigma\gamma^{-1}$ locally has values in W. Since $\mu_{x_1} \sim \mu_{x_1}$, μ and μ^{-1} both locally have values in W and applying Lemma 4.6 we conclude that $\tau\sigma\gamma^{-1}\mu^{-1}$ locally has values in W. Thus, ψ does not depend on the choice of sections and $\Gamma^r|W_x$ is an R_x-sheaf for each $x \in X$. This defines an r-structure on Γ^r making it into an r-sheaf. ◊

(Recall that $\Gamma^r \underset{\beta}{\overset{\alpha}{\rightrightarrows}} X$ denote the structure maps of Γ^r as a groupoid.)

If s is the r-structure on Γ^r described above, then s is locally defined over W_x by equivalence relations S_x, where

$$[\sigma_{x_1}] \sim [\gamma_{x_3}] \mod S_x$$

iff :

(1) $\beta(\sigma_{x_1}(x_1)) = \beta(\gamma_{x_3}(x_3))$, i.e. if

$$\sigma_{x_1}(x_1) = \langle x_1, x_2 \rangle \quad \text{and} \quad \gamma_{x_3}(x_3) = \langle x_3, x_4 \rangle$$

then $x_2 = x_4$;

(2) there is $[\tau_{x_3}] \in \Gamma^r$ with

$$\tau_{x_3} \in F \;,\; \tau_{x_3}(x_3) = \langle x_3, x_1 \rangle \text{ and } (\tau\sigma)_{x_1} \in [\gamma_{x_3}] .$$

Let S be the global equivalence relation on Γ^r, whose equivalence classes are given by $\{\beta^{-1}(x)\}_{x \in X}$. (Note that since Γ^r is a groupoid, S could also be given by left multiplication in Γ^r, as

$$[\sigma_{x_1}] \sim [\gamma_{x_3}] \mod S$$

iff there is

$$\mu_{x_3} \in \Gamma \quad \text{with} \quad \mu_{x_3}(x_3) = \langle x_3, x_1 \rangle \quad \text{and} \quad (\mu\sigma)_{x_3} \in [\gamma_{x_3}].)$$

Clearly, we have that $s \leq loc(S)$. The following proposition will show that in fact $s = loc(S)$, i.e., there is an open cover of Γ^r such that on each piece of the cover, the relations locally defining s agree with the restrictions of S.

Proposition 4.11. *Let s be the r-structure on Γ^r described in Theorem*

4.9. Then, s = loc(S), where S is the equivalence relation on Γ^r with equivalence classes $\{\beta^{-1}(x)\}_{x \in X}$.

Proof. Let $[\gamma_{x_1}] \in \Gamma^r$. Suppose it is locally defined by a section

$$\gamma : U \to R \qquad \text{with} \qquad U \subset W_{x_1} .$$

V denotes the basic open set $\{\gamma_z \mid z \in U\}$ in Γ'. From Proposition 4.8, it follows that $\tilde{V}$, the Γ_{x_1}-saturation of V, is open in Γ'. Since the quotient map $\rho : \Gamma' \to \Gamma^r$ is open, $\rho(\tilde{V})$ is an open neighborhood of $[\gamma_{x_1}]$ in Γ^r. We shall show that on $\rho(\tilde{V})$, S coincides with S_{x_1}. Suppose

$$[\mu_{x_2}] \sim [\tau_{x_3}] \mod S$$

in $\rho(\tilde{V})$ with representatives $\mu_{x_2}, \tau_{x_3} \in \tilde{V}$. Then,

$$\beta(\mu_{x_2}(x_2)) = \beta(\tau_{x_3}(x_3)).$$

By the definition of Γ_{x_1}-saturation, it follows that

$$\mu_{x_2} = (\omega\gamma)_{x_2} \quad \text{and} \quad \tau_{x_3} = (\nu\gamma)_{x_3} \quad \text{with} \quad \omega_{x_2}, \nu_{x_3} \in \Gamma_{x_1} .$$

Let

$$\omega_{x_2}(x_2) = \langle x_2, z_2 \rangle \quad \text{and} \quad \nu_{x_3}(x_3) = \langle x_3, z_3 \rangle .$$

Then, $z_2, z_3 \in U$, since V is the saturation of

$$V = \{\gamma_z \mid z \in U\} .$$

Furthermore, $z_2 = z_3$, since γ is an invertible local section and

$$\beta(\gamma_{z_2}(z_2)) = \beta(\mu_{x_2}(x_2)) = \beta(\tau_{x_3}(x_3)) = \beta(\gamma_{z_3}(z_3)).$$

Now,

$$(\mu\tau^{-1})x_2 = ((\omega\gamma)(\gamma^{-1}\nu^{-1}))x_2 = (\omega\nu^{-1})x_2 .$$

From the proof of Lemma 4.7, it follows that $(\omega\nu^{-1})x_2 \in \Gamma_{x_1}$, since

$$(\omega\nu^{-1})_{x_2}(x_2) = \langle x_2, x_3 \rangle \in R_{x_1}$$

and each section locally has values in R_{x_1}. Thus,

$$((\omega\nu^{-1})\tau)x_2 = \mu_{x_2}$$

and therefore

$$[\mu_{x_2}] \sim [\tau_{x_3}] \mod S_{x_1} .$$

We have proven that

$$S_{x_1} | \rho(\tilde{V}) = S | \rho(\tilde{V}) \qquad \text{and so} \qquad s = loc(S),$$

since at any element of Γ^r, they locally agree on some open neighborhood of that element. ◊

We immediately obtain the following facts about the local equivalence relation s on Γ^r, which will allow us to prove our main result.

Proposition 4.12. *s is a globally coherent local equivalence relation on Γ^r.*

Proof. Since $s = loc(S)$, it suffices to show that S is coherent. Since (R, W) is α-connected (see page 20), it follows that Γ^r is α-connected from Pradines' Theorem. From this and the fact that Γ^r is a groupoid, we can conclude that $\beta^{-1}(x)$ is connected in Γ^r for each $x \in X$. But, these are the equivalence classes of S, so by proposition 1.7, S is coherent. ◊

Proposition 4.13. $sh(\Gamma^r; s) \simeq sh(X)$.

Proof. First, we observe that Γ^r/S is homeomorphic to X. It is easy to check that

$$\Gamma^r_{(2)} \underset{\pi_2}{\overset{\mu}{\rightrightarrows}} \Gamma^r \xrightarrow{\ \beta\ } X$$

is a coequalizer diagram, where $\Gamma^r_{(2)}$ is all composable elements of Γ^r, μ is the groupoid multiplication and π_2 is the second projection. The equivalence relation S on Γ^r is the image of

$$\Gamma^r_{(2)} \xrightarrow{(\mu,\ \pi_2)} \Gamma^r \times \Gamma^r$$

and thus, $\Gamma^r/S \simeq X$. Since β is a local homeomorphism, it locally admits section and that together with the fact that s is globally coherent (Proposition 4.11) allows us to apply Theorem 3.7 to obtain

$$sh(\Gamma^r; s) \simeq sh(\Gamma^r/S).$$

Hence, we have

$$sh(\Gamma^r; s) \simeq sh(X). \qquad ◊$$

Now, we can state our main result.

Theorem 4.14. *Let r be a strictly regular local equivalence relation on X. Then, the topos $sh(X; r)$ of r-sheaves is an étendue and*

$$sh(X; r) \simeq sh(X; \Gamma^r),$$

where Γ^r is the holonomy groupoid of r.

Proof. By Lemma 3.5, the objects of $sh(X; r)/(\Gamma^r; s)$ can be thought of as s-sheaves over Γ^r. Conversely, if (Y, t) is an s-sheaf over Γ^r, it follows from Propositions 4.12 and 4.13 that (Y, t) is isomorphic as an s-sheaf to a pullback $\Gamma^r \times_X Z$ of a sheaf $\rho: Z \to X$ along the map β:

$\Gamma^r \to X$. It is clear that the composite sheaf

$$\Gamma^r \underset{X}{\times} Z \longrightarrow \Gamma^r \xrightarrow{\alpha} X$$

inherits an r-structure from Γ^r making it into an r-sheaf. Thus, we can obtain an equivalence

$$sh(X\ ;\ r)/(\Gamma^r\ ; s) \simeq sh(\Gamma^r\ ;\ s).$$

Since $sh(\Gamma^r\ ;\ s) \simeq sh(X)$, by Proposition 4.13, we can conclude that $sh(X\ ;\ r)$ is an étendue. From the proof of Theorem 4.5 in [16] it follows that

$$sh(X\ ;\ r) \simeq sh(X\ ;\ \Gamma^r). \qquad \Diamond$$

Hopefully, the above shows that there is some connection between the notions of local equivalence relation and r-sheaf and the holonomy groupoid construction. This should give some insight into what motivated Grothendieck's thinking regarding the various rather technical definitions about local equivalence relations and their connections with the concept of an étendue.

Department of Mathematics
Union College
SCHENECTADY, N.Y. 12308
U. S. A.

References.

1. N. BOURBAKI, Variétés différentielles et analytiques (Fasc. de Rés.), Hermann, Paris 1967.
2. R. BROWN, Holonomy and monodromy groupoids, Univ. College of North Wales, Pure Math., Preprint n° 82-2, 1982.
3. R. BROWN, G. DANESH-NARUIE & J.P.L. HARDY, Topological groupoids II : Covering morphisms and G-spaces, Math. Nachr. 74 (1976), 143-156.
4. C. EHRESMANN, Structures locales, Ann. di Mate. Pura e appl. 36 (1954) ; re-published in "Charles Ehresmann : Oeuvres complètes et commentées", Partie I, Amiens, 1984.
5. C. EHRESMANN, Les connexions infinitésimales dans un espace fibré différentiable, Coll. Top. Bruxelles 1951, in "Charles Ehresmann : Oeuvres complètes et commentées", Partie I, Amiens, 1984.
6. C. EHRESMANN, Structures feuilletées, Proc. 5th Canad. Math. Cong. Montréal 1961 ; in "Charles Ehresmann : Oeuvres complètes et commentées", Partie II-2, Amiens 1982.
7. C. EHRESMANN, Catégories topologiques, catégories différentiables, Coll. Géom. Diff. Glob. Bruxelles, CBRM 1958 ; in "Charles Ehresmann : Oeuvres complètes et commentées", Partie I, Amiens 1984.
8. C. EHRESMANN, Catégories topologiques III, Indig. Math. 28-1, 1966 ; in "Charles Ehresmann : Oeuvres complètes et commentées", Partie II-2, Amiens 1982.
9. C. EHRESMANN, Prolongements des catégories différentiables, Cahiers Top. et Géom. Diff. VI, 1964 ; in "Charles Ehresmann : Oeuvres complètes et commentées" Partie I, Amiens 1984.
10. C. EHRESMANN, Propriétés infinitésimales des catégories différentiables, Cahiers Top. et Géom. Diff. IX-1, 1967 ; in "Charles Ehresmann : Oeuvres complètes et commentées", Partie I, Amiens, 1984.
11. A. GROTHENDIECK & J.L. VERDIER, Théorie des topos (SGA 4, Vol. 1), Lecture Notes in Math. 269, Springer (1972).
12. P.T. JOHNSTONE, Topos Theory, Academic Press, 1977.
13. J. PRADINES, Théorie de Lie pour les groupoïdes différentiables. Relations entre propriétés locales et globales, C.R.A.S. A 26 (1966), 907-910.
14. G. REEB, Sur la théorie générale des systèmes dynamiques, Ann. Inst. Fourier VI, Grenoble (1955), 89-116.
15. K. ROSENTHAL, Aspects of Etendues, Ph.D. Thesis, State Univ. of N.Y. at Buffalo, 1979.
16. K. ROSENTHAL, Etendues and categories with monic maps, J. Pure and Appl. Algebra 22 (1981), 193-212.
17. K. ROSENTHAL, Local equivalence relations, Top. and its Appl. 13 (1982), 167-176.
18. K. ROSENTHAL, Quotient systems in Grothendieck topoi, Caniers Top. et Géom. Diff. XXIII-4 (1982), 425-438.