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A CONVENIENT SETTING FOR DIFFERENTIAL GEOMETRY AND GLOBAL ANALYSIS II

by Peter MICHOR

RÉSUMÉ. On développe une théorie des variétés différentiables et des espaces fibrés vectoriels, où les courbes différentiables prennent la place des cartes et atlas, de sorte que la catégorie correspondante soit cartésienne fermée. Dans le cas de dimension finie, on retrouve les variétés usuelles.

CONTENTS.

Sections 1-4 have been published in Volume XXV-1.

5. PRE-VECTOR BUNDLES IN MORE DETAIL.

5.1. **Lemma.** *Let (E, p, M) be a pre-vector bundle, let $0_E : M \to E$ denote the zero-section, $0_E(x) = 0_x \in E_x$. Then $0 = 0_E$ is smooth.*

Proof. Consider the following diagram :

(a)

$$\begin{tikzcd}
 & S(\mathbf{R}, M) \times_M S_{const}(\mathbf{R}, E) \arrow[d, "Cart^E"] \\
S(\mathbf{R}, M) \arrow[ur, "{(Id, 0 \circ ev_0)}"] \arrow[r, "0_*"] \arrow[d, "\delta_0"'] & S(\mathbf{R}, E) \arrow[d, "\delta_0"] \\
TM \arrow[r, "T0"] \arrow[dr, "{(Id, 0 \circ \pi_M, 0 \circ \pi_M)}"'] & TE \arrow[d, "Dec^E"] \\
 & TM \times_M E \times_M E
\end{tikzcd}$$

So, 0 is S^1,

$$T(0) = (Dec^E)^{-1} \circ (Id,\ 0 \circ \pi_M, 0 \circ \pi_M)$$

is again S^1 by Theorem 4.13, so 0 is S^2 , so $T0$ is S^2, so 0 is S^3 and by recursion 0 is smooth. QED

5.2. Lemma. *Let (E, p, M), (F, q, N) be pre-vector bundles and consider a commuting diagram of the form*

$$\begin{tikzcd}
E \arrow[r, "f"] \arrow[d, "p"'] & F \arrow[d, "q"] \\
M \arrow[r, "g"] & N
\end{tikzcd}$$

such that g is smooth and f is fibrewise a C^∞-mapping between C^∞-complete bornological lcs and commutes with the parallel transports. Then f is smooth.

Proof. f commutes with the parallel transports means that, for all $g \in \mathbf{R}$, $f \in S(\mathbf{R}, M)$, we have

$$f|_{E_{c(0)}} \circ Pt^E(c, t) = Pt^F(g \circ c, t) \circ f|_{E_{c(0)}} : E_{c(0)} \to F_{gc(t)}.$$

Now consider the following diagram :

$$(a) \quad \begin{tikzcd}
S(\mathbf{R}, M) \underset{M}{\times} S_{const}(\mathbf{R}, E) \arrow[r, "g_* \times f_*"] \arrow[d, "Cart^E"] \arrow[ddd, bend right=60, "\delta_0 \times \tilde{\delta}_0"'] & S(\mathbf{R}, N) \underset{M}{\times} S_{const}(\mathbf{R}, F) \arrow[d, "Cart^F"] \arrow[ddd, bend left=60, "\delta_0 \times \tilde{\delta}_0"] \\
S(\mathbf{R}, E) \arrow[r, "f_*"] \arrow[d, "\delta_0"] & S(\mathbf{R}, F) \arrow[d, "\delta_0"] \\
TE \arrow[r, "Tf"] \arrow[d, "Dec^E"] & TF \arrow[d, "Dec^F"] \\
TM \underset{M}{\times} E \underset{M}{\times} E \arrow[r, "Tg \times (f \circ pr_2,\ d^V f \circ pr_{2,3})"'] & TN \underset{M}{\times} F \underset{M}{\times} F
\end{tikzcd}$$

The first line makes sense because f is fibrewise C^∞, so

$$f_* : C^\infty(\mathbf{R}, E) \to C^\infty(\mathbf{R}, F)$$

makes sense. The top quadrangle commutes :

$$Cart^F(g \circ c_1, f \circ c_2)(t) = Pt^F(g \circ c_1, t) \circ f(c_2(t))$$

$$= f \circ Pt^E(c_1, t)(c_2(t)) = (f \circ Cart^E(c_1, c_2))(t).$$

So the second line in the diagram makes sense. By $d^V f$ we mean the "vertical" derivative of f, given by

$$d^V f(v_x, w_x) = d(f|_E)(v_x)(w_x).$$

It is clear that the outermost quadrangle commutes. So f is S^1. Now consider the following diagram :

(b)

$$\begin{tikzcd}
TE \arrow[rrr, "Tf"] \arrow[ddd, "Dec^E"'] \arrow[dr] & & & TF \arrow[dl] \arrow[ddd, "Dec^F"] \\
& TM \arrow[r, "Tg"] & TN & \\
& & & \\
TM \underset{M}{\times} E \underset{M}{\times} E \arrow[uur] \arrow[rrr, "Tg \circ (f \circ pr_2, d^V f \circ pr_{2,3})"'] & & & TN \underset{M}{\times} F \underset{M}{\times} F \arrow[uul]
\end{tikzcd}$$

We see that

$$Tf : (TE, Tp, TM) \to (TF, Tq, TN)$$

is a fibre-respecting mapping which is fibrewise C^∞ and we claim that it commutes with the parallel transports : For

$$c \in S(\mathbf{R}, M), \quad t \in \mathbf{R} \quad \text{and} \quad (v_x, w_x) \in E \underset{M}{\times} E$$

where $x = c(0)$, we have

$$Pt^F(g \circ c, t) \circ d^V f(v_x, w_x) = Pt^F(g \circ c, t) \circ d(f|_{E_x})(v_x)(w_x) =$$
$$= d(Pt^F(g \circ c, t) \circ f|_{E_x})(v_x)(w_x) = d(f|_{E_{c(t)}} \circ Pt^E(c, t))(v_x)(w_x) =$$
$$= d(f|_{E_{c(t)}})(Pt^E(c, t).v_x)(Pt^E(c, t).w_x).$$

If we now take $c \in (\mathbf{R}, TM)$ with

$$c(0) = u_x \in T_x M \quad \text{and} \quad (u_x, v_x, w_x) \in TM \underset{M}{\times} E \underset{M}{\times} E,$$

then we have :

$$Dec^F \circ Tf \circ Pt^{(TE,Tp,TM)}(c, t) \circ (Dec^E)^{-1}(u_x, v_x, w_x) =$$
$$= Dec^F \circ Tf \circ (Dec^E)^{-1}(c(t), Pt^E(\pi \circ c, t).v_x, Pt^E(\pi \circ c, t).w_x) =$$
$$= (Tg \circ c(t), f \circ Pt^E(\pi \circ c, t).v_x, d^V f(Pt^E(\pi \circ c, t).v_x, Pt^E(\pi \circ c, t).w_x))=$$
$$= (Tg \circ c(t), Pt^F(g \circ \pi \circ c, t) \circ f(v_x), Pt^F(g \circ \pi \circ c, t) \circ d^V f(v_x, w_x)) =$$
$$= (Tg \circ c(t), Pt^F(\pi \circ Tg \circ c, t) \circ f(v_x), Pt^F(\pi \circ Tg \circ c, t) \circ d^V f(v_x, w_x)) =$$
$$= Dec^F \circ Pt^{(TF,Tq,TN)}(Tg \circ c, t) \circ (Dec^E)^{-1}(Tg \circ c(0), f(v_x), d^V f(v_x, w_x)) =$$
$$= Dec^F \circ Pt^{(TF,Tq,TN)}(Tg \circ c, t) \circ Tf \circ (Dec^E)^{-1}(u_x, v_x, w_x).$$

So Tf commutes with the parallel transports, so we may apply the argument above to show that Tf is S^1; but then f is S^2 and in the same token Tf is S^2, so f is S^3 and so on. QED

5.3. Corollary. *Let (E, p, M) be a pre-vector bundle. Then the fibre addition $+_p : E \times E \to E$ and the fibre scalar multiplication $m_p : \mathbf{R} \times E \to E$ are smooth.*

Proof. $+_p$ is fibrewise linear and continuous and commutes with the parallel transport, since the parallel transport is fibrewise linear. So by 5.2, $+_p$ is smooth. The same argument applies to

$$\begin{tikzcd}
\mathbf{R} \times E \arrow[rr, "m_p"] \arrow[dr, "p \circ pr_2"'] & & E \arrow[dl, "p"] \\
& M &
\end{tikzcd}$$

QED

5.4. Let F be a functor from the category BCS of bornological C^∞-complete lcs and linear mappings into BCS, of one or several variables, even infinitely many (but less than the least inaccessible cardinal), co- or contravariant.

Examples. Let V, W etc. denote objects in BCS.

$L(V, W)$, the space of continuous linear mappings, with the bornological topology described in §1, is a contra- covariant bifunctor.

$V \hat{\otimes} W$, the bornological projective tensor product, described in §1, is a co-covariant bifunctor.

The last two functors describe the cartesian closed category BCS.

$V' = L(V, \mathbf{R})$, the bornological dual space, is a contravariant functor.

$\overset{n}{\hat{\otimes}} V$, the n-fold bornological tensor product, is a covariant n-functor.

$\Lambda^n V$, the n-fold bornological exterior product, i.e., the closed subspace of all antisymmetric elements in $\overset{n}{\hat{\otimes}} V$, is a covariant n-functor.

Definition. A functor as described above is called a C^∞-functor, if for all objects the mappings

$$L(V, W) \to L(F(V), F(W)), \quad f \longmapsto F(f)$$

(here expressed for a functor F with one covariant variable) are C^∞ in the sense of §1, i.e., map C^∞-curves to C^∞-curves.

All the examples above are C^∞-functors, since the morphism-mappings are bounded multilinear mappings or polynomial mappings derived from bounded multilinear mappings.

C^∞-functors will play an important role for the theory of vector bundles. For pre-vector bundles just functors suffice.

5.5. Theorem. *Let F be a functor on the category BCS as described above in 5.4. Let (E^i, p_i, M) be pre-vector bundles over a fixed premanifold M, one for each variable of F. Then $(F((E^i)_{i\in I}), p, M)$ is a pre-vector bundle in a canonical way, where the fibre*

$$F((E^i)_{i\in I})_x = F((E^i_x)_{i\in I}).$$

Proof. For the sake of simplicity and clearness let us assume that F has two variables, one contra- and one covariant, $F(V, W)$, contravariant in V. Then we put

$$F(E^1, E^2) = \bigcup_{x\in M} F(E^1_x, E^2_x),$$

so each fibre is a bornological C^∞-complete space. So (VB1) holds. Now, define the parallel transport by

$$Pt^{F(E^1, E^2)}(c, t) := F(Pt^{E^1}(c, t)^{-1}, Pt^{E^2}(c, t)) :$$

$$F(E^1, E^2)_{c(0)} = F(E^1_{c(0)}, E^2_{c(0)}) \to F(E^1_{c(t)}, E^2_{c(t)}) = F(E^1, E^2)_{c(t)}.$$

Clearly we have

$$Pt^{F(E^1, E^2)}(c, 0) = F(Id, Id) = Id$$

and for $f \in C^\infty(\mathbf{R}, \mathbf{R})$:

$$Pt^{F(E^1, E^2)}(c, f(t)) = F(Pt^{E^1}(c, f(t))^{-1}, Pt^{E^2}(c, f(t))) =$$

$$= F((Pt^{E^1}(c \circ f, t) \circ Pt^{E^1}(c, f(0)))^{-1}, Pt^{E^2}(c \circ f, t) \circ Pt^{E^2}(c, f(0))) =$$

$$= F(Pt^{E^1}(c, f(0))^{-1} \circ Pt^{E^1}(c \circ f, t)^{-1}, Pt^{E^2}(c \circ f, t) \circ Pt^{E^2}(c, f(0))) =$$

$$F(Pt^{E^1}(c \circ f, t)^{-1}, Pt^{E^2}(c \circ f, t)) \circ F(Pt^{E^1}(c, f(0))^{-1}, Pt^{E^2}(c, f(0))) =$$

$$Pt^{F(E^1, E^2)}(c \circ f, t) \circ Pt^{F(E^1, E^2)}(c, f(0)). \qquad \text{QED}$$

5.6. Example. Consider the functor $C^\infty(\mathbf{R}, \ ) : BCS \to BCS$, assigning to each bornological space V the space $C^\infty(\mathbf{R}, V)$ of all C^∞-curves in V. This is a C^∞-functor. Let $(E, p, \mathbf{R})$ be a pre-vector bundle. Applying the functor $C^\infty(\mathbf{R}, \ )$ to this vector bundle we get the vector bundle

$$(S_{const}(\mathbf{R}, E), p, M)$$

with the parallel transport

$$Pt^{S_{const}(\mathbf{R}, E)}(c, t) = Pt\,(c, t)_* :$$
$$S_{const}(\mathbf{R}, E)_{c(0)} = C^\infty(\mathbf{R}, E_{c(0)}) \to C^\infty(\mathbf{R}, E_{c(t)}) = S_{const}(\mathbf{R}, E)_{c(t)}.$$

More generally, we may take any C^∞-complete bornological space instead of **R** to get the bundles $(S_{const}(V, E), p, M)$.

5.7. **Theorem.** *Let F be a functor on the category BCS as described in 5.4. Let (E^i, p_i, M_i) be pre-vector bundles, one for each variable of F. Then*

$$(F((E^i)_{i\in I}), (p_i), \prod_{i\in I} M)$$

is a pre-vector bundle in a canonical way, where for $x = (x_i)$ the fibre is given by $F((E^i))_x = F((E^i_{x_i}))$.

Proof. Let us assume that F is purely covariant here. Note first that the product $\prod_{i\in I} M_i$ is a premanifold by 4.1. $c \in S(\mathbf{R}, \prod_{i\in I} M_i)$ is given by

$$c = (c_i), \quad c_i \in S(\mathbf{R}, M_i) \text{ for all } i.$$

Define parallel transport by

$$Pt^{F((E^i))}(c, t) = F((Pt^{E^i}(c_i, t))),$$

then by the functor property of F it is clear that (VB2) is satisfied. (VB1) is clear by construction. If F has contravariant variables too, then we put $Pt^{E^i}(c\ , t)^{-1}$ into each contravariant variable. QED

5.8. **Example.** Let (E^i, p_i, M_i) be pre-vector bundles for $i = 1, 2$, and consider the functor $L(V, W)$. Then we get the vector bundle

$$(L(E^1, E^2), (p_1, p_2), M_1 \times M_2),$$

the fibre over (x_1, x_2) being given by

$$L(E^1, E^2)_{(x_1, x_2)} = L(E^1_{x_1}, E^2_{x_2}),$$

and the parallel transport being given by

$$Pt^{L(E^1, E^2)}((c_1, c_2), t).g = Pt^{E^2}(c_2, t) \circ g \circ Pt^{E^1}(c_1, t)^{-1} \in L(E^1_{c_1(t)}, E^2_{c_2(t)})$$

for $g \in L(E^1_{c_1(0)}, E^2_{c_2(0)})$.

5.9. **Lemma.** *Let (E, p, M) be a pre-vector bundle, then*

$$(E \underset{(p,M,pr_1\circ p)}{\times} L(E, E), (p, p), M \times M)$$

is a pre-vector bundle again, since it can be written as a pullback in the form

$$pr_1{}^* E \underset{M\times M}{\times} L(E, E)$$

(cf. 5.8, 4.3, 4.2). So it is a premanifold by 2.6. Then the mapping

$$ev : E \times_M L(E, E) \to E$$

is smooth.

Proof.

$$\begin{tikzcd}
E \times_{(p,M,pr_1 \circ p)} L(E, E) \arrow[r, "ev"] \arrow[d] & E \arrow[d, "p"] \\
M \times M \arrow[r, "pr_2"] & M
\end{tikzcd}$$

and in this fibration ev is fibrewise bilinear and continuous (note that on each fibre $E_x \times L(E_x, E_y)$ we have the bornologicalization of the product topology), so a C^∞-mapping. We claim that ev commutes with the parallel transports. For let $(c_1, c_2) \in S(\mathbf{R}, M \times M)$,

$$ev \circ Pt^{E \times_M L(E, E)}((c_1, c_2), t)(v, \lambda) =$$

$$= ev(Pt^E(c_1, t).v, Pt^E(c_2,t) \circ \lambda \circ Pt^E(c_1, t)^{-1}) =$$

$$= Pt^E(c_2, t) \circ \lambda \circ Pt^E(c_1, t)^{-1} \circ Pt^E(c_1, t).v =$$

$$= Pt^E(c_2, t)(\lambda.v) = Pt^E(pr_2 \circ (c_1, c_2), t) \circ ev(v, \lambda).$$

So we may use Lemma 5.2 to conclude that ev is smooth. The form of $T(ev)$ can be read off the diagrams in 5.2. QED

5. 10. **Lemma.** *If (E^i, p_i, M_i) are pre-vector bundles for $i = 1, 2, 3$, then:*

$$(L(E^1, E^2) \times_{M_2} L(E^2, E^3), (p_1, p_2, p_3), M_1 \times M_2 \times M_3)$$

is again a pre-vector bundle since we may write it in the form :

$$pr_{1,2}^* L(E^1, E^2)_{M_1 \times M_2 \times M_3} pr_{2,3}^* L(E^2, E^3).$$

Then the composition

$$L(E^1, E^2) \times_{M_2} L(E^2, E^3) \to L(E^1, E^3)$$

is smooth.

Proof.

$$\begin{tikzcd}
L(E^1, E^2) \times_{M_2} L(E^2, E^3) \arrow[r, "comp"] \arrow[d] & L(E^1, E^3) \arrow[d] \\
M_1 \times M_2 \times M_3 \arrow[r, "pr_{1,3}"] & M_1 \times M_3
\end{tikzcd}$$

commutes and in this fibration *comp* is fibrewise bilinear and continuous, so C^∞. A computation similar to that in the proof of 5.9 shows that *comp* commutes with the parallel transports. So by Lemma 5.2 *comp* is smooth. QED

5. 11. *The covariant derivative.* Let (E, p, M) be a pre-vector bundle, let $s : N \to E$ be a smooth mapping and put $f := p \circ s$. Then we have the following commutative diagram

(a)
$$\begin{tikzcd} & E \arrow[d, "p"] \\ N \arrow[ur, "s"] \arrow[r, "f"] & M \end{tikzcd}$$

Consequently s is called a *section* over f. From this we get the following commutative diagram :

(b)
$$\begin{tikzcd} & TE \arrow[d, "Tp"] \\ TN \arrow[ur, "Ts"] \arrow[r, "Tf"] \arrow[d, "\pi_N"] & TM \arrow[d, "\pi_M"] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

Definition. In the situation above the *covariant derivative of* s is defined by :

$$\nabla s := pr_3 \circ Dec^E \circ Ts : TN \to E.$$

Then for $u_x \in T_xN$ we have

$$Dec^E(Ts.u_x) = (Tf.u_x, s(x), \nabla s.u_x).$$

Of course $s : T_xN \to E_{f(x)}$ is linear and even continuous, since it is smooth, so maps C^∞-curves into C^∞-curves, so is bounded by §1. This notion of covariant derivative of course depends heavily on the parallel transport of E.

5.12. **Lemma.** *∇ has all the properties of the classical linear covariant derivative, as there are :*

1. Let s_1, $s_2 : N \to E$ be two sections over $f : N \to M$. Then

$$s_1 \underset{p}{+} s_2 : N \to E$$

is again a section over f by 5.2 and we have

$$\nabla(s_1 \underset{p}{+} s_2) = \nabla s_1 +_p \nabla s_2.$$

2. If s is a section over f and $g \in S(N, \mathbf{R})$, *then* $g.s = m_p(s, g)$ *is again a section over f by 5.3 and we have*

$$\nabla(g.s) = dg.s + g.\nabla s, \quad \text{where} \quad dg = pr_2 \circ Tg : TN \to T\,\mathbf{R} = \mathbf{R}\times\mathbf{R} \to \mathbf{R}\ .$$

3. $\nabla s : TN \to E$ *induces a continuous linear mapping* $T_xN \to E_{f(x)}$ *for each* $x \in N$.

Proof. 1. Writing $Dec = Dec^E$, we have

$$\nabla(s_1 + s_2) = pr_3 \circ Dec \circ T(s_1 + s_2) = pr_3 \circ Dec \circ T(+_p) \circ T(s_1, s_2)$$

$$= pr_3 \circ (Id \times (+_p) \times (+_p)) \circ Dec^{E\times E}_M \circ T(s_1, s_2)$$

$$\underset{4.7}{=} (+_p) \circ (pr_3 \times pr_3) \circ (Dec \times Dec)(Ts_1, Ts_2) \quad = \nabla s_1 + \nabla s_2.$$

2. $$\nabla(g.s) = pr_3 \circ Dec \circ T(g.s) = pr_3 \circ Dec \circ T(m_p \circ (s, g))$$

$$= pr_3 \circ Dec \circ T(m_p) \circ (Ts, Tg) \underset{5.3}{=} pr_3 \circ A \circ (Dec \times Id) \circ (Ts, Tg)$$

$$\underset{5.11}{=} pr_3 \circ A(Tp \circ Ts, s \circ \pi_N, \nabla s, g \circ \pi_N, dg)$$

$$\underset{5.3}{=} pr_3 \circ (Tp \circ Ts, (g.s) \circ \pi_N, (g \circ \pi_N).s + dg.(s \circ \pi_N))$$

$$= (g \circ \pi_N).s + dg.(s \circ \pi_N) = g.\nabla s + dg.s$$

for short. Here, A is given by $Dec \circ T(m_p) = A \circ (Dec \times Id)$. See 5.2 for the form of A .

3 has already been proved in 5.11. QED

5. 13. Let $c \in S(\mathbf{R}, E)$, then

$$c = Cart(c_1, c_2) \quad \text{for} \quad (c_1, c_2) \in S(\mathbf{R}, M) \times S_{const}(\mathbf{R}, E)$$

i.e.,

$$c(t) = Pt(c_1, t).c_2(t).$$

Let $\partial/\partial t = (t, 1)$ denote the unit tangent vector at t in $\mathbf{R}$.

Lemma.

$$\nabla c.(\frac{\partial}{\partial t}) = \nabla(Cart(c_1, c_2))(\frac{\partial}{\partial t}) = Pt(c_1, t).\frac{d}{dt}c_2(t) = Cart(c_1, c_2')(t).$$

Proof.

$$\nabla c.(\frac{\partial}{\partial t}) = \nabla(Cart(c_1, c_2))(\frac{\partial}{\partial t}) =$$

$$= pr_3 \circ Dec \circ T(Cart(c_1, c_2)) \circ \delta_t(Id_{\mathbf{R}}) = pr_3 \circ Dec \circ \delta_t \circ Cart(c_1, c_2)_*(Id_{\mathbf{R}}) =$$

$$= pr_3 \circ Dec \circ \delta_t \circ Cart(c_1, c_2) =$$

$$\underset{2.12}{=} pr_3(\delta_t c_1, Pt(c_1, t).c_2(t), Pt(c_1, t).\frac{d}{dt}c_2(t))$$

$$= Pt(c_1, t).c_2'(t) = Cart(c_1, c_2')(t).$$

QED

5.14. **Lemma.** *If $s : N \to E$ is a section over $f : N \to M$ and $g : P \to N$ is a smooth mapping, then $\nabla(s \circ g) = \nabla s \circ Tg$.*

Proof.

$$\nabla(s \circ g) = pr_3 \circ Dec \circ T(s \circ g) = pr_3 \circ Dec \circ Ts \circ Tg = \nabla s \circ Tg.$$

QED

5.15. **Corollary.** *If $s : N \to E$ is a section over $f : N \to M$, then for any c in $S(\mathbb{R}, N)$, we have*

$$\nabla s.(\delta_0 c) = \frac{d}{dt}|_0(Pt(f \circ c, t)^{-1}.s(c(t))).$$

This is a convenient mean to compute covariant derivatives.

Proof.

$$\nabla s.(\delta_0 c) = \nabla s \circ Tc.(0, 1) = \nabla(s \circ c).(0, 1)$$

by 5.14 above. Put

$$c_2(t) := Pt(f \circ c, t)^{-1}.s(c(t)),$$

Then $c_2 : \mathbb{R} \to E_{f(c(0))}$. Since

$$\nabla Cart(f \circ c, c_2) = s \circ c \in S(\mathbb{R}, E),$$

c_2 is a C^∞-curve. Then by 5.13 we have :

$$\nabla(s \circ c).(0, 1) = \nabla(s \circ c).(\frac{\partial}{\partial t}|_0) = \nabla(Cart(f \circ c, c_2)).(\frac{\partial}{\partial t}|_0) =$$

$$\nabla Cart(f \circ c, \frac{d}{dt}c_2)(0) = \frac{d}{dt}c_2(0).$$

QED

5.16. **Theorem.** *Let (E, p, M) be a pre-vector bundle with parallel transport Pt and covariant derivative $\nabla = \nabla^E$. Then for any $c \in S(\mathbb{R}, M)$ and $v_x \in E_x$ with $x = c(0)$ the smooth curve*

$$t \mapsto Pt(c, t).v_x = Cart(c, const(v_x))(t)$$

in $S(\mathbb{R}, E)$ is the unique solution of the "ordinary differential equation"

$$p \circ s = c, \quad s(0) = v \ , \quad \nabla s = 0 = 0_E \circ c \circ \pi_{\mathbb{R}}$$

for $s \in S(\mathbb{R}, E)$.

Definition. Let us call a smooth section $s : N \to E$ over $f : N \to M$ *paral-*

lel with respect to Pt iff

$$\nabla s = 0 = 0_E \circ f \circ \pi_N;$$

then this theorem says that $t \mapsto Pt(c, t).v_x$ is the unique parallel section over c with initial value v_x.

Proof.

$$\nabla(Pt(c,\ ).v_x)(\frac{\partial}{\partial t}) = \nabla(Cart(c, const(v_x))(\frac{\partial}{\partial t}) =$$

$$= Cart(c, const(0_x))(t) = Pt(c, t).0_x = 0_{c(t)}$$

by 5.13. So $t \mapsto Pt(c, t).v_x$ is a solution. Now suppose that $s \in S(\mathbf{R}, E)$ is any other solution. Since $p \circ s = c$ we have $s = Cart^E(c, \bar{c})$ for unique $\bar{c} \in S(\mathbf{R}, E_x)$. But then we have

$$0_{c(t)} = \nabla s.(\frac{\partial}{\partial t}) = \nabla(Cart(c, \bar{c}))(\frac{\partial}{\partial t}) = Pt(c, t).\frac{d}{dt}\bar{c}(t)$$

by 5.13 again. But then $\frac{d}{dt}\bar{c}(t) = 0$ in E_x for all t, since each $Pt(c, t)$ is an isomorphism. So

$$\bar{c} = const = \bar{c}(0) = s(0) = v_x , \quad \text{so} \quad s(t) = Pt(c, t).v_x .$$

QED

Remark. This theorem might one lead to suspect that the condition

$$Pt(c, f(t)) = Pt(c \circ f, t) \circ Pt(c, f(0))$$

in (VB2) is equivalent to the following weaker condition :

$$Pt(c, t + s) = Pt(c(. + t), s) \circ Pt(c, t)$$

for all s , t in **R**. But we have used the stronger condition in the given proof of Theorem 2.6 in a very essential way (in 2.9 and 2.13) and the whole differential structure on E depends on this.

5.17. **Definition.** Let (E, p, M) be a pre-vector bundle. Denote by

$$\Gamma(E) = \Gamma(E, p, M)$$

the space of all smooth sections of p, i.e.,

$$(E, p, M) = \{s \in S(M, E) : p \circ s = Id_M\} .$$

Likewise denote by $\Gamma^r(E) = \Gamma^r(E, p, M)$ the space of all S^r-sections of p .

Lemma. *If (E, p, M) is a pre-vector bundle, then $\Gamma^1(E, p, M)$, the space of all S^1-sections of p, is a bornological C^∞-complete lcs in the point-*

linear structure and with a naturally given topology.

Proof. By Lemmas 5.1, 5.2, 5.3, the space $\Gamma^1(E)$ is a vector space in the pointwise linear topology. Now we put a topology on $\Gamma^1(E)$. For each $c \in S(\mathbf{R}, M)$ consider the linear mapping

$$B(c) : \Gamma^1(E) \to C^\infty(\mathbf{R}, E_{c(0)}) \quad \text{given by} \quad B(c)(s)(t) = Pt(c, t)^{-1}.s(c(t)).$$

Since $s \circ c$ is in $S(\mathbf{R}, E)$ and

$$Cart(c, B(c)(s)) = s \circ c$$

we see that $B(c)(s)$ is, indeed, a C^∞-curve in $E_{c(0)}$. Now equip $\Gamma^1(E)$ with the initial topology with respect to all mappings

$$B(c) : \Gamma^1(E) \to C^\infty(\mathbf{R}, E_{c(0)}) \quad \text{for all } c \in S(\mathbf{R}, M).$$

This topology is not bornological in general, so we take the associated bornological topology. The initial topology is locally convex since all the mappings $B(c)$ are linear and the topologies on the spaces $C^\infty(\mathbf{R}, E_{c(0)})$ are locally convex. It remains to show that $\Gamma^1(E)$ is C^∞-complete. Let s_n be a Mackey Cauchy sequence in $\Gamma^1(E)$. Then s_n is a Mackey Cauchy sequence in the initial topology too, since any locally convex topology and its bornologicalized topology have the same Mackey sequences (see §1). Since all the mappings $B(c)$ are bounded, they map s_n to Mackey sequences. For $c = const(x)$, $x \in M$,

$$B(c)(s_n) = const(s_n(x))$$

in E_x, so $s_n(x)$ is a Mackey Cauchy sequence in E_x and converges to some element $s(x)$ in E_x, because E_x is C^∞-complete. Clearly $s : M \to E$ is a section.

Claim. For $c \in S(\mathbf{R}, M)$, $s \circ c \in S(\mathbf{R}, E)$.

For $B(c)(s_n)$ is a Mackey Cauchy sequence in $C^\infty(\mathbf{R}, E_{c(0)})$, so it converges uniformly on compacts in $\mathbf{R}$, in each derivative separately, and it converges to

$$B(c)(s) = Pt(c, .)^{-1}.s(c(.))$$

since it converges to this limit in the weaker topology of pointwise convergence. So

$$B(c)(s) \in C^\infty(\mathbf{R}, E_{c(0)})$$

and the claim is proved.

$$Dec \circ \delta_0(s_n \circ c) = (\delta_0 \times \tilde{\delta}_0) \circ (Cart)^{-1}(s_n \circ c) =$$

$$= (\delta_0 \times \tilde{\delta}_0)(c, Pt(c, .)^{-1}(s_n \circ c)) = (\delta_0 \times \tilde{\delta}_0)(c, B(c)(s)) =$$

$$= (\delta_0 c, B(c)(s_n)(0), \frac{d}{dt}(B(c)(s_n))(0))$$

So this last expression depends only on $\delta_0 c$, since the sequence above depends only on $\delta_0 c$ (all s_n are S^1). But this means exactly that s is S^1, and an element of $\Gamma^1(E)$. By a standard argument $s_n - s$ converges to 0 in the initial topology and is even a Mackey sequence in the initial topology, so it is a Mackey sequence in the bornologicalized topology of $\Gamma^1(E)$; so $s_n - s$ converges to 0 in $\Gamma^1(E)$, so $s_n \to s$ in $\Gamma^1(E)$.

QED

5.18. **Lemma.** *If (E, p, M) is a pre-vector bundle, then the space $\Gamma(E, p, M)$ of all smooth sections of p is a bornological, C^∞-complete lcs in the pointwise linear structure and a canonically given topology.*

Proof. By 5.1 - 5.3, $\Gamma(E)$ is a vector space in the pointwise linear structure.

Claim. The mapping

$$T : \Gamma(E, p, M) \to \Gamma(TE, Tp, TM), \qquad s \mapsto Ts,$$

is linear and injective. Injective is clear.

$$Tp \circ Ts = T(p \circ s) = T(Id_M) = Id_{TM}, \quad \text{so} \quad Ts \in \Gamma(TE, Tp, TM).$$

Note that (TE, Tp, TM) is a pre-vector bundle, it is isomorphic (via Dec) to

$$(TM \times_M E \times_M E, pr_1 , TM) = \pi_M{}^*E \ {}_{T}\!\times_M \pi_M^*E = \pi_M{}^*(E \times_M E).$$

For any $s \in \Gamma(E, p, M)$ we have

$$Dec \circ Ts(u_x) = (u_x, s(x), \nabla s.u_x)$$

by 5.11. So

$$Dec \circ T(s_1 +_p s_2) = (Id_{TM}, (s_1 +_p s_2) \circ \pi_M, \nabla(s_1 +_p s_2)) =$$

$$= (Id_{TM}, s_1 \circ \pi_M + s_2 \circ \pi_M, \nabla s_1 + \nabla s_2) =$$

$$= (Id_{TM}, s_1 \circ \pi_M, \nabla s_2) +_{Tp} (Id_{TM}, s_1 \circ \pi_M, \nabla s_2) = Dec(Ts_1 +_{Tp} Ts_2)$$

by 5.2 (a) and the claim follows. Now consider the mappings

$$\Gamma(E, p, M) \xrightarrow{T^n} \Gamma(T^nE, T^np, T^nM) \hookrightarrow \Gamma^1(T^nE, T^np, T^nM),$$

which give a linear embedding

$$\Gamma(E, p, M) \to \prod_{n=1}^{\infty} \Gamma^1(T^nE, T^np, T^nM).$$

The latter space is a bornological C^∞-complete lcs by Lemma 5.17 above and the categorical properties of §1. We equip $\Gamma(E)$ with the subspace topology induced by this embedding. This need not be bornological in general, so we consider the bornologicalized topology. It remains to show that it is C^∞-complete. So let s_n be a Mackey-Cauchy sequence in $\Gamma(E)$; then it is a Mackey-Cauchy sequence in the weaker subspace topology, but this means that $T^m(s_n)$ is a Mackey-Cauchy sequence in

$$\Gamma^1(T^mE,\ T^mp,\ T^mM) \quad \text{for each} \quad m.$$

By 5.17 $T^m(s_n)$ converges to an element T^ms in $\Gamma^1(T^mE, T^mp, T^mM)$ for each m and from the proof of 5.17 it follows that

$$T(T^ms) = T^{m+1}s \quad \text{for all} \quad m.$$

So s is smooth, is in $\Gamma(E)$. By the same argument as in the end of the proof of 5.17 we see that s_n converges to s in $\Gamma(E)$. QED

5.19. **Lemma.** *Let $(E,\ p,\ M)$, $(F,\ q,\ M)$ be pre-vector bundles over the same premanifold M. Let $f : E \to F$ be a fibre respecting mapping which is fibrewise linear and continuous and which commutes with the respective parallel transports (i.e.,*

$$f \circ Pt(c,\ t) = Pt^F(c,\ t) \circ f,$$

for all c and t). Then the induced mapping

$$f_* : \Gamma(E,\ p,\ M) \to \Gamma(F,\ q,\ M)$$

is linear and continuous.

Proof. Let $c \in S(\mathbf{R}, M)$. Then $B(c) : \Gamma(F) \to C^\infty(\mathbf{R},\ F_x)$ is one of the generating mappings for the Γ^1-topology, where $x = c(0)$. For $s \in \Gamma(E)$ we have

$$(B(c) \circ f_*(s))(t) = Pt^F(c,\ t)^{-1} \circ f \circ s \circ c(t) =$$

$$= f \circ Pt(c,\ t)^{-1} \circ s \circ c(t) = (f_* \circ B(c))(s))(t),$$

where

$$f_* : C^\infty(\mathbf{R},\ E_x) \to C^\infty(\mathbf{R},\ F_x)$$

is clearly linear and continuous. So the following diagram commutes for all $c \in S(\mathbf{R},\ M)$:

$$\begin{tikzcd}
\Gamma^1(E) \arrow[r, "B(c)"] \arrow[d, "f_*"'] & C^\infty(\mathbf{R},\ E_x) \arrow[d, "(f|E_x)_*"] \\
\Gamma^1(F) \arrow[r, "B(c)"] & C^\infty(\mathbf{R},\ F_x)
\end{tikzcd}$$

This implies that $f_* : \Gamma^1(E) \to \Gamma^1(F)$ is continuous.

Now by Lemma 5.2, $f : E \to F$ is smooth and a glance at the diagrams in 5.2 shows that

$$\begin{tikzcd} TE \arrow[r, "Tf"] \arrow[d, "Tp"'] & TF \arrow[d, "Tq"] \\ TM \arrow[r, "Id"] & TM \end{tikzcd}$$

is again fibre respecting, fibrewise linear and continuous, and commutes with the respective parallel transports (given in 4.15). So by the first part of the proof we conclude that

$$(Tf)_* : \Gamma^1(TE, Tp, TM) \to \Gamma^1(TF, Tq, TM)$$

is linear and continuous. The following diagram obviously commutes :

$$\begin{tikzcd} \Gamma(E, p, M) \arrow[r, "T"] \arrow[d, "f_*"'] & \Gamma^1(TE, Tp, TM) \arrow[d, "(Tf)_*"] \\ \Gamma(F, q, M) \arrow[r, "T"] & \Gamma^1(TF, Tq, TM) . \end{tikzcd}$$

We can repeat the last argument and conclude by recursion that

$$f_* : \Gamma(E) \to \Gamma(F)$$

is continuous indeed. QED

5.20. **Corollary.** *Let (E, p, M) be a pre-vector bundle. Then the covariant derivative is a linear and continuous mapping*

$$\nabla : \Gamma(E, p, M) \to \Gamma(\pi_M^*E, \pi_M^*p, TM).$$

Proof. $\nabla s = pr_3 \circ Dec \circ Ts$, so

$$\nabla = (pr_3 \circ Dec)_* \circ T : \Gamma(E, p, M) \to \Gamma(TE, Tp, TM) \to \Gamma(\pi_M^*E, \pi_M^*p, TM).$$

T is linear and continuous by the definition of the topology on $\Gamma(E, p, M)$ in 5.18, $(pr_3 \circ Dec)_*$ is linear and continuous by Lemma 5.19 above, since

(a)
$$\begin{tikzcd} TE \arrow[r, "Dec"] \arrow[d, "Tp"] & TM \times_M E \times_M E \arrow[r, "pr_3"] \arrow[d, "pr"] & \pi_M^*E \arrow[d, "\pi_M^*p"] \\ TM \arrow[r, equal] & TM \arrow[r, equal] & TM \end{tikzcd}$$

is fibrewise linear and continuous and commutes with the respective parallel transports. Note that

$$(\pi_M{}^*E,\ \pi_M{}^*p,\ TM) = (TM \underset{M}{\times} E,\ pr_1,\ TM)$$

and the mapping pr_3 above is actually $pr_{1,3}$. This comes from the fact that $\nabla s : TM \to E$ is a section over π_M, but we look at ∇s as a section of the pre-vector bundle $\pi_M{}^*E$ using a universal pullback property. QED

5.21. Lemma. *Let (E, p, M), $(F, q,, N)$ be pre-vector bundles and let $f : P \to M$, $g : Q \to N$ be smooth mappings, where P, Q are premanifolds. Then we have an (parallel transport respecting) isomorphism of pre-vector bundles over $P \times Q$:*

$$L(f^*E, g^*F) = (f \times g)^*L(E, F) = P \times_M L(E, F) \times_M Q.$$

Proof. All three sets are pre-vector bundles over $P \times Q$, they coincide fibrewise, and they have the same parallel transports by 5.7. By Lemma 5.2 it follows that the identity mapping is then smooth. QED

5.22. Lemma. *Let*

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "p"'] & F \arrow[d, "q"] \\ M \arrow[r, "g"] & N \end{tikzcd}$$

be smooth mappings, f fibre linear, where (E, p, M) and (F, q, N) are pre-vector bundles. Define the mapping

$$\hat{f} : M \to L(E, F) \quad \text{by} \quad \hat{f}(x) = f|_{E_x} \in L(E_x, F_{g(x)}).$$

Then $\hat{f}$ is smooth.

Proof. Let

$$A(c)(t) := Pt^F(g \circ c, t)^{-1} \circ (f|_{E_{c(0)}}) \circ Pt(c, t)$$

for $c \in S(\mathbf{R}, M)$.

Claim. For $c \in S(\mathbf{R}, M)$, $A(c) \in S_{const}(\mathbf{R}, L(E, F))$.
By §1 it suffices to show that for each $v \in E_{c(0)}$ the mapping

$$t \mapsto A(c)(t)(v)$$

is in $C^\infty(\mathbf{R}, F_{gc(0)})$. But this is the case since we have :

$$A(c)(t)(v) = Pt^F(g \circ c, t)^{-1} \circ f \circ Pt(c, t).v =$$

$$= Pt^F(g \circ c, t)^{-1} \circ f \circ Cart(c, const(v))(t).$$

So the top triangle of the following diagram makes sense :

(a)

$$\begin{tikzcd}
 & S(\mathbf{R}, M\times N) \underset{M\times N}{\times} S_{const}(\mathbf{R}, L(E, F)) \arrow[d, "Cart^{L(E, F)}"] \arrow[ddd, bend left, "\delta_0 \times\tilde{\delta}_0"] \\
S(\mathbf{R}, M) \arrow[ur, "(Id_M \times g)_* \times A"] \arrow[r, "(\tilde{f})_*"] \arrow[d, "\delta_0"'] & S(\mathbf{R}, L(E, F)) \arrow[d, "\delta_0"] \\
TM \arrow[dr, dashed, "B"'] & TL(E, F) \arrow[d, "Dec^{L(E, F)}"] \\
 & (TM\ TN) \underset{M\times N}{\times} L(E, F) \underset{M\times N}{\times} L(E, F)
\end{tikzcd}$$

We claim that a mapping B fits commutingly into this diagram. Let $c \in S(\mathbf{R}, M)$. Then we have :

$$Dec^{L(E, F)} \circ \delta_0 \circ (\tilde{f})_*(c) = (\delta_0 \times\tilde{\delta}_0)((c, g \circ c), A(c)) =$$

$$= ((\delta_0 c, Tg.(\delta_0 c)), A(c)(0), \frac{d}{dt}|_0 A(c)(t)).$$

$$(\frac{d}{dt}|_0 A(c)(t))(v) = \frac{d}{dt}|_0 (A(c)(t)(v)) = \frac{d}{dt}|_0 (Pt^F(g \circ c, t)^{-1} \circ f \circ Pt(c, t).v)$$

$$\underset{5.15}{=} \nabla f.(\delta_0(Pt(c,).v) \underset{5.11\ \&\ 5.16}{=} \nabla f \circ (Dec)^{-1}(\delta_0 c, v, 0_{c(0)})$$

So

$$Dec^{L(E, F)} \circ \delta_0 \circ (\tilde{f})_*(c) =$$

$$= ((\delta_0 c, Tg.(\delta_0 c)), \tilde{f}(c(0)), \nabla f \circ (Dec)^{-1}(\delta_0 c, ., 0_{c(0)})),$$

which depends only on $\delta_0 c$. Now put

$$h := \nabla f \circ (Dec)^{-1} \circ (Id_{TM} \times Id_M, 0_E \circ p) : TM \underset{M}{\times} E = \pi_M{}^*E \to F,$$

then we have a commuting diagram

(b)

$$\begin{tikzcd}
TM \times_M E = \pi_M{}^*E \arrow[r, "h"] \arrow[d, "pr_1"'] & F \arrow[d, "q"] \\
TM \arrow[r, "g \circ \pi_M"] & N
\end{tikzcd}$$

of smooth mappings such that h is fibre linear in the fibration given by the diagram. We can write B as the following sequence of mappings :

$$\begin{tikzcd}
TM \arrow[r, "(\tilde{f}, \tilde{h}, Tg)"] & L(E, F) \underset{M\times N}{\times} L(\pi_M{}^*E, F) \underset{N}{\times} TN \arrow[d, equal, "\text{by Lemma 5.21 above}"] \\
 & L(E, F) \underset{M\times N}{\times} (TM \underset{M}{\times} L(E, F)) \times_N TN \arrow[d, equal] \\
 & (TM \times TN) \underset{M\times N}{\times} L(E, F) \underset{M\times N}{\times} L(E, F) .
\end{tikzcd}$$

So we see by diagram (a) that f is S^1. The mapping B (and so $T(f)$) is again S^1 since its components f, h are, so f is S^2 and so on. QED

5.23. **Corollary.** *Let* (E^i, p_i, M_i) *be pre-vector bundles for* $i = 1, 2, 3$. *Let*

$$\begin{tikzcd}
E^2 \arrow[r, "f"] \arrow[d, "p_2"'] & E^3 \arrow[d, "p_3"] \\
M_2 \arrow[r, "g"] & M_3
\end{tikzcd}$$

be smooth mappings with f fibre linear. Then the mapping

$$\begin{tikzcd}
L(E^1, E^2) \arrow[r, "{L(E^1, f)}"] \arrow[d] & L(E^1, E^3) \arrow[d] \\
M_1 \times M_2 \arrow[r, "Id \times g"] & M_1 \times M_3
\end{tikzcd}$$

is smooth.

Proof. The following diagram clearly commutes, so by Lemma 5.22 and 5.10 the mapping $L(E^1, f)$ is smooth.

$$\begin{tikzcd}
L(E^1, E^2) \arrow[r, "{(Id_{L(E^1, E^2)}, f \circ p_2)}"] \arrow[dr, "{L(E^1, f)}"'] & L(E^1, E^2) \times_M L(E^2, E^3) \arrow[d, "comp"] \\
 & L(E^1, E^3)
\end{tikzcd}$$

QED

5.24. **Lemma.** *1. Let* $(f, g) : N \to E \times_M L(E, E)$ *be a smooth mapping, where* N *is a premanifold and* (E, p, M) *is a pre-vector bundle. Then*

$$\nabla^E (g.f) = \nabla^{L(E, E)} g.f + g.\nabla^E f .$$

2. Let $(f, g) : N \to L(E^1, E^2) \times_{M_2} L(E^2, E^3)$ *be a smooth mapping, where* N *is a premanifold and* (E^i, p_i, M_i) *are pre-vector bundles for* $i = 1, 2, 3$, *then we have*

$$\nabla^{L(E^1, E^3)} (g.f) + (\nabla^{L(E^2, E^3)} g).f + g.(\nabla^{L(E^1, E^2)} f).$$

Proof. By 5.9 and 5.2 we have the following commutative diagram :

$$\begin{tikzcd}
T(E \times_M L(E, E)) \arrow[r, "T(ev)"] \arrow[d, "Dec^{E \times_M L(E, E)}"'] & TE \arrow[d, "Dec^E"] \\
(TM \times TM) \times_{M \times M} (E \times_M L(E, E)) \times_{M \times M} (E \times_M L(E, E)) \arrow[r, "h"] & TM \times_M E \times_M E
\end{tikzcd}$$

where

$$h = (pr_2,\ ev \circ pr_{3,4}, d^V(ev) \circ pr_{3,4,5,6})$$

and d^V is the vertical derivative. Thus

$$\nabla^E (ev) = pr_3 \circ Dec^E \circ T(ev) = d^V(ev) \circ pr_{3,4,5,6} \circ Dec^{E \times_M L(E,\ E)} .$$

So for

$$(a_x,\ b_y\ ;\ v_x,\ {}_xh_y\ ;\ w_x,\ {}_xk_y)$$

we have

$$(\nabla^E ev)(Dec^{E \times_M L(E,\ E)})^{-1} (a_x, b_y\ ;\ v_x, {}_xh_y\ ;\ w_x,\ {}_xk_y) =$$

$$= d^V(ev)(v_x,\ {}_xh_y\ ;\ w_x, {}_xk_y) = {}_xk_y(v_x) + {}_xh_y(w_x)$$

since ev is fibrewise bilinear and bounded. Now consider the smooth mappings :

$$\begin{tikzcd}
N \arrow[r, "{(f,\ g)}"] \arrow[dr, "{(h,\ k)}"'] & E \times_M L(E,\ E) \arrow[d] \\
 & M \times M
\end{tikzcd}$$

$$Dec^E \circ Tf = (Th,\ f \circ \pi_N,\ \nabla^E f) : TN \to TM \times_M E \times_M E$$

and

$$Dec^{L(E,\ E)} \circ Tg = (Th,\ Tk\ ;\ g \circ \pi_N,\ \nabla^{L(E,\ E)} g\ ) :$$

$$TN \to (TM \times_M TM) \times_{M \times M} L(E,\ E) \times_{M \times M} L(E,\ E).$$

So we can compute as follows :

$$\nabla^E (g.f) = \nabla^E(ev \circ (f,\ g)) = \nabla^E(ev) \circ T(f,\ g) \underset{5.14}{=}$$

$$= \nabla^E (ev) \circ (Dec^{E \times_M L(E,\ E)})^{-1} \circ Dec^{E \times_M L(E,\ E)} \circ T(f,\ g) =$$

$$= \nabla^E(ev) \circ (Dec^{E \times_M L(E,\ E)})^{-1} (Th,\ Tk\ ;\ f \circ \pi_N,\ g \circ \pi_N\ ;\ \nabla^E f,\ \nabla^{L(E,\ E)} g)$$

$$= (\ \nabla^{L(E,\ E)} g).(f \circ \pi_N) + (g \circ \pi_N).(\nabla^E f) = \nabla^{L(E,\ E)} g.f + g.\ \nabla^E f$$

in short hand. So assertion 1 is proved. Assertion 2 is completely similar.
QED

5.25. **Corollary.** *Let $f : N \to L(E,\ E)$ be a smooth mapping, N a premanifold, $(E,\ p,\ M)$ a pre-vector bundle. Suppose furthermore that $f(n)$ is invertible in $L(E_{p_1f(n)},\ E_{p_2f(n)})$ for each $n \in N$, and that the mapping*

$$inv\ f : N \to L(E,\ E), \quad inv\ f(n) = f(n)^{-1},$$

is again smooth. Then we have :

$$\nabla^{L(E,\ E)}(\text{inv } f) = -(\text{inv } f).(\nabla^{L(E,\ E)} f).(\text{inv } f),$$

or

$$\nabla^{L(E,\ E)}(\text{inv } f)(u_n) = -f(n)^{-1}.\ \nabla^{L(E,\ E)} f(u_n).f(n) \quad .$$

Proof.

$$(\text{inv } f)(n).f(n) = Id_{p_1 f(n)} = (Id_E)^{\sim}(p_1 g(n))$$

in the notation of 5.22.

$$\nabla^{L(E,\ E)}((Id_E)^{\sim} \circ p_1 \circ f)(u_n) \underset{5.14}{=} \nabla^{L(E,\ E)}(\tilde{Id_E}) \circ T(p_1 \circ f)(u_n) =$$

$$= \nabla^E(Id_E) \circ (Dec^E)^{-1}(T(p_1 \circ f)(u_n),\ .\ ,\ 0_{p_1 f(n)})$$

by the proof of 5.22; this is an element in $L(E_{p_1 f(n)}, E_{p_1 f(n)})$ which equals the constant mapping $0_{p_1 f(n)}$. So

$$\nabla^{L(E,\ E)}(\tilde{Id_E} \circ p_1 \circ f) = 0_{L(E,\ E)} \circ (p_1, p_2) \circ f.$$

So

$$0 = \nabla^{L(E,\ E)}((\text{inv } f).f) = \nabla^{L(E,\ E)}(\text{inv } f).f + (\text{inv } f).\ \nabla^{L(E,E)} f$$

by 5.24.2, and

$$\nabla^{L(E,\ E)}(\text{inv } f) = -(\text{inv } f).\nabla^{L(E,\ E)} f.(\text{inv } f).$$

QED

6. FIRST STEPS TOWARDS CARTESIAN CLOSEDNESS.

6.1. **Proposition.** *If M, N are premanifolds, then $S(M, N)$, the set of all smooth mappings from M to N, satisfies axioms (M1)-(M3) in a natural way.*

Proof. Put

(a)
$$\begin{tikzcd}
TS(M, N) \arrow[dr, "\pi_{S(M,\ N)}"'] & := & S(M, TN) \arrow[dl, "S(M,\ \pi_N) = (\pi_N)_*"] \\
& S(M, N) &
\end{tikzcd}$$

Then it remains to prove that

$$\pi_{S(M,\ N)}^{-1}(f) = (\pi_N)_*^{-1}(f) =$$

$$= \{ s \in S(M, TN) \mid \pi_N \circ s = f \} =: S_f(M, TN),$$

the space of all sections over f, is a bornological C^∞-complete lcs. But we have a canonical isomorphism

$$S_f(M, TN) = \Gamma(f^*TN, f^*\pi_N, M)$$

induced by the universal property of the pullback :

(b)
$$\begin{tikzcd} f^*TN \arrow[r, "\pi^*f"] \arrow[d, "f^*\pi_N"'] & TN \arrow[d, "\pi_N"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

$\Gamma(f^*TN, f^*\pi_N, M)$ is a bornological C^∞-complete lcs by 5.18, and we carry its structure over to $S_f(M, TN)$ - the pointwise linear structures coincide. So we have proved that (M1) holds.

(M 2) Let $S(\mathbf{R}, S(M, N))$ consist of all mappings $c : \mathbf{R} \to S(M, N)$ such that the associated mapping

$$\hat{c} : \mathbf{R} \times M \to N \quad \text{given by} \quad \hat{c}(t, m) = c(t)(m)$$

is in $S(\mathbf{R} \times M, N)$. So we have bijections :

$$S(\mathbf{R}, S(M, N)) \underset{\vee}{\overset{\wedge}{\rightleftarrows}} S(\mathbf{R} \times M, N)$$

given by

$$\hat{c}(t, m) = c(t)(m) \quad \text{and} \quad \check{g}(t)(m) = g(t, m).$$

For $f \in C^\infty(\mathbf{R}, \mathbf{R})$ we have

$$(g^\vee \circ f(t))(m) = g(f(t))(m) = g(f(t), m) = g \circ (f \times Id_M)(t, m),$$

so

$$c \circ f = (\hat{c} \circ (f \times Id_M)) \in S(\mathbf{R}, S(M, N)).$$

Here $\mathbf{R} \times M$ bears the premanifold structure of 4.1. $S(\mathbf{R}, S(M, N))$ contains all constant mappings, since for $g \in S(M, N)$ we have :

$$const(g) = (g \circ pr_2)\ , \quad \text{where} \quad g \circ pr_2 \in S(\mathbf{R} \times M, N).$$

(M3) For $t \in \mathbf{R}$ put

$$A_t : M \to T(\mathbf{R} \times M) = T\mathbf{R} \times TM = \mathbf{R}^2 \times TM, \quad A_t(m) = (t, 1\ ;\ 0_M),$$

or

$$A_t = (const(t, 1)\ ;\ 0_T) : M \to T\mathbf{R} \times TM.$$

Then $A_t \in S(M, T\mathbf{R} \times TM)$. Then define

$$\delta_t(c) := T\hat{c} \circ A_t \in S(M, TN) = TS(M, N),$$

for $c \in S(\mathbf{R}, S(M, N))$.

Claim : $\pi_{S(M,N)}(\delta_t c) = c(t)$.

$$\pi_{S(M,N)}(\delta_t c)(m) = (\pi_N)_*(\delta_t c)(m) = \pi_N \circ T\hat{c} \circ A_t(m) = \hat{c} \circ \pi_{\mathbf{R} \times M} \circ A_t(m)$$

$$= \hat{c}(t,\ m) = c(t)(m).$$

Claim : If $f \in C^{\infty}(\mathbf{R},\ \mathbf{R})$ and $c \in S(\mathbf{R},\ S(M,\ N))$, then

$$\delta_t(c \circ f) = f'(t).\delta_{f(t)}c.$$

$$(\delta_t(c \circ f))(m) = T((c \circ f)\hat{\ }) \circ A_t(m) = T(\hat{c} \circ (f \times Id_M))(t,\ 1\ ;\ 0_M) =$$

$$\doteq T\hat{c} \circ (Tf \times Id_{TM})(t,\ 1\ ;\ 0_M) = T\hat{c}(f(t),\ f'(t)\ ;\ 0_M) =$$

$$= T\hat{c}(f'(t).(f(t),\ 1\ ;\ 0_M)) = f'(t).T\hat{c}(f(t),\ 1\ ;\ 0_M) =$$

$$= f'(t).(T\hat{c} \circ A_{f(t)}(m)) = (f'(t).\delta_{f(t)}\ c)(m).$$

Claim : If $c \in S(\mathbf{R},\ S(M,\ N))$ with $\delta_t c = 0_{c(t)}$ for all t, then c is constant.

$$0_{c(t,\ m)} = (\delta_t\ c)(m) = T\hat{c} \circ A_t(m) = T\hat{c}(\delta_t(ins(m))),$$

$$= \delta_t\ (\hat{c} \circ (ins(m))),$$

where $ins\ (m) : \mathbf{R} \to \mathbf{R} \times M$ is given by $ins(m)(t) = (t,\ m)$. Now

$$\hat{c} \circ (ins(m)) \in S(\mathbf{R},\ N), \quad (\hat{c} \circ (ins(m)))(t) = \hat{c}(t,\ m).$$

So by (M3) for the premanifold N we conclude that

$$\hat{c} \circ (ins(m)) = const \text{ in } N, \quad \text{i.e., } \hat{c}(t,\ m) = \hat{c} \circ (ins(m))(t)$$

does not depend on t. So $c(t)$ does not depend on t,

$$c = const \text{ in } S(\mathbf{R},\ S(M,\ N)).$$

QED

6.2. If M, N are premanifolds, then $TS(M,\ N) = S(M,\ TN)$ is again of the form $S(M,\ P)$, so it satifies (M1)-(M3) too. Therefore we may continue and get the whole sequence of iterated tangent bundles :

$$\begin{tikzcd}
\cdots \arrow[r] & T^{n+1}\ S(M,\ N) \arrow[r] \arrow[d, equal] & T^nS(M;\ N) \arrow[d, equal] & \ldots \arrow[r] & TS(M,\ N) \arrow[r] \arrow[d, equal] & S(M,\ N) \\
\cdots \arrow[r] & S(M,\ T^{n+1}\ N) \arrow[r] & S(M,\ T^nN) & \ldots \arrow[r] & S(M,\ TN) \arrow[ur] &
\end{tikzcd}$$

So we may speak of smooth mappings between objects of the form $S(M,N)$ for premanifolds M, N , by just using Definition 3.1.

6.3. **Lemma.** *If M, N, P are premanifolds, then the set $S(M,\ N) \times P$ satisfies (M1)-(M3) in a canonical way, and*

$$T(S(M,\ N) \times P) = S(M,\ TN) \times TP$$

is of the same form.

Proof. Look at the proof of Theorem 4.1 and proceed in the same manner.

6.4. **Lemma.** *If M, N, P are premanifolds and $f \in S(M, N)$ then the following two mappings are smooth in the sense of 6.2 :*

$$f^* = S(f, P) : S(N, P) \to S(M, P), \quad g \mapsto g \circ f,$$

$$f_* = S(P, f) : S(P, M) \to S(P, N), \quad g \mapsto f \circ g.$$

Proof. Consider the following diagram :

(a)

$$\begin{tikzcd}
S(\mathbf{R} \times N, P) \arrow[rr, "S(Id \times f,\, P)"] \arrow[ddd, bend right=60, "T(\ )\circ A_0"'] & & S(\mathbf{R} \times M, P) \arrow[ddd, bend left=60, "T(\ )\circ A_0"] \\
S(\mathbf{R}, S(N, P)) \arrow[rr, "S(f,\, P)_*"] \arrow[u, "\wedge"] \arrow[d, "\delta_0"] & & S(\mathbf{R}, S(M, P)) \arrow[u, "\wedge"'] \arrow[d, "\delta_0"'] \\
TS(N, P) \arrow[rr, dashed, "TS(f,\, P)"] \arrow[d, equal] & & TS(M, P) \arrow[d, equal] \\
S(N, TP) \arrow[rr, "S(f,\, TP)"] & & S(M, TP)
\end{tikzcd}$$

This diagram commutes :

$$(\delta_0 \circ S(f, P)_*(c))(m) = T(\hat{c} \circ (Id_{\mathbf{R}} \times f)) \circ A_0(m) = T\hat{c} \circ (Id_{T\mathbf{R}} \times Tf)(0, 1 ; 0_{\pi})$$

$$= T\hat{c}(0, 1 ; 0_{f(m)}) = T\hat{c} \circ A_0(f(m)) = (S(f, TP) \circ \delta_0 (c))(m).$$

Therefore $Ts(f, P) = S(f, TP)$ and we may iterate and conclude that $S(f, P)$ is smooth. Now consider the following diagram :

(b)

$$\begin{tikzcd}
S(\mathbf{R} \times P, X) \arrow[rr, "f_*"] \arrow[ddd, bend right=60, "T(\ )\circ A_0"'] & & S(\mathbf{R} \times P, Y) \arrow[ddd, bend left=60, "T(\ )\circ A_0"] \\
S(\mathbf{R}, S(P, X)) \arrow[rr, "S(P,\, f)_*"] \arrow[u, "\wedge"] \arrow[d, "\delta_0"] & & S(\mathbf{R}, S(P, Y)) \arrow[u, "\simeq", "\wedge"'] \arrow[d, "\delta_0"'] \\
TS(P, X) \arrow[rr, dashed, "TS(P,\, f)"] \arrow[d, equal] & & TS(P, Y) \arrow[d, equal] \\
S(P, TX) \arrow[rr, "S(P,\, Tf)"] & & S(P, TY)
\end{tikzcd}$$

Let us check that (b) commutes :

$$\delta_0 \circ S(P, f)_*(c) = \delta_0((f \circ \hat{c})) = T(f \circ \hat{c}) \circ A_0 =$$

$$= Tf \circ T\hat{c} \circ A_0 = Tf \circ \delta_0 c = (Tf)_*(\delta_0 c).$$

Therefore $TS(P, f) = S(P, Tf)$, and by recursion we see that $S(P, f)$ is smooth. QED

6.5. Lemma. *If M, N are premanifolds, then the insertion mapping*

$$ins : M \to S(N, M \times N), \quad ins(m)(n) = (m, n)$$

is smooth.

Proof. We use the diagram :

(a)

$$\begin{tikzcd}
 & S(\mathbf{R} \times N, M \times N) \\
S(\mathbf{R}, M) \arrow[ur, "(. \times Id)"] \arrow[r, "ins_*"] \arrow[d, "\delta_0"'] & S(\mathbf{R}, S(N, M \times N)) \arrow[u, "\simeq"', "\wedge"] \arrow[d, "\delta_0"] \\
TM \arrow[r, dashed, "T(ins)"] \arrow[d, "ins"] & TS(N, M \times N) \arrow[d, equal] \\
S(N, TM \times N) \arrow[r, "{S(N, Id_{TM} \times 0_N)}"] & S(N, TM \times TN)
\end{tikzcd}$$

This diagram commutes : $(ins \circ c)\hat{} = c \times Id_N$,

$$(\delta_0(ins_*(c)))(n) = T((ins \circ c)\hat{}) \circ A_0(n) = T(c \times Id_N)(0, 1 ; 0_n)$$

$$= (Tc \times Id_{TN})(0, 1 ; 0_n) = (Tc (0, 1), 0_n) \simeq (\delta_0 c, 0_m)$$

$$= ((Id_{TM} \times 0_N) \circ ins(\delta_0 (c)))(n).$$

Therefore ins is S^1 and

$$T(ins) = S(N, Id_{TM} \times 0_N) \circ ins$$

is again S^1 by 6.3 and induction, so we may iterate and conclude that ins is smooth. QED

6.6. Lemma. *If M, N are premanifolds, then for any $m \in M$ the mapping $ev_m : S(M, N) \to N$ is smooth in the sense of 6.2.*

Proof. Consider the following diagram :

(a)

$$\begin{tikzcd}
S(\mathbf{R} \times M, N) \arrow[drr, "{S(ins(m), N)}"] \arrow[ddd, bend right=60, "T(\ )\circ A_t"'] & & \\
S(\mathbf{R}, S(M, N)) \arrow[u, "\wedge"'] \arrow[rr, "(ev_m)_*"] \arrow[d, "\delta_t"] & & S(\mathbf{R}, N) \arrow[dd, "\delta_t"] \\
TS(M, N) \arrow[d, equal] & & \\
S(M, TN) \arrow[rr, "ev_m"] & & TN
\end{tikzcd}$$

Let us check that this diagram commutes :

$$\delta_t \circ (ev_m)_*(c) = \delta_t\,(ev_m \circ c) = \delta_t\,(\hat{c}(\;, m)) = \delta_t\,(\hat{c} \circ ins(m))$$

$$= \delta_t \circ \hat{c}_*(ins(m)) = T\hat{c} \circ \delta_t(ins(m)) = T\hat{c}(t, 1\,;\, 0_m) = T\hat{c} \circ A_t(m)$$

$$= ev_m(T\hat{c} \circ A_t) = ev_m \circ \delta_t(c).$$

Note that we used only $ins(m) : \mathbf{R} \to \mathbf{R} \times M$ and not ins, and 6.4. If we put $t = 0$ in (a) we see that ev_m is S^1 and $T(ev_m) = ev_m$. So by recursion ev_m is smooth. QED

6.7. Lemma. *Let M, N be premanifolds, and let (E, p, M) be a pre-vector bundle. Then for any $n \in N$ the space*

$$S(N, M)\times_{(ev_n, M, p)} E$$

satisfies (M1)-(M3) in a canonical way, and the tangent space is of the same form.

Proof. (M1) Put

$$\begin{tikzcd}
T(S(N, M)\times_{(ev_n, M, p)} E) \arrow[drr, "\pi"'] & = & TS(N, M)\times_{(T(ev_n), TM, Tp)} TE & = & S(N, TM)\times_{(ev_n, TM, Tp)} TE \arrow[dll, "S(N, \pi_M) \times \pi_E"] \\
 & & S(N, M)\times_{(ev_n, M, p)} E & &
\end{tikzcd}$$

Then

$$\pi^{-1}(f, v_{f(n)}) = (S(N, \pi_M) \times \pi_E)^{-1}(f, v_{f(n)}) =$$

$$= \{(g, w) \in S_f(N, TM) \times T_vE \mid ev_n(g) = T_v(p).w\},$$

where $v = v_{f(n)}$. This is a closed linear subspace of $S_f(N, TM) \times T_vE$, since it is the kernel of the linear continuous mapping

$$ev_n \circ pr_1 - T_vp \circ pr_2 : S_f(N, TM) \times T_vE \to T_{f(n)}M.$$

So it is a bornological C^∞-complete vector space with the bornologized subspace topology.

(M2)+(M3) We choose the following setting :

$$\begin{tikzcd}
S(\mathbf{R}, S(N, M))\times_{(ev_n, M, p)} E \arrow[d, "\delta_t"] & = & S(\mathbf{R}, S(N, M))\times_{((ev_n)_*, S(\mathbf{R},M), p_*)} S(\mathbf{R}, E) \arrow[d, "\delta_t \times \delta_t"] & = & S(\mathbf{R}\times N, M)\times_{(ins(n)^*, S(\mathbf{R},M), p_*)} S(\mathbf{R}, E) \arrow[d, "(T(\;) \circ A_t) \times \delta_t"'] \\
T(S(N, M)\times_{(ev_n, M, p)} E) & = & TS(N, M))\times_{(T(ev_n), TM, Tp)} TE & = & S(N, TM)\times_{(ev_n, TM, Tp)} TE
\end{tikzcd}$$

It is easy to check that all requirements of (M2) and (M3) are satisfied.
QED

Remark. Since the tangent space is of the same form again we have the whole tower of iterated tangent bundles and the notion of smooth mappings makes sense, as in 6.2.

Note that the proof of (M1) does not work for a general smooth $f : S(N, P) \to M$ instead of ev_n , since we do not know that Tf is fibre linear (δ_0 surjective depends on *Geo* in 2.1 and was used in 3.1).

6.8. **Lemma.** *In the setting of Lemma 6.6 the fibred product*

$$S(N, M)_{(ev_n, \times_M, p)} E$$

has the universal property of a pullback with respect to smooth mappings.

Proof. Let X be a premanifold or of the form $S(N, M)$, or even itself a fibred product as above, and consider smooth mappings f, g in the situation of the following diagram :

(a)
$$\begin{tikzcd}
X \arrow[drr, bend left, "f"] \arrow[dr, dashed, "{(g, f)}"] \arrow[ddr, bend right, "g"'] & & \\
& S(N, M) \underset{(ev_n, M, p)}{\times} E \arrow[r, "pr_2"] \arrow[d, "pr_1"] & E \arrow[d, "p"] \\
& S(N, M) \arrow[r, "ev_n"] & M
\end{tikzcd}$$

Now look at diagram (b) below. It shows that the projections pr_1 , pr_2, and (g, f) are of class S^1 and their tangent mappings are of the same form. So by recursion all these mappings are smooth. QED

6.9. **Lemma.** *Let (E, p, M) be a pre-vector bundle. Then the mapping*

$$\begin{tikzcd}
L(TM, TM) \underset{M \times M}{\times} L(E, E) \arrow[r, "\mathcal{J}"] \arrow[d] & L(TM \underset{M}{\times} E, TM \underset{M}{\times} E) \arrow[d] \\
M \times M \arrow[r, equal] & M \times M
\end{tikzcd}$$

given by $(f, g) \mapsto f \times g$, is smooth.

Proof. $\mathcal{J}$ is fibrewise continuous and linear, so fibrewise a C^∞ -mapping, and clearly commutes with the parallel transports. So we may use 5.2.
QED

6.10. **Lemma.** *Let M, N, P, Q be premanifolds. Let $f : P \to Q$ be*

$$\begin{tikzcd}
 & S(\mathbf{R}\times N, M) \underset{(ins(n)^*, S(\mathbf{R},M), p_*)}{\times} S(\mathbf{R}, E) \arrow[dr, "pr_1"] \arrow[ddrr, bend left, "pr_2"] & & & \\
 & & S(\mathbf{R}\times N, M) \arrow[ddrr, bend left, "ins(m)*"] & & \\
S(\mathbf{R}, X) \arrow[uur, dashed] \arrow[r, dashed, "(g, f)_*"'] \arrow[rrr, bend left, "f_*"] \arrow[drr, bend right, "g_*"] \arrow[dd, "\delta_0"] & S(\mathbf{R}, S(N, M) \underset{(ev_n, M, p)}{\times} E) \arrow[uu, "\simeq"] \arrow[rr, "(pr_2)_*"] \arrow[dr, "(pr_1)_*"] \arrow[dd, "\delta_0"] & & S(\mathbf{R}, \, , E) \arrow[dr, "p_*"] \arrow[dd, "\delta_0"] & \\
 & & S(\mathbf{R}, S(N, M)) \arrow[uu, "\simeq"] \arrow[rr, "(ev_n)_*"] \arrow[dd, "\delta_0"] & & S(\mathbf{R}, M) \arrow[dd, "\delta_0"] \\
TX \arrow[r, dashed, "T(g, f)"] \arrow[rrr, bend left, "Tf"] \arrow[drr, bend right, "Tg"] \arrow[ddr, dashed, "(Tg, Tf)"'] & T(S(N, M) \underset{(ev_n, M, p)}{\times} E) \arrow[rr, "T(pr_2)"] \arrow[dr, "T(pr_1)"] \arrow[dd, "\simeq"] & & TE \arrow[dr, "Tp"] & \\
 & & TS(M, N) \arrow[rr] \arrow[dd, "\simeq"] & & TM \\
 & S(N, TM) \underset{(ev_n, TM, Tp)}{\times} TE \arrow[uurr, bend right, "pr_2"'] \arrow[dr, "pr_1"] & & & \\
 & & S(N, TM) \arrow[uurr, bend right, "ev_n"'] & &
\end{tikzcd}$$

Diagram (b) of Lemma 6.8.

smooth. Then the mapping

$$(.\ \times f) : S(M, N) \to S(M \times P, N \times Q)$$

is smooth.

Proof. Consider the following diagram :

(a)

$$\begin{tikzcd}
S(\mathbf{R} \times M, N) \arrow[r, "(.\ \times f)"] \arrow[dd, bend right=60, "T(\ )\circ A0"'] & S(\mathbf{R} \times M \times P, N \times Q) \arrow[dd, bend left=60, "T(\ )\circ A0"] \\
S(\mathbf{R}, S(M, N)) \arrow[r, "(.\ \times f)_*"] \arrow[u, "\wedge"'] \arrow[d, "\delta_0"'] & S(\mathbf{R}, S(M \times P, N \times Q)) \arrow[u, "\wedge"'] \arrow[d, "\delta_0"] \\
S(M, TN) \arrow[r, "(.\ \times 0_Q \circ f)"] & S(M \times P, TN \times TQ)
\end{tikzcd}$$

A little computation shows that this diagram commutes. Then $(.\ \times f)$ is S^1 and

$$T(.\ \times f) = (.\ \times 0_Q \circ f)$$

which is S^1 too, so $(.\ \times f)$ is S^2 and so on. QED

7. Manifolds, Vector bundles and cartesian closedness.

7.1. **Definition.** A premanifold M is called a *manifold,* if its defining parallel transport Pt^{TM} and geodesic structure Geo^M satisfy the following further requirements (besides (M1)-(M6) of 2.1) :

(M7) $Pt^{TM} : S(\mathbf{R}, M) \times \mathbf{R} \to L(TM, TM)$ is smooth.
(M8) $Geo^M : TM \to S(\mathbf{R}, M)$ is smooth.

Condition (M8) is equivalent to either one of the two following conditions :

$$exp^M = ev_1 \circ Geo^M : TM \to M \quad \text{and} \quad (Geo^M)\hat{} : TM \times \mathbf{R} \to M$$

are smooth.

The mapping $exp = exp^M$ is called the *exponential mapping* for the geodesic structure $Geo = Geo^M$.

Proof. If Geo is smooth, then by the formula above exp is smooth, and in turn

$$(Geo)\hat{} = exp \circ m^{TM} : TM \times \mathbf{R} \to TM \to M$$

is smooth, since by (M6) we have

$$(Geo)\hat{}(v_x, t) = Geo(v_x)(t) = Geo(t.v_x)(1) = exp \circ m^{TM}(v_x, t).$$

On the other hand

$$Geo = S(\mathbf{R}, (Geo)\hat{}\,) \circ ins : TM \to S(\mathbf{R}, TM \times \mathbf{R}) \to S(\mathbf{R}, M)$$

so Geo is smooth if $(Geo)\hat{}$ is it. QED

7.2. **Definition.** A pre-vector bundle (E, p, M) is called a *vector bundle* if M is a manifold and the following condition holds for the defining parallel transport of the pre-vector bundle :

(VB3) $Pt^E : S(\mathbf{R}, M) \times \mathbf{R} \to L(E, E)$ is smooth.

Remark. We have shown in 5.16 that

$$t \mapsto Pt^E(c, t).v_{c(0)}$$

is the unique smooth solution in E of the ordinary differential equation $\nabla s = 0$. This may be a way to show that some premanifolds are already manifolds. I have no results in this direction.

7.3. **Theorem.** *Let (E, p, M) be a vector bundle over a manifold M. Then the total space E (with its premanifold structure from 2.6) is actually a manifold.*

Proof. (M1)-(M6) hold by Theorem 2.6.

(M7) $$Pt^E : S(\mathbf{R}, M) \times \mathbf{R} \to L(E, E)$$

is smooth by (VB3) for (E, p, M). Define

$$\tilde{P}t^E : S(\mathbf{R}, M) \times \mathbf{R} \underset{(ev_0, M, p)}{\times} E \to E$$

by

$$\tilde{P}t^E(c, t, v_{c(0)}) = Pt^E(c, t).v_{c(0)} \ .$$

Then $\tilde{P}t^E$ is smooth in the sense of 6.7, since

$$\tilde{P}t^E = ev \circ (Pt^E \times Id_E) : S(\mathbf{R}, M) \times \mathbf{R} \times_M E \to L(E, E) \times_M E \to E$$

and the fibre linear evaluation is smooth by Lemma 5.9. Likewise

$$\tilde{P}t^{TM} : S(\mathbf{R}, M) \times \mathbf{R} \underset{M}{\times} TM \to TM$$

is smooth. By (2.15) we have

$$\tilde{P}t^{TE}(c, t, w) =$$

$$= (Dec^E)^{-1}(\tilde{P}t^{TM}(p \circ c, t, pr_1 \circ Dec^E(w)), c(t), \tilde{P}t^E(p \circ c, t, pr_3 \circ Dec^E(w)),$$

so $\tilde{P}t^{TE}$ is smooth and we have the diagram :

(a)
$$\begin{tikzcd}
S(\mathbf{R}, E) \times \mathbf{R} \underset{E}{\times} TE \arrow[r, "\tilde{Pt}^{TE}"] \arrow[d, "pr_{1,2}"'] & TE \arrow[d, "\pi_E"] \\
S(\mathbf{R}, E) \times \mathbf{R} \arrow[r, "ev"] & E
\end{tikzcd}$$

$\tilde{Pt}^{TE}$ is fibre linear in the fibration given by (a). Note that diagram (a) implies that $ev : S(\mathbf{R}, E) \times \mathbf{R} \to E$ is smooth for

$$ev = \pi_E \circ \tilde{Pt}^{TE} \circ (Id_{S(\mathbf{R},E)\times\mathbf{R}} , 0_{TE} \circ ev_0 \circ pr_1)$$

and the mapping $(Id, 0_{TE} \circ ev_0 \circ pr_1)$ is smooth by Lemma 6.8.

It is rather complicated to show that smoothness of $\tilde{Pt}^{TE}$ implies smoothness of Pt^{TE} directly. Consider the isomorphism of pre-vector bundles and differentiable structures

$$(TE, \pi_E, E) = (TM \underset{M}{\times} E \underset{M}{\times} E, pr_2, E).$$

This implies an isomorphism of pre-vector bundles over $E \times E$:

$$(L(TE, TE), (\pi_E, \pi_E), E\times E) =$$
$$= (E \underset{(p,M,p_1)}{\times} L(TM \underset{M}{\times} E, TM \underset{M}{\times} E) \underset{(p_2,M,p)}{\times} E, pr_{1,4} , E \times E).$$

By (2.15) the parallel transport on TE is given by the following sequence of mappings :

$$\begin{tikzcd}
S(\mathbf{R}, E) \arrow[d, "{(ev_0 \circ pr_1, Pt^{TM} \circ (p_* \times Id_{\mathbf{R}}), Pt^E \circ (p_* \times Id_{\mathbf{R}}), ev)}"] \\
E \underset{M}{\times} (L(TM, TM) \underset{M\times M}{\times} L(E, E)) \underset{M}{\times} E \arrow[d, "Id_E \times \mathcal{J} \times Id_E"] \\
E \underset{M}{\times} L(TM \underset{M}{\times} E, TM \underset{M}{\times} E) \underset{M}{\times} E \arrow[d, "Iso"] \\
L(TE, TE)
\end{tikzcd}$$

(ev is smooth, see above, ev_0 is smooth by 6.5, $\mathcal{J}$ is smooth by 6.8)

So Pt^{TE} is smooth.

(M8) $Geo^E : TE \to S(\mathbf{R}, E)$ is given by (2.16),

$$Geo^E((Dec^E)^{-1}(u_x, v_x, w_x))(t) = Pt^E(Geo^M(u_x), t).(v_x + t.w_x) =$$
$$= Pt^E(Geo^M(u_x), t, v_x + t.w_x).$$

The last expression is smooth in all appearing variables, since Pt^E and Geo^M are smooth and by 5.3. So

$$(Geo^E)^\wedge \circ ((Dec^E)^{-1} \times Id\ )$$

is smooth and thus $(Geo^E) : TE \times \mathbf{R} \to E$ too. By the Lemma in 7.1 this suffices. QED

7.4. **Proposition.** *Let* (E, p, M) *be a vector bundle. Then the space* $C^\infty(\mathbf{R}, \Gamma(E))$ *of all* C^∞*-curves in the bornological* C^∞*-complete space* $\Gamma(E)$ *corresponds exactly to the space* $S_{pr_2}(\mathbf{R} \times M, E)$ *of all smooth mappings* $g : \mathbf{R} \times M \to E$ *with* $p \circ g(t, x) = x$.

Proof. Let $g : \mathbf{R} \to \Gamma(E)$ be a C^∞-curve. Then

$$\hat{g} : \mathbf{R} \times M \to E, \quad \hat{g}(t, x) = g(t)(x),$$

is a mapping satisfying $p \circ \hat{g}(t, x) = x$.

Claim : $\hat{g}$ is smooth. Let $c \in S(\mathbf{R}, M)$, put $c(0) = x$. Consider the mapping $B(c) : \Gamma(E) \to C^\infty(\mathbf{R}, E_x)$ from the proof of 5.17 -this is one of the generating mappings for the Γ^1-topology. So $B(c)$ is linear and continuous. For $s \in \Gamma(E)$ we had

$$(B(c)(s))(t) = Pt^E(c, t)^{-1}(s(c(t))).$$

So

$$B(c) \circ g : \mathbf{R} \to C^\infty(\mathbf{R}, E_x)$$

is a C^∞-curve in $C^\infty(\mathbf{R}, E_x)$, given by

$$r \mapsto (t \mapsto Pt^E(c, t)^{-1}g(r)(c(t))).$$

But now we are in the setting of §1, and by the cartesian closedness of the category of C^∞-mappings and bornological C^∞-complete vector spaces the mapping

$$(r, t) \mapsto Pt^E(c, t)^{-1}g(r)(c(t))$$

is a C^∞-mapping $\mathbf{R}^2 \to E_x$. Thus the mapping

$$H : S(\mathbf{R}, \mathbf{R}) \times S(\mathbf{R}, M) \to S(\mathbf{R}, M) \times S_{const}(\mathbf{R}, E)$$

given by

$$H(f, c) = (c, t \mapsto Pt^E(c, t)^{-1}g(f(t))(c(t)))$$

makes sense. Now consider the following diagram (a) on page 78, in which the mapping R is given by

$$R(a, b\ ; u_x) = (u_x, \hat{g}(a, x), b.g'(a)(x) + \nabla(g(a)).u_x).$$

(a)

$$\begin{tikzcd}
\mathcal{S}(\mathbf{R}, \mathbf{R}) \times \mathcal{S}(\mathbf{R}, M) \arrow[r, "H"] \arrow[d, equal] \arrow[dd, bend right=60, "\delta_0 \times \delta_0"'] & \mathcal{S}(\mathbf{R}, M) \times_M \mathcal{S}_{const}(\mathbf{R}, E) \arrow[d, "Cart^E"] \arrow[ddd, bend left=60, "\delta_0 \times \delta_0"] \\
\mathcal{S}(\mathbf{R}, \mathbf{R} \times M) \arrow[r, "(\hat{g})_*"] \arrow[d, "\delta_0"'] & \mathcal{S}(\mathbf{R}, E) \arrow[d, "\delta_0"] \\
T\mathbf{R} \times TM \arrow[r, "T\hat{g}"] \arrow[dr, "R"'] & TE \arrow[d, "Dec^E"] \\
 & TM \times_M E \times_M E
\end{tikzcd}$$

A little computation shows that this diagram commutes. So $\hat{g}$ is S^1. By 5. 20 the mapping

$$\nabla : \Gamma(E) \to \Gamma(\pi_M^* E)$$

is linear and continuous, so

$$\nabla \circ g : \mathbf{R} \to \Gamma(\pi_M^* E)$$

is again a C^∞-curve. If we apply the argument above to the curve $\nabla \circ g$ we see that the mapping

$$\mathbf{R} \times TM \to E, \quad (a, u_x) \mapsto \nabla(g(a)).u_x ,$$

is S^1. Likewise $g' : \mathbf{R} \to \Gamma(E)$ is a C^∞-curve, so by the argument above, the mapping

$$\mathbf{R} \times M \to E, \quad (a, x) \mapsto g'(a)(x)$$

is S^1. So all the ingredients of the mapping R are S^1 so R is S^1 and Tg is S^1. Now by the whole argument above the two critical mappings in R turn out to be S^2, so g is S^3. This can be repeated, so g is smooth and the claim is proved.

Conversely, let $h : \mathbf{R} \times M \to E$ be smooth with $p \circ h = pr$. Then

$$h^\vee(t) = h(t, \ ) = h \circ ins(t) : M \to M \times \mathbf{R} \to M \to E$$

is smooth, so $h^\vee(t) \in \Gamma(E)$. We have to show that $h^\vee : \mathbf{R} \to \Gamma(E)$ is a C^∞-curve. Let $c \in \mathcal{S}(\mathbf{R}, M)$, put again $x = c(0)$. Then

$$(B(c)(h^\vee(t)))(r) = Pt^E(c, r)^{-1} h^\vee(t)(c(s)) = \tilde{Pt}^E(c(. + s), -s, h(t, c(s))) \in E_x .$$

The last expression, viewed as a function of s, t, is smooth $\mathbf{R}^2 \to E$ (the mapping

$$t \mapsto c(. +t), \quad \mathbb{R} \to S(\mathbb{R}, M),$$

is smooth, since it coincides with

$$S(\mathbb{R}, c \circ +) \circ ins : \mathbb{R} \to S(\mathbb{R}, \mathbb{R} \times \mathbb{R}) \to S(\mathbb{R}, M))$$

and takes its values only in the fibre E_x. So it maps C^∞-curves in $\mathbb{R}^2$ to smooth curves in E lying in the fibre E_x, but these latter are exactly the C^∞-curves in E_x by the definition of $Cart^E$ in 2.8. By §1 this suffices to see that

$$(t, r) \mapsto (B(c)(h^\vee(t)))(r)$$

is a C^∞-mapping $\mathbb{R}^2 \to E_x$. This means that

$$(B(c) \circ h^\vee)^\wedge : \mathbb{R}^2 \to E_x$$

is C^∞. By cartesian closedness the mapping

$$B(c) \circ h^\vee : \mathbb{R} \to C^\infty(\mathbb{R}, E_x)$$

is C^∞ then. So $h^\vee : \mathbb{R} \to \Gamma(E)$ is C^∞ if $\Gamma(E)$ bears the initial topology with respect to all mappings of the form $B(c)$, and is C^∞ too if we bornologize this topology. So $h^\vee : \mathbb{R} \to \Gamma^1(E)$ is C^∞. Now finally

$$T \circ h^\vee : \mathbb{R} \to \Gamma(E) \to \Gamma^1(TE, Tp, TM)$$

coincides with the mapping

$$(Th \circ (0_{\mathbb{R}} \times Id_{TM})) : \mathbb{R} \to \Gamma^1(TE, Tp, TM),$$

which is C^∞ by the argument above. This can be repeated and shows that $h^\vee : \mathbb{R} \to \Gamma(E)$ is C^∞. QED

7.5. **Lemma.** *Let M be a premanifold, let N be a manifold. Then S(M, N) is a premanifold.*

Proof. (M1)-(M3) have been checked in 6.1.

(M4) Construction of the parallel transport for $S(M, N)$. For $c \in S(\mathbb{R}, S(M, N))$ we define

$$Pt^{TS(M,N)}(c, t) : S_{c(0)}(M, TN) \to S_{c(t)}(M, TN)$$

by

$$(Pt^{TS(M,N)}(c, t).s)(m) = Pt^{TN}(\hat{c}(,m), t).s(m),$$

$s \in S_{c(0)}(M, TN)$, $m \in M$.

Claim. For $s \in S_{c(0)}(M, TN)$ we have

$$Pt^{TS(M,\ N)}(c,\ t).s \in S_{c(t)}(M,\ TN).$$

$$Pt^{TS(M,\ N)}(c,\ t).s = \tilde{Pt}^{TN} \circ (\bar{c},\ const(t),\ s) : M \to S(\mathbf{R},\ N) \times \mathbf{R} \underset{N}{\times} TN \to TN.$$

That $\tilde{Pt}^{TN}$ is smooth was shown in the proof of Theorem 7.3.

$$\bar{c} : M \to S(\mathbf{R},\ N) \text{ given by } \bar{c}(m)(t) = c(t)(m) = c(t,\ m)$$

is smooth since

$$\bar{c} = (\hat{c} \circ flip)^{\vee} = S(\mathbf{R},\ \hat{c} \circ flip) \circ ins : M \to S(\mathbf{R},\ M \times \mathbf{R}) \to S(\mathbf{R},\ N),$$

and all components are smooth by 6.4 and 6.5. The claim follows.

Claim.

$$Pt^{TS(M,\ N)}(c,\ t) : S_{c(0)}(M,\ TN) \to S_{c(t)}(M,\ TN)$$

is continuous and linear, where

$$S_f(M,\ TN) = \Gamma\ (f^*TN,\ f^*\pi_N,\ M)$$

as bornological C^{∞}-complete vector spaces.

$Pt^{TS(M,\ N)}(c,\ t)$ is clearly linear since the linear structure is the pointwise one. To show that it is continuous (= bounded) it suffices to show that it maps C^{∞}-curves to C^{∞}-curves by §1. By Proposition 7.4

$$C^{\infty}(\mathbf{R},\ S_f(M,\ TN)) = C^{\infty}(\mathbf{R},\ \Gamma\ (f^*TN)) = S_{f \circ pr_2}(\mathbf{R} \times M,\ TN).$$

So let $g : \mathbf{R} \to S_{c(0)}(M,\ TN)$ be a C^{∞}-curve, then $\hat{g} : \mathbf{R} \times M \longrightarrow TN$ is smooth, and :

$$(Pt^{TS(M,\ N)}(c,\ t) \circ g(s))(m) = Pt^{TN}(\hat{c}(\ ,\ m),\ t).g(s)(m) =$$

$$\tilde{Pt}^{TN}(\bar{c}(m),\ t,\ \hat{g}(s,\ m)) = \tilde{Pt}^{TN} \circ (\bar{c} \circ pr_2,\ const(t),\ \hat{g})(s,\ m)\ ;$$

this is a smooth function of $(s,\ m)$. So

$$Pt^{TS(M,\ N)}(c,\ t) \circ g : \mathbf{R} \to S_{c(t)}(M,\ TN)$$

is a C^{∞}-curve.

Claim. $Pt^{TS(M,\ N)}(c,\ 0) = Id.$

$$(Pt^{TS(M,\ N)}(c,\ 0).s)(m) = Pt^{TN}(\hat{c}(\ ,\ m),\ 0).s(m) = s(m).$$

Claim. For $f \in C^{\infty}(\mathbf{R},\ \mathbf{R})$ we have

$$Pt^{TS(M,\ N)}(c,\ f(t)) = Pt^{TS(M,\ N)}(c \circ f,\ t) \circ Pt^{TS(M,\ N)}(c,\ f(0)).$$

$$(Pt^{TS(M,\ N)}(c,\ f(t)).s)(m) = Pt^{TN}(\hat{c}(\ ,\ m),\ f(t)).s(m) =$$

$$= Pt^{TN}(\hat{c}(f(\ ), m), t) \circ Pt^{TN}(\hat{c}(\ , m), f(0)).s(m) =$$

$$= (Pt^{TS(M,\ N)}(c \circ f, t) \circ Pt^{TS(M,\ N)}(c, f(0)).s)(m).$$

All requirements of (M4) are satisfied.

(M5) Let $c \in S(\mathbf{R}, S(M, N))$. We have that

$$Pt^{TS(M,\ N)}(c, t)^{-1}.\delta_t c$$

as a function of t is in $C^\infty(\mathbf{R}, S_{c(0)}S(M, TN))$. By Proposition 7.4 it suffices to show that

$$(t, m) \mapsto (Pt^{TS(M,\ N)}(c, t)^{-1}. \delta_t c)(m)$$

is a smooth mapping $\mathbf{R} \times M \to T\,N$

$$(Pt^{TS(M,\ N)}(c, t)^{-1}. \delta_t c)(m) = Pt^{TN}(\hat{c}(\ , m), t)^{-1} (\delta_t c)(m) =$$

$$= Pt^{TN}(\hat{c}(.+t, m), -t) \circ T\hat{c} \circ A_t(m) = \tilde{P}t^{TN}(\hat{c}(.+t, m), -t, T\hat{c}(t, 1 ; 0_N(m))).$$

This last expression is smooth in (t, m) ; that

$$(t, m) \mapsto c(.+t, m)$$

is a smooth mapping $\mathbf{R} \times M \to S(\mathbf{R}, N)$ can be checked similarly as smoothness of c at the beginning of this proof.

(M6) We define

$$Geo^{S(M,\ N)} : TS(M, N) = S(M, TN) \to S(\mathbf{R}, S(M, N))$$

by the formula

$$(Geo^{S(M,\ N)}(s)(t))(m) = Geo^N(s(m))(t)$$

for $s \in S(M, TN)$, $t \in \mathbf{R}$, $m \in M$.

Claim : $Geo^{S(M,\ N)}(s) \in S(\mathbf{R}, S(M, N))$.

$$(Geo^{S(M,\ N)}(s)(t))(m) = Geo^N(s(m))(t) = (Geo^N)\hat{\ }(s(m), t) =$$

$$= (Geo^N)\hat{\ } \circ (s \times Id)(m, t),$$

which is a smooth mapping $\mathbf{R} \times M \longrightarrow TN$ by (M8) for N. This suffices by the definition of $S(\mathbf{R}, S(M, N))$ in 6.1.

Claim : $Geo^{S(M,,N)}(t.s)(r) = Geo^{S(M,\ N)}(s)(tr)$.

$$(Geo^{S(M,\ N)}(t.s)(r))(m) = Geo^N(t.s(m))(r) = Geo^N(s(m))(tr) =$$

$$= Geo^{S(M,\ N)}(s)(tr))(m).$$

Claim : $\delta_t \mathit{Geo}^{S(M,N)}(s) = Pt^{TS(M,N)}(\mathit{Geo}^{S(M,N)}(s), t).s.$

$(\delta_t \mathit{Geo}^{S(M,N)}(s))(m) = ev_m \circ \delta_t(\mathit{Geo}^{S(M,N)}(s)) = T(ev_m) \circ \delta_t(\mathit{Geo}^{S(M,N)}(s)),$

since $T(ev_m) = ev_m$ by 6.6,

$$= \delta_t(ev_m \circ \mathit{Geo}^{S(M,N)}(s)) = \delta_t(\mathit{Geo}^{S(M,N)}(s)(\ )(m)) =$$

$$= \delta_t(\mathit{Geo}^N(s(m))) = Pt^{TN}(\mathit{Geo}^N(s(m)), t).s(m)$$

by (M6) for N

$$= (Pt^{TS(M,N)}(\mathit{Geo}^{S(M,N)}(s), t).s)(m).$$

Claim : $\mathit{Geo}^{S(M,N)}(\delta_t(\mathit{Geo}^{S(M,N)}(s))) = \mathit{Geo}^{S(M,N)}(s)(.+t) =$

$$(\mathit{Geo}^{S(M,N)}(\delta_t(\mathit{Geo}^{S(M,N)}(s)))(r))(m) = \mathit{Geo}^N((\delta_t(\mathit{Geo}^{S(M,N)}(s)))(m))(r)$$

$$= \mathit{Geo}^N(\delta_t(\mathit{Geo}\ (s(m))))(r)$$

as we just saw,

$$= \mathit{Geo}^N(s(m))(r+t)$$

by (M6) for N,

$$= (\mathit{Geo}^{S(M,N)}(s)(r+t))(m).$$

So (M6) is satisfied. QED

7.6. **Remark.** If N is a manifold, and if M, P are premanifolds, then $S(P, S(M, N))$ satisfies (M1) - (M3) by 6.1 and the tangent space is of the same form. So we may iterate and may consider smooth mappings from and into $S(P, S(M, N))$, as in 6.2. This will be essential for our next steps.

7.7. **Theorem.** *Let M be a manifold. Then there is a unique mapping*

$$\kappa_M : T^2M \to T^2M$$

(called the canonical flip mapping) with the following property :

(1) The following diagram commutes :

(a)

$$\begin{tikzcd}
& S(\mathbf{R}, S(\mathbf{R}, M)) \arrow[dl, "\delta_0"'] \arrow[dr, "S(\mathbf{R}, \delta_0)"] & \\
S(\mathbf{R}, TM) \arrow[d, "\delta_0"] & & S(\mathbf{R}, TM) \arrow[d, "\delta_0"] \\
T^2M \arrow[rr, "\kappa_M"] & & T^2M
\end{tikzcd}$$

Furthermore κ_M is smooth and has the following properties :

(2) If M, N are manifolds, then for any smooth mapping $f : M \to N$ we

have $T^2 f \circ \varkappa_M = \varkappa_N \circ T^2 f$.

(3) $\varkappa_M \circ \varkappa_N = Id_{T^2 M}$

(4) $\varkappa_{M \times N} = \varkappa_M \times \varkappa_N$.

(5) $\varkappa_M$ *is an isomorphism between the two vector bundles*

$$(T^2 M,\ T(\pi_M),\ TM) \quad \text{and} \quad (T^2 M,\ \pi_{TM},\ TM)$$

(i.e. a fibrewise linear diffeomorphism, not necessarily commuting with the parallel transports), so in particular we have

$$\varkappa_M \circ T(0_M) = 0_{TM} : TM \to T^2 M, \quad T(\pi_M) \circ \varkappa_M = \pi_{TM}, \quad \pi_{TM} \circ \varkappa_M = T(\pi_M).$$

(6) $\varkappa_{\mathbf{R}^2} : T^2 \mathbf{R} \to T^2 \mathbf{R}$ *is given by*

$$\varkappa_{\mathbf{R}^2}(x_1, x_2 ; x_3, x_4) = (x_1, x_3 ; x_2, x_4)$$

and for each $f \in S(\mathbf{R}^2, M)$ *we have*

$$\varkappa_M \circ T^2 f = T^2 f \circ \varkappa_{\mathbf{R}^2} .$$

This property characterizes $\varkappa_M$ *uniquely.*

Proof. First of all we recall that

$$\widetilde{Pt}^{TM} : S(\mathbf{R}, M) \times \mathbf{R} \underset{M}{\times} TM \to TM$$

is smooth. We saw this in the beginning of the proof of Theorem 7.3. The following diagram commutes :

(b)
$$\begin{tikzcd}
S(\mathbf{R}, M) \times \mathbf{R} \underset{M}{\times} TM \arrow[r, "\widetilde{Pt}^{TM}"] \arrow[d, "pr_{1,2}"'] & TM \arrow[d, "\pi_M"] \\
S(\mathbf{R}, M) \times \mathbf{R} \arrow[r, "ev"] & M
\end{tikzcd}$$

Then $ev : S(\mathbf{R}, M) \times \mathbf{R} \to M$ is smooth, since

$$ev = \pi_M \circ Pt^{TM} \circ (Id_{S(\mathbf{R}, M) \times \mathbf{R}}, 0_M \circ ev_0 \circ pr_1)$$

and the mapping

$$(Id_{S(\mathbf{R}, M) \times \mathbf{R}}, 0_M \circ ev_0 \circ pr_1)$$

is smooth by 6.6 and 5.1. Note that we know already that $S(\mathbf{R}, M)$ is a premanifold. We will see that smoothness of

$$ev : S(\mathbf{R}, M) \times \mathbf{R} \to M$$

suffices to construct $\varkappa_M$. First we want to investigate $T(ev)$. Consider the following commutative diagram (c). We have

$$\begin{tikzcd}
 & S(\mathbf{R}^2, M) \times S(\mathbf{R}, \mathbf{R}) \arrow[r, "Id\times (Id_{\mathbf{R}},.)"] \arrow[d, equal] \arrow[dddd, bend right=40, "(T(\ )\circ A_0)\times\delta_0"'] & S(\mathbf{R}^2, M) \times S(\mathbf{R}, \mathbf{R}^2) \arrow[dd, "composition"] \\
 & S(\mathbf{R}, S(\mathbf{R}, M)) \times S(\mathbf{R}, \mathbf{R}) \arrow[d, equal] & \\
(c) & S(\mathbf{R}, S(\mathbf{R}, M) \times \mathbf{R}) \arrow[r, "(ev)_*"] \arrow[d, "\delta_0"'] & S(\mathbf{R}, M) \arrow[d, "\delta_0"] \\
 & T(S(\mathbf{R}, M) \times \mathbf{R}) \arrow[r, "T(ev)"] \arrow[d, equal] & TM \\
 & S(\mathbf{R}, TM) \times \mathbf{R} &
\end{tikzcd}$$

$$\delta_0(ev_*(c_1, c_2)) = \delta_0(\hat{c}_1 \circ (id_{\mathbf{R}}, c_2)) = T(\hat{c}_1 \circ (id_{\mathbf{R}}, c_2))(0, 1)) =$$

$$= T\hat{c}_1 \circ (id_{T\mathbf{R}}, Tc_2)(0, 1) = T\hat{c}_1(0, 1 ; \delta_0 c) =$$

$$= T\hat{c}_1((0, 1 ; 0_{c_2(0)}) + (0, 0 ; \delta_0 c_2)) = T\hat{c}_1 \circ A_0(c_2(0)) + T(c_1(0))(\delta_0 c_2)$$

$$= (\delta_0 c_1)(c_2(0)) + T(c_1(0))(\delta_0 c_2)$$

$$= (ev_1 \circ (id_{S(\mathbf{R}, TM)} \times \pi_{\mathbf{R}}) \underset{\pi_M}{+} ev_2 \circ (T \circ S(\mathbf{R}, \pi_M) \times Id_{T\mathbf{R}}))(\delta_0 c_1, \delta_0 c_2),$$

where $ev_1 : S(\mathbf{R}, TM) \times \mathbf{R} \to TM$ is smooth by the argument above applied to the manifold TM (by 7.3) and where

$$ev_2 : S(T\mathbf{R}, TM) \times T\mathbf{R} \to TM$$

is *not* known to be smooth. So the smooth mapping $T(ev)$ can be written as

$$T(ev) = ev_1 \circ (Id_{S(\mathbf{R}, TM)} \times \pi_{\mathbf{R}}) \underset{\pi_M}{+} ev_2 \circ (T \circ S(\mathbf{R}, \pi_M) \times Id_{T\mathbf{R}}).$$

It follows that

$$ev_2 \circ (T \circ S(\mathbf{R}, \pi_M) \times Id_{T\mathbf{R}}) : S(\mathbf{R}, TM) \times T\mathbf{R} \to TM$$

is smooth. But then

$$T \circ S(\mathbf{R}, \pi_M) = S(T\mathbf{R}, ev_2 \circ (T \circ S(\mathbf{R}, \pi_M) \times Id_{T\mathbf{R}})) \circ ins :$$

$$S(\mathbf{R}, TM) \to S(T\mathbf{R}, S(\mathbf{R}, TM) \times T\mathbf{R}) \to S(T\mathbf{R}, TM)$$

is smooth by 7.5 and 6.4. Finally we conclude that

$$T : S(\mathbf{R}, M) \to S(T\mathbf{R}, TM)$$

is smooth, since

$$T = (T \circ S(\mathbf{R}, \pi_M)) \circ S(\mathbf{R}, 0_M) : S(\mathbf{R}, M) \to S(\mathbf{R}, TM) \to S(T\mathbf{R}, TM).$$

This implies in turn that for all $t \in \mathbf{R}$ the mapping $\delta_t : S(\mathbf{R}, M) \to TM$ is smooth, since

$$\delta_t(c) = Tc(0, 1) = (ev_{(0,\ 1)} \circ T)(c)$$

and $ev_{(0,\ 1)}$ is smooth by 6.6.
Now we investigate the following diagram, which clearly commutes :

(d)

$$\begin{tikzcd}
 & S(\mathbf{R}^2, M) \arrow[d, equal] \arrow[dddl, bend right, "T(\ )\circ A_0"'] \arrow[ddr, bend left, "T(\ )\circ B_t"] & \\
 & S(\mathbf{R}, S(\mathbf{R}, M)) \arrow[dl, "\delta_0"'] \arrow[dr, "(\delta_t)_*"] & \\
TS(\mathbf{R}, M) \arrow[d, equal] \arrow[drr, "T(\delta_t)"] & & S(\mathbf{R}, TM) \arrow[d, "\delta_0"] \\
S(\mathbf{R}, TM) \arrow[dr, "\delta_t"'] & & TM \\
 & TM \arrow[ur, dashed, "\varkappa_M"'] &
\end{tikzcd}$$

where

$$A_t : \mathbf{R} \to T\mathbf{R}^2, \quad A_t(u) = (t, 1 ; 0_u) \in T\mathbf{R} \times T\mathbf{R}$$
$$= (t, u, 1, 0) \in T\mathbf{R}^2 \qquad \text{as in 6.1}$$
$$B_t : \mathbf{R} \to T\mathbf{R}^2, \quad B_t(u) = (u, 0 ; t, 1) \in T\mathbf{R} \times T\mathbf{R}$$
$$= (u, t, 0, 1) \in T\mathbf{R}^2.$$

We have to show that, in diagram (d), the mapping $T(\delta_t)$ factors over δ_t . We know that δ_t is surjective (Remark following 2.1). So let c be in $S(\mathbf{R}, TM)$. We have to show that $T(\delta_t)(c)$ depends only on $\delta_t c$. This will follow by a diagram chase. Consider $Geo^{S(\mathbf{R}, M)}(c)$ in $S(\mathbf{R}, S(\mathbf{R}, M))$, constructed as in 7.5. Then of course

$$\delta_0 (Geo^{S(\mathbf{R}, M)}(c)) = c.$$

$$\delta_0 \circ (\delta_t)_*(Geo^{S(\mathbf{R}, M)}(c)) = \delta_0(T((Geo^{S(\mathbf{R}, M)}(c))\hat{}) \circ B_t) =$$

$$= \delta_0(T((Geo^M)\hat{} \circ (c \times Id_{\mathbf{R}}) \circ flip) \circ B_t),$$

since

$$(Geo^M)\hat{} \circ (c \times Id_{\mathbf{R}}) \circ flip(t, r) = (Geo^M)\hat{}(c(r), t) = Geo^M(c(r))(t) =$$

$$= (Geo^{S(\mathbf{R}, M)}(c)(t))(r) = (Geo^{S(\mathbf{R}, M)}(c))\hat{}(t, r).$$

So we may continue

$$\delta_0 \circ (\delta_t)_*(Geo^{S(\mathbf{R}, M)}(c)) = \delta_0(T((Geo^M)\hat{}) \circ (Tc \times Id_{T\mathbf{R}}) \circ T(flip)(\ , 0 ; t, 1))$$

$$= \delta_0(T((Geo^M)\hat{}) \circ (Tc \times Id_{\mathbf{R}})(t, 1 ; \ , 0)) = \delta_0 (T((Geo^M)\hat{})(Tc(t, 1) ; \ , 0))$$

$$= T^2((Geo^M)\hat{}) \circ \delta_0 (\delta_t c ; \ , 0) = T^2((Geo^M)\hat{})(0_{TM}(\delta_t c) ; 0, 0, 1, 0).$$

This last expression depends only on $\delta_t c$. So we see that in diagram (d) the mapping $T(\delta_t)$ factors over δ_t to a mapping $\varkappa_M$: $T^2M \to T^2M$. If we put $t = 0$ we get diagram (a). The mapping $\varkappa_M$ is uniquely given by (1) since it is the unique mapping fitting into diagram (d) ($T(\delta_t)$ is unique and δ_t is surjective) for $t = 0$. It is easy to see that $\varkappa_M$ fits into diagram (d) for all t. Note the formula for $\varkappa_M$ which we derived above :

$$\varkappa_M(\alpha) = T^2((Geo^M)^\frown)(0_\alpha; 0, 0, 1, 0).$$

This shows that $\varkappa_M$ is smooth.

(2) Let $f \in S(M, N)$. Then the following diagram commutes :

(e)
$$\begin{tikzcd}
S(\mathbf{R}, M) \arrow[r, "S(\mathbf{R}, f)"] \arrow[d, "\delta_0"'] & S(\mathbf{R}, N) \arrow[d, "\delta_0"] \\
TM \arrow[r, "Tf"] & TN
\end{tikzcd}$$

Put

$$\alpha := (0, 0 ; 1, 0 ; 0, 1 ; 0, 0) = T(A_0)(0, 1) \in T^2\mathbf{R}^2,$$

and apply the functor $S(\mathbf{R}, \ \ )$ to diagram (e) to get the following diagram :

(f)
$$\begin{tikzcd}
& S(\mathbf{R}^2, M) \arrow[r, "f_*"] \arrow[d, equal] \arrow[ddddl, bend right=60, "T^2(\ ).\alpha"'] & S(\mathbf{R}^2, N) \arrow[d, equal] \arrow[ddddr, bend left=60, "T^2(\ ).\alpha"] & \\
TS(\mathbf{R}, M) \arrow[d, equal] & S(\mathbf{R}, S(\mathbf{R}, M)) \arrow[l, "\delta_0"'] \arrow[r, "(f_*)_*", "S(\mathbf{R}, S(\mathbf{R}, f))"'] \arrow[dd, "S(\mathbf{R}, \delta_0)"'] & S(\mathbf{R}, S(\mathbf{R}, N)) \arrow[r, "\delta_0"] \arrow[dd, "S(\mathbf{R}, \delta_0)"] & TS(\mathbf{R}, N) \arrow[d, equal] \\
S(\mathbf{R}, TM) \arrow[dd, "\delta_0"'] & & & S(\mathbf{R}, TN) \arrow[dd, "\delta_0"] \\
& S(\mathbf{R}, TM) \arrow[r, "S(\mathbf{R}, Tf)"] \arrow[d, "\delta_0"'] & S(\mathbf{R}, TN) \arrow[d, "\delta_0"] & \\
T^2M \arrow[r, "\varkappa_M"] \arrow[rrr, bend right=30, "T^2f"'] & T^2M \arrow[r, "T^2f"] & T^2N & T^2N \arrow[l, "\varkappa_N"']
\end{tikzcd}$$

The outermost quadrangle commutes :

$$T^2f \circ (T^2(\ ).\alpha)(g) = T^2f \circ T^2g.\ \alpha = T(Tf \circ Tg).\alpha$$
$$= T^2(f \circ g).\ \alpha = (T^2(\ ).\alpha)(f_*(g)).$$

The mapping

$$\delta_0 : S(\mathbf{R}, S(\mathbf{R}, M)) \to S(\mathbf{R}, TM)$$

is surjective since $S(\mathbf{R}, M)$ is a premanifold by 7.5, so the mapping

$$\delta_0 \circ \delta_0 : S(\mathbf{R}, S(\mathbf{R}, M)) \to T^2M$$

is surjective. Thus we may conclude that the lowest quadrangle in diagram (f) commutes :

(g)
$$\begin{tikzcd}
T^2M \arrow[r, "T^2f"] \arrow[d, "\varkappa_M"'] & T^2N \arrow[d, "\varkappa_N"] \\
T^2M \arrow[r, "T^2f"] & T^2N
\end{tikzcd}$$

So (2) holds.

(3) *Claim :* The following diagram commutes :

(h)
$$\begin{tikzcd}
S(\mathbf{R}^2, M) \arrow[rr, "S(flip, M)"] \arrow[dr, bend right=80, "T(\ )\circ B_0"'] & & S(\mathbf{R}^2, M) \arrow[dl, bend left=80, "T(\ )\circ A_0"] \\
S(\mathbf{R}, S(\mathbf{R}, M)) \arrow[u, "\wedge"'] \arrow[rr, "(\bar{\ })"] \arrow[dr, "S(\mathbf{R}, \delta_0)"'] & & S(\mathbf{R}, S(\mathbf{R}, M)) \arrow[u, "\wedge"'] \arrow[dl, "\delta_0"] \\
& S(\mathbf{R}, TM) &
\end{tikzcd}$$

where

$$\bar{c}(t)(r) = c(r)(t).$$

For we have

$$\delta_0(\bar{c}) = T(\hat{c} \circ flip) \circ A_0 = T\hat{c} \circ T(flip) \circ (0, 1 ; 0_{\mathbf{R}}(\ )) =$$

$$= T\hat{c} \circ (0_{\mathbf{R}}(\ ) ; 0, 1) = T\hat{c} \circ B_0 = S(\mathbf{R}, \delta_0)(c).$$

Now consider the following diagram :

(i)
$$\begin{tikzcd}
& S(\mathbf{R}^2, M) \arrow[rr, "(flip)*"] \arrow[rrrr, bend left, "Id"] & & S(\mathbf{R}^2, M) \arrow[rr, "(flip)*"] & & S(\mathbf{R}^2, M) \\
& S(\mathbf{R}, S(\mathbf{R}, M)) \arrow[u, "\wedge"'] \arrow[dl, "\delta_0"'] \arrow[dr, "S(\mathbf{R}, \delta_0)"] & & S(\mathbf{R}, S(\mathbf{R}, M)) \arrow[u, "\wedge"'] \arrow[dl, "\delta_0"] \arrow[dr, "S(\mathbf{R}, \delta_0)"'] & & S(\mathbf{R}, S(\mathbf{R}, M)) \arrow[u, "\wedge"'] \arrow[dl, "\delta_0"] \\
S(\mathbf{R}, TM) \arrow[d, "\delta_0"'] & & S(\mathbf{R}, TM) \arrow[d, "\delta_0"] & & S(\mathbf{R}, TM) \arrow[d, "\delta_0"] & \\
T^2M \arrow[rr, "\varkappa_M"] & & T^2M \arrow[rr, "\varkappa_M"] & & T^2M &
\end{tikzcd}$$

Here

$$(flip)* \circ (flip)* = Id, \quad so \ \varkappa_M \circ \varkappa_M = Id$$

since $\delta_0 \circ \delta_0$ is surjective.

(4) That $\varkappa_{M\times N} = \varkappa_M \times \varkappa_N$ follows from the diagram (j) (on page 88).

(5) Consider the diagram (k) (on page 88). From this diagram it follows that

$$T(\pi_M) \circ \varkappa_M = \pi_{TM} \ .$$

Since $\varkappa_M^{-1} = \varkappa_M$ by (3), we conclude that

(j)

$$\begin{tikzcd}
& S(\mathbb{R}^2, M\times N) \arrow[rr] & & S(\mathbb{R}^2, M)\times S(\mathbb{R}^2,N) \\
& S(\mathbb{R}, S(\mathbb{R}, M\times N)) \arrow[u, "\wedge"] \arrow[rr] \arrow[dl, "\delta_0"'] \arrow[dd, "S(\mathbb{R}, \delta_0)"] & & S(\mathbb{R}, S(\mathbb{R}, M))\times S(\mathbb{R}, S(\mathbb{R}, N)) \arrow[u, "\wedge\times\wedge"'] \arrow[dl, "\delta_0"'] \arrow[dd, "S(\mathbb{R}, \delta_0)"] \\
S(\mathbb{R}, T(M\times N)) \arrow[rr] \arrow[dd, "\delta_0"'] & & S(\mathbb{R}, TM)\times S(\mathbb{R}, TN) \arrow[dd, "\delta_0"] & \\
& S(\mathbb{R}, T(M\times N)) \arrow[rr] \arrow[dd, "\delta_0"] & & S(\mathbb{R}, TM)\times S(\mathbb{R}, TN) \arrow[dd, "\delta_0"] \\
T^2(M\times N) \arrow[rr] \arrow[dr, "\varkappa_{M\times N}"'] & & T^2M\times T^2N \arrow[dr, "\varkappa_M\times\varkappa_N"'] & \\
& T^2(M\times N) \arrow[rr] & & T^2M\times T^2N
\end{tikzcd}$$

(k)

$$\begin{tikzcd}
& & S(\mathbb{R}, S(\mathbb{R}, M)) \arrow[d, "S(\mathbb{R}, ev_0)"] \arrow[ddll, bend right, "\delta_0"'] \arrow[ddrr, bend left, "S(\mathbb{R}, \delta_0)"] & & \\
& & S(\mathbb{R}, M) \arrow[dd, "\delta_0"'] & & \\
S(\mathbb{R}, TM) \arrow[drr, "T(ev_0)=ev_0"'] \arrow[dd, "\delta_0"'] & & & & S(\mathbb{R}, TM) \arrow[ull, "S(\mathbb{R}, \pi_M)"'] \arrow[dd, "\delta_0"] \\
& & TM & & \\
T^2M \arrow[urr, "\pi_{TM}"] \arrow[rrrr, "\varkappa_M"] & & & & T^2M \arrow[ull, "T(\pi_M)"']
\end{tikzcd}$$

$$\pi_{TM}\circ\varkappa_M = T(\pi_M).$$

So $\varkappa_M$ is fibre preserving for the two fibrations

$$(T^2M, \pi_{TM}, TM), \qquad (T^2M, T(\pi_M), TM).$$

It remains to show that $\varkappa_M$ is fibre linear. Isomorphism follows then since $\varkappa_M$ is a diffeomorphism. We consider the following diagram :

(l)

$$\begin{tikzcd}
S(\mathbb{R}, TM) \arrow[rr, "T(\delta_0)"] \arrow[dr, "\delta_0"] \arrow[dd, "\pi_{S(\mathbb{R}, M)} = S(\mathbb{R}, \pi_M)"'] & & T^2M \arrow[dd, "\pi_{TM}"] \\
& T^2M \arrow[ur, "\varkappa_M"'] \arrow[dr, "T(\pi_M)"] & \\
S(\mathbb{R}, M) \arrow[rr, "\delta_0"] & & TM
\end{tikzcd}$$

Here the top triangle commutes because it is part of diagram (d). From 5.3 we know that

$$+_{\pi_M} : TM \times TM \to TM$$

is smooth. $+_{\pi_M}$ is fibre linear, so a glance at diagram (a) in 5.2 shows that

$$T\,(+_{\pi_M}) = +_{T(\pi_M)}$$

Let $c \in S(\mathbf{R}, M)$ and let

$$c_1\,,\ c_2 \in S_c(\mathbf{R},\ TM) = \pi_{S(\mathbf{R},\ M)}{}^{-1}(c).$$

$T(\delta_0)$ is fibre linear by 3.3 and 7.5, so we have :

$$T(\delta_0)(c_1 +_{\pi_{S(\mathbf{R},\ M)}} c_2) = T(\delta_0).c_1 +_{\pi_{TM}} T(\delta_0).c_2$$

$$= \varkappa_M\delta_0\,(c_1) +_{\pi_{TM}} \varkappa_M\,\delta_0\,(c_2).$$

$$\delta_0(c_1 +_{\pi_{S(\mathbf{R},M)}} c_2) = \delta_0\,(+_{\pi_{S(\mathbf{R},M)}}(c_1,\ c_2))$$

$$= \delta_0 \circ S(\mathbf{R},\ +_{\pi_M})(c_1,\ c_2) = T(\ +_{\pi_M}) \circ \delta_0\,(c_1,\ c_2)$$

$$= +_{T(\pi_M)}\,(\delta_0 c_1,\ \delta_0 c_2) = \delta_0\, c_1 +_{T(\pi_M)} \delta_0\, c_2\ .$$

So finally we get :

$$\varkappa_M(\delta_0 c_1) +_{\pi_{TM}} \varkappa_M(\delta_0 c_1) = T(\delta_0\ )(c_1 +_{\pi_{S(\mathbf{R},\ M)}} c_2)$$

$$= \varkappa_M \circ \delta_0\,(c_1 +_{\pi_{S(\mathbf{R},\ M)}} c_2) = \varkappa_M(\delta_0\, c_1 +_{T(\pi_M)} \delta_0\, c_2).$$

So $\varkappa_M$ is fibre additive. Fibre linearity follows then by fibre continuity for the C^∞-curve topology.

(6) $\varkappa_{\mathbf{R}^2}$: $T^2\mathbf{R} \to T^2\mathbf{R}$ is given by

$$\varkappa_{\mathbf{R}^2}(x_1,\ x_2,\ x_3,\ x_4) = (x_1,\ x_3,\ x_2,\ x_4)$$

since diagram (a) says in this case :

$$\begin{tikzcd}
 & & f \in C^\infty(\mathbf{R}^2,\ \mathbf{R}^2) = S(\mathbf{R},\ S(\mathbf{R},\ \mathbf{R}^2)) \arrow[dll, mapsto, "\delta_0"'] \arrow[drr, mapsto, "{S(\mathbf{R},\ \delta_0)}"] & & \\
(f(0,\ ), \frac{\partial f}{\partial x_1}(0,\ \ )) \arrow[d, mapsto, "\delta_0"'] & & & & (f(\ ,\ 0), \frac{\partial f}{\partial x_2}(\ ,\ 0)) \arrow[d, mapsto, "\delta_0"] \\
(f(0,\ 0), \frac{\partial f}{\partial x_1}(0,\ 0),\ \frac{\partial f}{\partial x_2}(0,\ 0),\ \frac{\partial^2 f}{\partial x_1\,\partial x_2}(0,\ 0)) \arrow[rrrr, mapsto, bend left, "\varkappa_M"] & & & & (f(0,\ 0),\ \frac{\partial f}{\partial x_2}(0,\ 0),\ \frac{\partial f}{\partial x_1}(0,\ 0),\ \frac{\partial^2 f}{\partial x_1\,\partial x_2}(0,\ 0))
\end{tikzcd}$$

Then

$$\varkappa_M \circ T^2 f = T^2 f \circ \varkappa_{\mathbf{R}^2}$$

is a special case of property (2). It remains to show that $\varkappa_M$ is the only mapping $\lambda : T^2M \to T^2M$ with the property

$$\lambda \circ T^2f = T^2f \circ \varkappa_{\mathbb{R}^2} \quad \text{for all} \quad f \in S(\mathbb{R}^2, M),$$

where $\varkappa_{\mathbb{R}^2}$ is given by the formula above. We will show that any λ with this property fits commutingly into the following diagram, which is diagram (a). So by uniqueness in (1) the assertion follows then.

(m)

$$\begin{tikzcd}
& S(\mathbb{R}^2, M) \arrow[d] \arrow[ddl, bend right, "T(\ )\circ A_0"'] \arrow[ddr, bend left, "T(\ )\circ B_0"] & \\
& S(\mathbb{R}, S(\mathbb{R}, M)) \arrow[dl, "\delta_0"'] \arrow[dr, "S(\mathbb{R}, \delta_0)"] & \\
S(\mathbb{R}, TM) \arrow[d, "\delta_0"'] & & S(\mathbb{R}, TM) \arrow[d, "\delta_0"] \\
T^2M \arrow[rr, dashed, "\lambda"] & & T^2M
\end{tikzcd}$$

Let $c \in S(\mathbb{R}, S(\mathbb{R}, M))$.

$$\delta_0 \circ S(\mathbb{R}, \delta_0)(c) = \delta_0(T\hat{c} \circ B_0) = T(T\hat{c} \circ B_0)(0, 1) = T^2\hat{c} \circ T(B_0)(0, 1)$$

$$= T^2\hat{c}(0, 0 ; 0, 1 ; 1, 0 ; 0, 0) = T^2\hat{c} \circ \varkappa_{\mathbb{R}^2}(0, 0 ; 1, 0 ; 0, 1 ; 0, 0)$$

$$= \lambda \circ T^2\hat{c}(0, 0 ; 1, 0 ; 0, 1 ; 0, 0) = \lambda \circ T^2\hat{c} \circ T(A_0)(0, 1) =$$

$$= \lambda \circ T(T\hat{c} \circ A_0)(0, 1) = \lambda \circ \delta_0 \circ \delta_0(c).$$

QED

7.8. **Theorem.** *If M is a premanifold and N is a manifold, then the mapping* $T : S(M, N) \to S(TM, TN)$ *is smooth.*

Proof. Consider the following diagram :

(a)

$$\begin{tikzcd}
S(\mathbb{R}\times M, N) \arrow[r, "S(0_{\mathbb{R}}\times Id_{TM}\,,\ TN)\circ T"] \arrow[d, equal] \arrow[dd, bend right=60, "T(\ )\circ A_0"'] & S(\mathbb{R}\times TM, TN) \arrow[d, equal] \arrow[dd, bend left=60, "T(\ )\circ A_0"] \\
S(\mathbb{R}, S(M, N)) \arrow[r, "S(\mathbb{R},\ T)"] \arrow[d, "\delta_0"'] & S(\mathbb{R}, S(TM, TN)) \arrow[d, "\delta_0"] \\
S(M, TN) \arrow[r, "S(TM,\ \varkappa_N)\circ T"] & S(TM, T^2N)
\end{tikzcd}$$

Let us check that this diagram commutes. Let

$$c \in S(\mathbb{R}, S(M, N)), \qquad \xi \in TM.$$

$$(\delta_0 \circ S(\mathbb{R}, T)(c))(\xi) = T(\widehat{(S(\mathbb{R}, T)(c))}) \circ A_0(\xi) =$$

$$= T(T\hat{c} \circ (0_{\mathbb{R}} \times Id_{TM})) \circ A_0(\xi) = T^2\hat{c} \circ (T0_{\mathbb{R}} \times Id_{T^2M})(0, 1 ; 0_\xi)$$

$$= T^2\hat{c}((0, 0 ; 1, 0), 0_\xi).$$

$$(\varkappa_N \circ T(\delta_0 c))(\xi) = \varkappa_N \circ T(T\hat{c} \circ A_0)(\xi) = \varkappa_N \circ T^2\hat{c} \circ TA_0(\xi)$$

$$= \varkappa_N \circ T^2\hat{c}(0, 1 ; 0, 0 ; T(0_M).\xi) = T^2\hat{c} \circ \varkappa_{\mathbf{R}\times M}(0, 1 ; 0, 0 ; T(0_M).\xi)$$
by 7.7.2

$$= T^2\hat{c}(\varkappa_{\mathbf{R}}(0, 1 ; 0, 0), \varkappa_M \circ T(0_M)(\xi)) \qquad \text{by 7.7.4}$$

$$= T^2\hat{c}(0, 0 ; 1, 0), 0_{TM}(\xi)) \qquad \text{by 7.7.5 and 6.}$$

So diagram (a) commutes. This says that

$$T : S(M, N) \to S(TM, TN)$$

is S^1 and $T(T) = (\varkappa_N)_* \circ T$, in more detail,

$$T(S(M, N) \xrightarrow{T} S(TM, TN))$$

$$= (S(M, TN) \xrightarrow{T} S(TM, T^2N) \xrightarrow{(\varkappa_N)_*} S(TM, T^2N)).$$

By 7.3, TN is a manifold again, so we may apply the proof up to now to see that

$$T : S(M, TN) \to S(TM, T^2N)$$

is S^1, but then $T : S(M, N) \to S(TM, TN)$ is S^2. By induction we see that T is smooth as claimed. QED

7.9. Theorem. *If M is a premanifold and N is a manifold, then the evaluation mapping $ev : M \times S(M, N) \to N$ is smooth.*

Proof. Consider the following diagram :

$$\begin{tikzcd}
S(\mathbf{R}, M) \times S(\mathbf{R}\times M, N) \arrow[rr, "(Id_{\mathbf{R}},\ ) \times Id"] \arrow[ddd, bend right=60, "\delta_0 \times (T(\ ) \circ A_0)"'] & & S(\mathbf{R}, \mathbf{R}\times M) \times S(\mathbf{R}\times M, N) \arrow[dd, "\text{composition}"] \\
S(\mathbf{R}, M) \times S(\mathbf{R}, S(M, N)) \arrow[u] \arrow[d, equal] & & \\
S(\mathbf{R}, M \times S(M, N)) \arrow[rr, "S(\mathbf{R}, ev)"] \arrow[d, "\delta_0"'] & & S(\mathbf{R}, N) \arrow[d, "\delta_0"] \\
TM \times S(M, TN) \arrow[rr, "T(ev)"] & & TN
\end{tikzcd}$$

where we put for $T(ev)$ the mapping

$$T(ev) = ev \circ (\pi_M \times Id_{S(M, TN)}) +_{\pi_N} ev \circ (Id_{TM} \times T \circ S(M, \pi_N)),$$

$$T(ev)(\xi_m, s) = s(m) +_{\pi_N} T(\pi_N \circ s)(\xi_m).$$

We show that diagram (a) commutes. Let

$$c = (c_1, c_2) \in S(\mathbf{R}, M \times S(M, N)).$$

$T(ev)(\delta_0 c) = T(ev)(\delta_0 c_1, T\hat{c}_2 \circ A) = T\hat{c}_2 \circ A_0(c_1(0)) +_{\pi_N} T(\pi_N \circ T\hat{c}_2 \circ A_0)(\delta_0 c_1)$,

where

$$T(\pi_N \circ T\hat{c}_2 \circ A_0)(\delta_0 c_1) = T(\hat{c}_2 \circ \pi_{\mathbf{R}\times M} \circ A_0)(\delta_0 c_1)$$
$$= T(\hat{c}_2 \circ (0, Id_M))(\delta_0 c_1) = T\hat{c}_2(0, 0 ; \delta_0 c_1).$$
$$\delta_0 \circ S(\mathbf{R}, ev)(c) = \delta_0(\hat{c}_2 \circ (Id , c_1)) = T(\hat{c}_2 \circ (Id , c_1))(0, 1)$$
$$= T\hat{c}_2 \circ (Id_{T\mathbf{R}}, Tc_1)(0, 1) = T\hat{c}_2 \circ (0, 1 ; \delta_0 c_1)$$
$$= T\hat{c}_2((0, 1 ; 0_{c_1(0)}) +_{\pi_{\mathbf{R}\times M}} (0, 0 ; \delta_0 c_1)$$
$$= T\hat{c}_2(0, 1 ; 0_{c_1(0)}) +_{\pi_N} T\hat{c}_2(0, 0 ; \delta_0 c_1)$$
$$= T\hat{c}_2 \circ A_0(c_1(0)) +_{\pi_N} T\hat{c}_2(0, 0 ; \delta_0 c_1).$$

So diagram (a) commutes, so ev is S^1. Since TN is again a manifold, $T(ev)$ is S^1, so ev is S^2 and so on. QED

7.10. **Lemma.** *If M, N are manifolds and P is a premanifold, then the mapping $S(P, \ ) : S(M, N) \to S(S(P, M), S(P, N))$ is smooth.*

Proof. By Lemma 7.5, the spaces $S(P, M)$ and $S(P, N)$ are premanifolds so $S(S(P, M), S(P, N))$ satisfies (M1)-(M3) by 6.1 and we can talk about smooth mappings in the sense of 6.2.

Claim : The mapping $const : \mathbf{R} \to S(P, \mathbf{R})$ is smooth and

$$T(const) = const : T\mathbf{R} \to S(P, T\mathbf{R}).$$

For let $f \in C^\infty(\mathbf{R}, \mathbf{R})$, then

$$\delta_0 \circ (const)_*(f) = \delta_0(const \circ f) = T((const \times f)\hat{\ }) \circ A_0$$
$$= T((const)\hat{\ } \circ (f \times Id_P)) \circ A_0 = T(pr_1 \circ (f \times Id))(0, 1 ; 0_P) =$$
$$= const(Tf(0, 1)) : P \to T\mathbf{R} = const(\delta_0 f).$$

Now consider the following diagram

(a)

$$\begin{tikzcd}
 & S(S(P, \mathbf{R}\times M), S(P, N) \arrow[dr, "S((const,\ Id_{S(P,\ M)}),\ S(P,\ N))"] & \\
S(\mathbf{R}\times M, N) \arrow[ur, "S(P,\ )"] \arrow[dd, bend right=60, "T(\ )\circ A_0"'] & & S(\mathbf{R}\times S(P, M), S(P, N)) \arrow[dd, bend left=60, "T(\ )\circ A_0"] \\
S(\mathbf{R}, S(M, N)) \arrow[u, "\wedge"] \arrow[d, "\delta_0"'] \arrow[rr, "(S(P,\ ))_*"] & & S(\mathbf{R}, S(S(P, M), S(P,N))) \arrow[u] \arrow[d, "\delta_0"] \\
S(M, TN) \arrow[rr, "S(P,\ )"] & & S(S(P, M), S(P, TN))
\end{tikzcd}$$

The top diagram commutes :

$$(S((\mathrm{const}, Id_{S(P,\ M)}), S(P, N))(S(P, \hat{c})))(t, g)$$

$$= S(P, \hat{c}) \circ (\mathrm{const}, id)(t, g) = S(P, \hat{c})(\mathrm{const}(t), g) = \hat{c}(t, g(\ ))$$

$$= ((S(P,\ \ ))_*(c))\hat{\ }(t, g).$$

The outer pentagon commutes :

$$(\ \delta_0 \circ (S(P,\ \ ))_*(c))(g)(p) =$$

$$= (T(S(\mathrm{const}, Id_{S(P,\ M)}), S(P, N)).S(P, \hat{c})) \circ A_0(g))(p)$$

$$= (T(S(P, \hat{c}) \circ (\mathrm{const}, Id_{S(P,\ M)})).(0, 1 ; 0_g))(p)$$

$$= (S(p, T\hat{c}) \circ (T(\mathrm{const}), Id_{TS(P,\ M)}).(0, 1 ; 0_g))(p)$$

$$= (S(P, T\hat{c}).(\mathrm{const}(0, 1), 0_g))(p) = T\hat{c}(0, 1 ; 0_{g(p)}) = T\hat{c} \circ A_0(g(p))$$

$$= (\delta_0 c)(g(p)) = ((\delta_0\, c) \circ g)(p) = (S(P, \delta_0\, c).(g))(p).$$

So diagram (a) commutes, so $S(P,\ \ )$ is S^1 and

$$T(S(P,\ \ )) = S(P,\ \ )$$

is also S^1, so $S(P,\ \ )$ is S^2 and so on. QED

7.11. **Lemma.** *If M, N are manifolds, then the mapping*

$$(\bar{\ }) : S(\mathrm{R}, S(M, N)) \to S(M, S(\mathrm{R}, N)),$$

given by $\bar{c}(m)(t) = c(t)(m)$, *is smooth.*

Proof. For $c \in S(\mathrm{R}, S(M, N))$ we have

$$c = (\hat{c} \circ flip)^{\vee} = S(\mathrm{R}, \hat{c} \circ flip) \circ ins :$$

$$M \longrightarrow S(\mathrm{R}, M \times \mathrm{R}) \longrightarrow S(\mathrm{R}, \mathrm{R} \times M) \longrightarrow S(\mathrm{R}, N).$$

Claim : $\hat{\ } : S(\mathrm{R}, S(M, N)) \to S(\mathrm{R} \times M, N)$ is smooth.

$$\hat{c} = ev \circ (c \times Id_M) : \mathrm{R} \times M \longrightarrow S(M, N) \times M \longrightarrow N,$$

so we have

$$\hat{\ } = S(\mathrm{R} \times M, ev) \circ (. \times Id_M) :$$

$$S(\mathrm{R}, S(M, N)) \longrightarrow S(\mathrm{R} \times M, S(M, N) \times M) \longrightarrow S(\mathrm{R} \times M, N)$$

which is smooth by 7.5, 6.10, 7.9 and 6.4.

Claim : $^\vee : S(M \times \mathbb{R}, N) \to S(M, S(\mathbb{R}, N))$ is smooth.

$$g^\vee = S(\mathbb{R}, g) \circ ins : M \dashrightarrow S(\mathbb{R}, M \times \mathbb{R}) \longrightarrow S(\mathbb{R}, N),$$

so we have

$$^\vee = S(ins, S(\mathbb{R}, N)) \circ S(\mathbb{R}, \ ) :$$

$$S(M \times \mathbb{R}, N) \longrightarrow S(S(\mathbb{R}, M \times \mathbb{R}), S(\mathbb{R}, N)) \longrightarrow S(M, S(\mathbb{R}, N))$$

which is smooth by 7.10, 7.5, 6.5 and 6.4. So finally

$$^- = {}^\vee \circ S(flip, N) \circ {}^\wedge :$$

$$S(\mathbb{R}, S(M, N)) \to S(\mathbb{R} \times M, N) \to S(M \times \mathbb{R}, N) \to S(M, S(\mathbb{R}, N))$$

which is smooth. QED

Note that in this lemma M has to be a manifold : otherwise we cannot form $S(S(\mathbb{R}, M \times \mathbb{R}), S(\mathbb{R}, N))$ without developing a lot more technical background as in §6.

7.12. **Lemma.** *Let M, P be premanifolds and let (E, p, N) be a vector bundle (so N is a manifold). Let $f : M \to N$ be a smooth mapping. Then we have a canonical identification of the following two spaces :*

$$S(P, M) \underset{(f_*, S(P,N), p_*)}{\times} S(P, E) = S(P, M \underset{(f,N,p)}{\times} E) = S(P, f^*E).$$

Proof. First note that $(S(P, E), p_*, S(P, N))$ is a pre-vector bundle by 7.5 (or its method of proof), so by 6.7 the space

$$S(P, M) \underset{(f_*, S(P,N), p_*)}{\times} S(P, E)$$

satisfies (M1)-(M3) ; $S(P, f^*E)$ does it, by 6.1. So it makes sense to ask whether the natural identification of the two spaces makes sense.

Claim : In the setting above we have a diffeomorphism

$$T(f^*E) = (Tf)^*TE$$

in more detail

$$T(M \underset{(f, N,p)}{\times} E) = TM \underset{(Tf,TN, Tp)}{\times} TE.$$

$$T(f^*E) \xrightarrow{\mathrm{Dec}^{f^*E}} TM \underset{M}{\times} f^*E \underset{M}{\times} f^*E = TM \underset{M}{\times} f^*(E \underset{N}{\times} E) =$$

$$= TM \underset{(Tf,TN, pr_1)}{\times} (TN \underset{N}{\times} E \underset{N}{\times} E) = TM \underset{(Tf,TN,Tp)}{\times} TE = (Tf)^*TE.$$

This bijection clearly gives a fibrewise linear and continuous mapping

(a)

$$\begin{tikzcd}
T(f^*E) \arrow[rr] \arrow[dr] & & (Tf)^*TE \arrow[dl] \\
& TM &
\end{tikzcd}$$

which obviously commutes with the parallel transports of the two pre-vector bundles over TM, so the identification above is smooth by Lemma 5.2.

Now we set out to prove the lemma. We have to show that $S(P, f^*E)$ is a pullback. So let X be a premanifold or of the form $S(P, Q)$ and consider a situation as in the following diagram :

(b)

$$\begin{tikzcd}
X \arrow[dr, dashed, "\alpha"] \arrow[drr, "h"] \arrow[ddr, "g"'] & & \\
& S(P, f^*E) \arrow[r] \arrow[d] & S(P, E) \arrow[d, "p_*"] \\
& S(P, M) \arrow[r, "f_*"] & S(P, N)
\end{tikzcd}$$

The mapping α exists by the pullback property of $f^*E = M \times_N E$ and is given by

$$\alpha(x) = (g(x), h(x)) \in S(P, M \times_N E).$$

We have to show that α is smooth. For that we consider the following diagram :

(c)

$$\begin{tikzcd}
& S(\mathbf{R}\times P, M \times_N E) \arrow[rr] \arrow[dr] & & S(\mathbf{R}\times P, E) \arrow[dr, "p_*"] & \\
& & S(\mathbf{R}\times P, M) \arrow[rr, "f_*"] & & S(\mathbf{R}\times P, N) \\
S(\mathbf{R}, X) \arrow[uur, dashed, "\beta"] \arrow[r, dashed, "\alpha_*"] \arrow[rrr, bend left, "h_*"] \arrow[drr, "g_*"'] \arrow[dd, dash, "\delta_0"'] & S(\mathbf{R}, S(P, M \times E)) \arrow[uu, "\wedge"] \arrow[rr] \arrow[dr] \arrow[dd, "\delta_0"] & & S(\mathbf{R}, S(P, E)) \arrow[uu, "\wedge"] \arrow[dr] \arrow[dd, "\delta_0"] & \\
& & S(\mathbf{R}, S(P, M)) \arrow[uu, "\wedge"] \arrow[rr] \arrow[dd, "\delta_0"] & & S(\mathbf{R}, S(P, N)) \arrow[uu, "\wedge"] \arrow[dd, "\delta_0"] \\
TX \arrow[r, dashed, "\gamma"] \arrow[rrr, bend left, "Th"] \arrow[drr, "Tg"'] & S(P, T(M \times_N E)) \arrow[rr, "(Tf)_*"] \arrow[dr] & & S(P, TE) \arrow[dr, "(Tp)_*"] & \\
& & S(P, TM) \arrow[rr, "(Tf)_*"] & & S(P, TN)
\end{tikzcd}$$

In diagram (c) the mapping β is given by the same formula as α above, and clearly $\hat{} \circ \alpha = \beta$. From the claim above we have

$$T(M \underset{N}{\times} E) = TM \underset{TN}{\times} TE,$$

so the mapping γ may be constructed in the same manner as α above. The diagram commutes by the universal properties of the pullbacks involved. So α is S^1 and $T(\alpha) \doteq \gamma$ is of the same form as α, so is S^1 too, so α is S^2 and so on. QED

Remark. In the beginning of the proof we have used a slightly more general version of Lemma 6.7. We used the mapping $f_* = S(P, f)$ instead of ev_n. But the main point in 6.7 is that $T(ev_n) = ev_n$ is fibrewise linear and continuous ;

$$T(f_*) = S(P, Tf)$$

is it too.

7.13. **Theorem.** *If M, N are manifolds, then the set S(M, N) of all smooth mappings from M to N is again a manifold.*

Proof. (M1)-(M6) have already been checked in 6.1 and 7.5.

(M7) We have to show that

$$Pt^{TS(M,N)} : S(R, S(M, N)) \times R \to L(TS(M, N), TS(M, N))$$

is smooth. Note that by 7.5

$$(TS(M, N), \pi_{S(M,N)}, S(M, N)) = (S(M, TN), (\pi_N)_*, S(M, N))$$

is a pre-vector bundle, so $L(TS(M, N), TS(M, N))$ is a premanifold and the question for smoothness makes sense. Let

$$\alpha : S(M \times R, N) \to S(M, N)$$

be given by

$$\alpha(g) = g(\ , 0) = S(M, ev_0) \circ {}^{\vee}(g)$$

which is smooth by 7.11. Then consider the pullback

$$(S(M \times R, N) \times R \underset{(\alpha,\ S(M,N),\ (\pi_N)_*)}{\times} S(M, TN), pr_{1,2}, S(M \times R, N) \times R)$$

which is a pre-vector bundle since α is smooth, $S(M, TN)$ is a pre-vector bundle over $S(M, N)$ and all spaces are premanifolds by 7.5. Then consider the mapping

$$Pt^1 : S(M \times R, N) \times R \underset{(\alpha,\ S(M,N),(\pi_N)_*)}{\times} S(M, TN) \to S(M, TN)$$

given by

$$Pt^1(g, t, s)(m) = \tilde{P}t^{TN}(g(\ , m), t, s(m)) = \tilde{P}t^{TN} \circ (g^{\vee}, const(t), s)(m).$$

Pt^1 is smooth since we may write it as the following sequence of smooth mappings :

$$\begin{tikzcd}
S(M\times \mathbb{R}, N)\times \mathbb{R} \underset{(\alpha,\, S(M,N),(\pi_N)_*)}{\times} S(M, TN) \arrow[d, "{}^{\vee}\times const \times Id_{S(M,TN)}"] & \text{smooth by 7.10, 7.11} \\
S(M, S(\mathbb{R}, N))\times S(M, \mathbb{R}) \underset{((ev_0)_*, S(M,N),(\pi_N)_*)}{\times} S(M, TN) \arrow[d] & \text{smooth by Lemma 7.12} \\
S(M, S(\mathbb{R}, N)\times \mathbb{R} \underset{(ev_0, N,\, \pi_N)}{\times} TN) \arrow[d, "{S(M, \tilde{P}t^{TN})}"] & \text{smooth by Lemma 6.4} \\
S(M, TN) &
\end{tikzcd}$$

Now consider the following mapping :

$$\beta : S(M \times \mathbb{R}, N)\times \mathbb{R} \longrightarrow S(M, N)$$

given by

$$\beta(g, t) = g(\ , t) = S(M, ev)(g^{\vee}, const(t)),$$

which is smooth by 6.4, 7.9 - 7.12, since

$$\beta = S(M, ev) \circ ({}^{\vee} \times const) :$$

$$S(M \times\mathbb{R}, N)\times \mathbb{R} \to S(M, S(\mathbb{R}, N)) \to S(M, \mathbb{R}) = S(M, S(\mathbb{R}, N)\times \mathbb{R}) \to S(M, N),$$

Then consider the mapping Pt^2 which is smoothly given by the following diagram :

$$\begin{tikzcd}
L(S(M, TN), S(M, TN)) \underset{(pr_2, S(M,N),\alpha)}{\times} S(M\times \mathbb{R}, N)\times \mathbb{R} \arrow[dd, equal, "\text{equal by 5.21}"'] \arrow[dr] & \\
& S(M, N)\times S(M\times \mathbb{R}, N)\times \mathbb{R} \arrow[dd, "Id_{S(M,N)}\times \beta"] \\
L(S(M, TN), S(M \times\mathbb{R}, N)\times \mathbb{R} \underset{(\alpha, S(M,N),\, (\pi_N)_*)}{\times} S(M, TN)) \arrow[ur] \arrow[dd, "{L(S(M, TN), Pt^1)}", "\text{smooth by 5.23}"'] & \\
& S(M, N)\times S(M, N) \\
L(S(M, TN), S(M, TN)) \arrow[ur] &
\end{tikzcd}$$

Finally consider the mapping $\widetilde{ins}$ as in the following diagram

$$\begin{tikzcd}
S(\mathrm{R}, S(M, N)) \times \mathrm{R} \arrow[r, "\widetilde{ins}"] \arrow[dr, "{(ev_0 \circ pr_1,\ S(flip, N) \circ \hat{} \circ pr_1,\ pr_2)}"'] & L(S(M, TN), S(M, TN)) \underset{(pr_2, S(M,N), \alpha)}{\times} S(M \times \mathrm{R}, N) \times \mathrm{R} \arrow[d] \\
 & S(M, N) \times S(M \times \mathrm{R}, N) \times \mathrm{R}
\end{tikzcd}$$

which is given by

$$\widetilde{ins}(c, t) = Id_{S_{c(0)}(M,TN)} \in L(S_{c(0)}(M, TN), S_{c(0)}(M, TN)) ;$$

the latter space is the fibre of the pre-vector bundle above over $(c(0), \hat{c} \circ flip, t)$. $\widetilde{ins}$ is smooth since it may be written as the following sequence of smooth mappings :

$$\begin{tikzcd}
S(\mathrm{R}, S(M, N)) \times \mathrm{R} \arrow[d, "{(ev_0 \circ pr_1,\ S(flip, N) \circ \hat{} \circ pr_1,\ pr_2),}"] \\
S(M, N) \times S(M \times \mathrm{R}, N) \times \mathrm{R} \arrow[d, "{(Id_{S(M,TN)})^{\sim} \times Id_{S(M \times \mathbf{R}, N) \times \mathbf{R}} \times Id_{\mathbf{R}}}"] \\
L(S(M, TN), S(M, TN)) \underset{(pr_2, S(M,N), \alpha)}{\times} S(M \times \mathrm{R}, N) \times \mathrm{R}
\end{tikzcd}$$

smooth by 7.11

The mapping

$$(Id_{S(M,\ TN)})^{\sim} : S(M, N) \to L(S(M, TN), S(M, TN))$$

is given by

$$(Id_{S(M,\ TN)})^{\sim}(f) = Id_{S_f(M,\ TN)}$$

as in Lemma 5.22, where we proved that any mapping of this form is smooth. We have now :

$$Pt^{TS(M,\ N)} = Pt^2 \circ \widetilde{ins} : S(\mathrm{R},\ S(M, N)) \times \mathrm{R}$$

$$\to L(S(M, TN), S(M, TN)) \underset{(pr_2, S(M,N), \alpha)}{\times} S(M \times \mathrm{R}, N) \times \mathrm{R}$$

$$\to L(S(M, TN), S(M, TN)),$$

as is easily checked, so $Pt^{TS(M,\ N)}$ is smooth as claimed.

(M8) We have to show that

$$Geo = Geo^{S(M,\ N)} : S(M, TN) \to S(\mathrm{R}, S(M, N))$$

is smooth. In 7.5, (M6), Geo was defined by the formula

$$(Geo(s)(t))(m) = Geo^N(s(m))(t)$$

for $s \in S(M, TN)$, $t \in \mathrm{R}$, $m \in M$. We have

$$((Geo)\hat{}(s,\ t))(m) = (Geo^M)\hat{}(s(m),\ t) = (Geo^M)\hat{} \circ (s,\ const(t))(m),$$

so $(Geo)\hat{}$ is given by the following sequence of mappings :

$$\begin{tikzcd}
S(M,\ TN) \times \mathbb{R} \arrow[d, "Id_{S(M,\ TN)} \times const", "\text{smooth by 7.10}"'] \\
S(M,\ TN) \times S(M,\ \mathbb{R}) \arrow[d, "\text{equal by 7.12}"'] \\
S(M,\ TN \times \mathbb{R}) \arrow[d, "S(M,\ (Geo^N)\hat{}),", "\text{smooth by 7.1 and 6.4}"'] \\
S(M,\ S(\mathbb{R},\ N)) \arrow[d, "\tilde{}", "\tilde{}\text{, smooth by 7.11}"'] \\
S(\mathbb{R},\ S(M,\ N))
\end{tikzcd}$$

So $(Geo)\hat{}$ is smooth. By the lemma in 7.1 this suffices. QED

7.14 **Theorem.** *The category Mf of manifolds and smooth mappings is cartesian closed. That means :*

$S(M,\ S(N,\ P)) = S(M \times N,\ P)$ *holds naturally in* $M,\ N,\ P \in Mf$.

Proof. This is a consequence of the fact that S is an internal hom-functor by 7.13 and 6.4 and that ev and *ins* are smooth in general. For define

$$\begin{tikzcd}
S(M,\ S(N,\ P)) \arrow[r, shift left, "\wedge"] & S(M \times N,\ P) \arrow[l, shift left, "\vee"]
\end{tikzcd}$$

by

$$\hat{f} = ev \circ (f \times Id) : M \times N \longrightarrow S(N,\ P) \times N \longrightarrow P$$

and

$$g^{\vee} = S(N,\ g) \circ ins : M \longrightarrow S(N,\ M \times N) \longrightarrow S(N,\ P).$$

These two mappings are natural and inverse to each other. QED

7.15. **Corollary.** *The following natural mappings are smooth :*

$$\hat{}: S(M,\ S(N,\ P)) \to S(M \times N,\ P), \qquad {}^{\vee}: S(M \times N,\ P) \to S(M,\ S(N,\ P)),$$

$$comp : S(M,\ N) \times S(P,\ M) \to S(P,\ N),$$

$$S(\ ,\ ) : S(M,\ M') \times S(N',\ N) \to S(S(M',\ N'),\ S(M,\ N)),$$

$$\Pi : \Pi\ S(M_i,\ N_i) \to S(\Pi\ M_i,\ \Pi\ N_i).$$

Prooof. It suffices to check that carefully chosen associated mappings are smooth, by cartesian closedness.

$$((\hat{\ }))^{\vee} = ev \circ (ev \times Id) : S(M, S(N, P)) \times M \times N \to S(N, P) \times N \to P,$$

$$((\check{\ }))^{\vee\vee} = ev : S(M \times N, P) \times M \times N \to P,$$

$$comp\hat{\ } = ev \circ (S(M, N) \times ev) : S(M, N) \times S(P, M) \times P \to S(M, N) \times M \to N,$$

$$S(\, , \,) = comp \circ (S(M, M') \times comp) :$$
$$S(M, M') \times S(N', N) \times S(M', N') \to S(M, M') \times S(M', N) \to S(M, N).$$

$(\Pi)\hat{\ }$ is given by the universal product property in the following diagram :

$$\begin{tikzcd}
\Pi S(M_i, N_i) \times \Pi M_i \arrow[r, dashed] \arrow[d, "pr_j \times pr_j"'] & \Pi N_i \arrow[d, "pr_j"] \\
S(M_j, N_j) \times M_j \arrow[r] & N_j
\end{tikzcd}$$

QED

8. Miscellany.

8.1. Let F be a smooth functor from the category of C^∞-complete locally convex spaces and continuous linear mappings into the same category, of one or several variables, even infinitely many, co- or contravariant, as described in 5.4. We recall that F is called C^∞, if

$$\Pi L(V_i, W_i) \to L(F((V_i)_i), F(W_i)_i))$$

is a C^∞-mapping in the sense of §1 (in this formulation F is assumed to be purely covariant).

Theorem. *Let F be a C^∞-functor as described above, let (E^i, p_i, M_i) be vector bundles, one for each variable of F. Then $(F((E^i)_i), (p_i), \Pi M_i)$ is a vector bundle.*

Proof. First note that ΠM_i is a manifold, by 4.1 and checking (M7), (M8) (use 1.21). By 5.7 we get a pre-vector bundle and since F is a C^∞-functor, the parallel transport described in 5.7 is smooth. QED

8.2. **Theorem.** *Consider the situation*

(a)
$$\begin{tikzcd}
f^*E \arrow[r, "p^*f"] \arrow[d, "f^*p"'] & E \arrow[d, "p"] \\
N \arrow[r, "f"] & M
\end{tikzcd}$$

If f is smooth, N is a manifold and (E, p, M) is a vector bundle, then *the pullback* (f^*E, f^*p, N) *is again a vector bundle.*

Proof. We just have to show that the parallel transport Pt^{f^*E} is smooth. This follows from the diagram :

$$\begin{tikzcd}
S(\mathbb{R}, N) \times \mathbb{R} \arrow[rrr, "f_* \times Id"] \arrow[dr, "Pt^{f^*E}"] \arrow[ddr, "(ev_0, ev)" swap] & & & S(\mathbb{R}, M) \times \mathbb{R} \arrow[dl, "Pt^E" swap] \arrow[ddl, "(ev_0, ev)"] \\
& L(f^*E, f^*E) \arrow[r] \arrow[d] & L(E, E) \arrow[d] & \\
& N \times N \arrow[r, "f \times f"] & M \times M &
\end{tikzcd}$$

since

$$L(f^*E, f^*E) = (f \times f)^*L(E, E)$$

is a pullback in the category *pMf* of premanifolds. QED

8.3. **Corollary.** *If in the situation of 8.1 all manifolds* M_i *coincide, we get a vector bundle* $(F((E^i)_i), p, M)$.

Proof. The pre-vector bundle structure has been described in 5.5. Here we use a simpler argument :

$$(F((E^i)_i), p, M) = diag^*(f((E^i)_i), (p_i), \prod_i M).$$

QED

8.4. **Theorem.** *Let* (E, p, M) *be a vector bundle. Let*

$$Q : S(\mathbb{R}, M) \times \mathbb{R} \to L(E, E)$$

be a smooth mapping satisfying all the functional equations of (VB2). In particular (E, Q) *is a pre-vector bundle, called* $(\bar{E}, p, M)$, *with the same fibres as* E. *Suppose furthermore that*

$$\bar{Q} : S(\mathbb{R}, M) \times_M E \to S(\mathbb{R}, E)$$

factors as follows :

(a)
$$\begin{tikzcd}
S(\mathbb{R}, M) \times_M E \arrow[r, "\bar{Q}"] \arrow[d, "\delta_0 \times Id_E" swap] & S(\mathbb{R}, E) \arrow[d, "\delta_0"] \\
TM \times_M E \arrow[r, dashed, "C"] & TE
\end{tikzcd}$$

Then the identity gives a diffeomorphism $J : E \to \bar{E}$.

Remark. 1. In some cases property (a) holds automatically, follows from the functional equations of Q.

In general I have only been able to show that the germ of $\bar{Q}(c, v_x)$ at 0 depends only on the germ of c at 0 and not on c.

2. This result shows that although the smooth structure of a manifold depends heavily on the parallel transport it is somehow independent from the particular parallel transport chosen.

Proof. $\bar{Q}$ is given by

$$\bar{Q}(c, v_x)(t) = Q(c, t)(v_x).$$

$\bar{Q}$ is clearly smooth.

$$\textit{Claim}: ((\nabla^{L(E,E)}Q)(0_M \circ c, 0, 1))(v) = (pr_3 \circ Dec^E \circ C)(\delta_0 c, v).$$

$$= ((\nabla^{L(E,E)}Q)(0_M \circ c, 0, 1))(v) = ((pr_3 \circ Dec \circ TQ)(0_M \circ c, 0, 1))(v)$$

$$= (ev_v \circ pr_3 \circ Dec \circ TQ)(0_M \circ c, 0, 1)$$

$$= (ev_v \circ pr_3 \circ Dec \circ TQ \circ \delta_0)(const(c), Id_{\mathbf{R}})$$

$$= (ev_v \circ pr_3 \circ Dec \circ \delta_0 \circ Q_*)(const(c), Id_{\mathbf{R}}) = (ev_v \circ pr_3 \circ Dec \circ \delta_0)(Q(c, \ ))$$

$$= pr_3 \circ (pr_2 \times ev_v \times ev_v) \circ Dec \circ \delta_0(Q(c, \ ))$$

$$= pr_3 \circ Dec^E \circ T.(ev_v) \circ \delta_0 (Q(c, \ )) = pr_3 \circ Dec^E \circ \delta_0 (Q(c, \ )(v))$$

$$= pr_3 \circ Dec^E \circ C(\delta_0 c, v),$$

where $Dec \doteq Dec^{L(E,E)}$. So the claim follows. Note that C is smooth, since

$$C(u_x, v_x) = \delta_0 (Q(Geo(u_x), \ )(v_x)).$$

Let $(c_1, c_2) \in S(\mathrm{R}, M) \underset{M}{\times} S_{const}(\mathrm{R}, E)$. Then the curve

$$t \mapsto Q(c_1, t)^{-1}.Pt^E(c_1, t) = comp \circ (Q(c_1(.+t), -t), Pt^E(c_1, t))$$

is a smooth curve in $L(E, E)$, and takes values only in the fibre

$$L(E_{c_1(0)}, E_{c_2(0)})$$

so it is a C^∞-curve in the C^∞-complete bornological locally convex space $L(E_{c_1(0)}, E_{c_2(0)})$. By the cartesian closedness proved in §1 the mapping

$$t \mapsto Q(c_1, t)^{-1}.Pt^E(c_1, t).c_2(t)$$

is C^∞ in $E_{c_1(0)}$. So the following diagram makes sense and the top quadrangle commutes, where

$$A(c_1,\ c_2)(t) = (c_1(t),\ Q(c_1,\ t)^{-1}.Pt^{\bar{E}}(c_1,\ t).c_2(t)).$$

Note that

$$S_{const}(\mathbb{R},\ E) = S_{const}(\mathbb{R},\ \bar{E})$$

and that A is invertible, the inverse being given by

$$A^{-1}(c_1,\ c_2)(t) = (c_1t(),\ Pt^{E}(c_1,\ t)^{-1}.Q(c_1,\ t).c_2(t)).$$

(b)

$$\begin{tikzcd}
S(\mathbb{R},\ M)\times_M S_{const}(\mathbb{R},\ E) \arrow[rrr, "A"] \arrow[dr, "Cart^E"] \arrow[dddr, bend right=40, "\delta_0\times\tilde{\delta}_0"'] & & & S(\mathbb{R},\ M)\times_M S_{const}(\mathbb{R},\ \bar{E}) \arrow[dl, "Cart^{\bar{E}}"'] \arrow[dddl, bend left=40, "\delta_0\times\tilde{\delta}_0"] \\
& S(\mathbb{R},E) \arrow[r, "J_*"] \arrow[d, "\delta_0"'] & S(\mathbb{R},\ \bar{E}) \arrow[d, "\delta_0"] & \\
& TE \arrow[d, "Dec^E"] & T\bar{E} \arrow[d, "Dec^{\bar{E}}"'] & \\
& TM\times_M E\times_M E \arrow[r, dashed, "B"] & TM\times_M \bar{E}\times_M \bar{E} &
\end{tikzcd}$$

We will show that a mapping B fits commutingly into this diagram and we will compute its form. Let

$$(c_1,\ c_2)\in S(\mathbb{R},\ M)\times_M S_{const}(\mathbb{R},\ E).$$

Then we have :

$$\begin{aligned}
\delta_0\times\tilde{\delta}_0 A(c_1,\ c_2) &= (\delta_0 c_1,\ Q(c_1,\ 0)^{-1}.Pt^{E}(c_1,\ 0).c_2(0),\\
&\quad \frac{d}{dt}\Big|_0 (Q(c_1,\ t)^{-1}.Pt^{E}(c_1,\ t).c_2(t)))\\
&= (\delta_0 c_1,\ c_2(0),\ c_2'(0) - pr_3\circ Dec^E\circ C(\delta_0 c_1,\ c_2(0))),
\end{aligned}$$

since we may compute as follows :

$$\begin{aligned}
&\frac{d}{dt}\Big|_0 (Q(c_1,\ t)^{-1}.Pt^{E}(c_1,\ t).c_2(t)) =\\
&= (\frac{d}{dt}\Big|_0 (Q(c_1,\ t)^{-1}.Pt^{E}(c_1,\ t)).c_2(0) + c_2'(0),
\end{aligned}$$

since

$$ev : L(E_x,\ E_x)\times E_x \to E_x$$

is bilinear and bounded, $x = c_1(0)$.

$$\frac{d}{dt}\Big|_0 (Q(c_1,\ t)^{-1}.Pt^{E}(c_1,\ t)) = \frac{d}{dt}\Big|_0 (Pt^{L(E,E)}(c_1,\ const(c_1(0)))^{-1}.Q(c_1,\ t)^{-1}) \qquad \text{by 5.8}$$

$$= \nabla^{L(E,E)}_{\frac{\partial}{\partial t}|_0}(Q(c_1,\ t)^{-1}) \qquad \text{by 5.15}$$

$$= -Q(c_1, 0)^{-1}.\nabla^{L(E,E)}_{\frac{\partial}{\partial t}|0}(Q(c_1, t)).Q(c_1, 0)^{-1} \qquad \textbf{by 5.25}$$

$$= -(\nabla^{L(E,E)}Q)(0_M \circ c, 0, 1) = -pr_3 \circ Dec^E \circ C(\delta_0 c_1, \ )$$

by the claim above. So there is a mapping B fitting commutingly into diagram (b) and may be written as

$$B(u_x, v_x, w_x) = (u_x, \mathcal{J}(v_x), \mathcal{J}(w_x - pr_3 \circ Dec^E \circ C(u_x, v_x))).$$

So $\mathcal{J}$ is S^1 and $T\mathcal{J}$ "=" B is S^1 too, so $\mathcal{J}$ is S^2 and by recursion $\mathcal{J}$ is smooth. Note that

$$B^{-1}(u_x, \bar{v}_x, \bar{w}_x) = (u_x, \mathcal{J}^{-1}(\bar{v}_x), \mathcal{J}^{-1}(\bar{w}_x) + pr_3 \circ Dec^E \circ C(u\ , \mathcal{J}^{-1}(v_x))).$$

So diagram (b) makes sense with the arrows A, $\mathcal{J}$, B inverted and the standard recursion argument shows that $\mathcal{J}^{-1}$ is smooth too. QED

8.5. **Proposition.** *Let V be a C^∞-complete bornological lcs. Then V is a manifold in a canonical way, where*

$$\pi_V = pr_1 : TV = V \times V \to V, \quad S(\mathbb{R}, V) = C^\infty(\mathbb{R}, V),$$

$$\delta_t c = (c(t), c'(t)), \quad Pt^{TV}(c, t)(c(0), v) = (c(t), v), \quad Geo^V(v, w)(t) = v + tw.$$

Furthermore the smooth maps between C^∞-complete bornological spaces, viewed at as manifolds, are exactly the C^∞-mappings in the sense of §1.

Proof. (M1)-(M6) is rather trivial. Now we check the last statement : smooth mappings are clearly C^∞ ; the converse holds by 1.25. Using this it is clear that Pt^{TV} is smooth since it respects C^∞-curves :

$$Pt^{TV} : C^\infty(\mathbb{R}, V) \times \mathbb{R} \to L(TV, TV) = V \times V \times L(V, V),$$

$$Pt^{TV}(c, t) = (c(0), c(t), Id_V). \qquad \text{QED}$$

8.6. Let M be a manifold. Note that

$$S(M, \mathbb{R}) = \Gamma(M \times \mathbb{R}, pr_1, M)$$

is a C^∞-complete bornological lcs by 5.18. Consider the mapping :

$$\varepsilon : M \to S(M, \mathbb{R})' = L(S(M, \mathbb{R}), \mathbb{R}), \quad \text{given by } \langle f, \varepsilon(x)\rangle = f(x).$$

Lemma. *$\varepsilon : M \to S(M, \mathbb{R})'$ is smooth, where $S(M, \mathbb{R})'$ is viewed as a manifold in the sense of 8.5.*

Proof. Let $c \in S(\mathbb{R}, M)$. Then

$$\langle f, (\varepsilon \circ c)(t)\rangle = f(c(t)),$$

so

$$ev_f \circ \varepsilon \circ c = f \circ c \in S(\mathbb{R}, \mathbb{R}) = C^\infty(\mathbb{R}, \mathbb{R}) ,$$

so $\varepsilon \circ c$ is a C^∞-curve in $L(S(M, \mathbb{R}), \mathbb{R})$ by 1.20.3. So

$$\varepsilon_* : S(\mathbb{R}, M) \to C^\infty(\mathbb{R}, S(M, \mathbb{R})')$$

makes sense.

Claim : Let

$$df := pr_2 \circ Tf : TM \to T\mathbb{R} = \mathbb{R} \times \mathbb{R} \to \mathbb{R} .$$

Then $d : S(M, \mathbb{R}) \to S(TM, \mathbb{R})$ is linear and continuous.

(a)

$$\begin{tikzcd}
S(M, \mathbb{R}) \arrow[r, equal] \arrow[d, "d"'] & \Gamma(M \times \mathbb{R}, pr_1, M) \arrow[dr, "\nabla^{M \times \mathbb{R}}"] & \\
S(TM, \mathbb{R}) \arrow[r, equal] & & \Gamma(TM \times \mathbb{R}, pr_1, TM) = \Gamma(\pi_M{}^*(M \times \mathbb{R}))
\end{tikzcd}$$

via $f \to (Id\ \ , f)$

It is easy to check that diagram (a) commutes, where $\nabla^{M \times \mathbb{R}}$ comes from the constant parallel transport, and $\nabla^{M \times \mathbb{R}}$ is linear and continuous by 5. 20. So the claim follows. Consider the following diagram

(b)

$$\begin{tikzcd}
S(\mathbb{R}, M) \arrow[r, "(\varepsilon^M)_*"] \arrow[d, "\delta_0"'] & C^\infty(\mathbb{R}, S(M, \mathbb{R})') \arrow[d, "\delta_0"] \\
TM \arrow[r, "{(\varepsilon^M \circ \pi_M , d' \circ \varepsilon^{TM})}"'] & S(M, \mathbb{R})' \times S(M, \mathbb{R})'
\end{tikzcd}$$

We claim that diagram (b) commutes :

$$\delta_0(\varepsilon^M_*(c)) = (\varepsilon^M(c(0)), \frac{d}{dt}\Big|_0 \varepsilon^M(c(t))),$$

$$<f, \frac{d}{dt}\Big|_0 \varepsilon^M(c(t))> = ev_f \frac{d}{dt}\Big|_0 \varepsilon^M(c(t)) = \frac{d}{dt}\Big|_0 (ev_f \varepsilon^M(c(t))),$$

$$= \frac{d}{dt}\Big|_0 (f \circ c)(t) = pr_2(Tf(\delta_0 c)) = df(\delta_0 c) =$$

$$= <df, \varepsilon^{TM}(\delta_0 c)> = <f, (d' \circ \varepsilon^{TM})(\delta_0 c) \ .$$

So diagram (b) commutes, ε^M is S^1 and $T(\varepsilon^M)$ is again S^1, so ε^M is S^2 and by recursion ε^M is smooth. QED

8.7. **Definition.** A manifold M is called *regular,* if the mapping

$$T(\varepsilon^M) : TM \to S(M, \mathbb{R})' \times S(M, \mathbb{R})'$$

is injective.

So we require that the functions in $S(M, \mathbb{R})$ separate points in M and that

$$T_x(\varepsilon_x^M) : T_x M \rightarrow S(M, \mathbb{R})'$$

is injective for each $x \in M$. The second condition means : if $c \in S(\mathbb{R}, M)$ and for all $f \in S(M, \mathbb{R})$ we have $(f \circ c)'(0) = 0$, then $\delta_0 c = 0_{c(0)}$.

8.8. **Theorem.** *Let M be a finite dimensional C^∞-manifold in the usual sense (with charts), paracompact and Hausdorff. Then M is a regular manifold in our sense.*

Proof. M admits a complete Riemannian metric, so it is a premanifold (see 2.3). The exponential map is clearly smooth, so Geo^M is smooth. It is not so easy to check that Pt^{TM} is smooth. This is done in Lemma 8.9 below. QED

8.9. **Lemma.** *Let (E, p, M) be a finite dimensional C^∞-vector bundle in the usual sense (with charts and locally trivial). If M is paracompact, then this bundle admits a connection, and the parallel transport Pt^E induced by this connection turns out to be smooth :*

$$S(\mathbb{R}, M) \times \mathbb{R} \longrightarrow L(E, E).$$

Proof. Let $C : TM \underset{M}{\times} E \rightarrow TE$ be any linear connection in the usual sense, i.e. C is C^∞,

$$(Tp, \pi_E) \circ C = Id, \qquad C(u_x, \ ) : E_x \rightarrow (Tp)^{-1}(u_x)$$

is linear in the (TE, Tp, TM) vector bundle structure for each $u_x \in T_x M$ and

$$C(\ , v_x) : T_x M \rightarrow (\pi_E)^{-1}(v_x)$$

is linear in the (TE, π_E, E) vector bundle structure for all $v_x \in E_x$. Then the parallel transport Pt^E corresponding to this connection is uniquely given by the following diagram :

(a)

$$\begin{tikzcd}
S(\mathbb{R}, M) \underset{M}{\times} E \arrow[r, "\bar{Pt}^E"] \arrow[d, "\delta_t \times Id"] & S(\mathbb{R}, E) \arrow[d, "\delta_t"] \\
TM \underset{M}{\times} E \arrow[r, "C"] & TE
\end{tikzcd}$$

where

$$Pt^E(c, t).v_{c(0)} = \bar{Pt}^E(c, v_{c(0)})(t).$$

For this diagram commutes for all t iff

$$t \mapsto \bar{Pt}^E(c, v_{c(0)})$$

is a horizontal curve in TE, so

$$\nabla^{C}(\bar{Pt}^{E}(c, v_{c(0)}))= 0$$

for the covariant derivative ∇^{C} induced by C. It is well known (and a standard fact of the theory of solutions of ordinary differential equations) that $\bar{Pt}^{E}$ maps smooth curves to smooth curves (note that

$$S(\mathbb{R}, S(\mathbb{R}, M)) = C^{\infty}(\mathbb{R}^{2}, M)).$$

So we have to construct $T(Pt^{E})$ or rather $T(\bar{Pt}^{E})$. We do this with a suitably chosen connection on (TE, Tp, TM). Consider the connection C^{1} given by the diagram :

$$\begin{tikzcd}
& T^2M \underset{(\pi_{TM},TM,Tp)}{\times} TE \arrow[rr, dashed, "C^1"] \arrow[d, "\varkappa_M \times Id"'] & & T^2E \arrow[dd, "\varkappa_E"] \\
\text{(b)} & T^2M \underset{(T(\pi_M),TM,Tp)}{\times} TE \arrow[d, equal] & & \\
& T(TM \underset{M}{\times} E) \arrow[rr, "TC"] & & T^2E
\end{tikzcd}$$

where $\varkappa_M$, $\varkappa_E$ are the canonical flip mappings which can be given in local coordinates and so exist and satisfy 7.7.

It is not so difficult now to show that the parallel transport $\bar{Pt}$ given by the connection C^{1} is exactly $T(\bar{Pt}^{E})$. This process can be repeated and shows that $\bar{Pt}^{E}$ is smooth and by general principles Pt^{E} itself is also smooth. QED

8.10. **Theorem.** *Let M be a regular manifold such that T_xM is a finite dimensional vector space for all $x \in M$. Then M is a C^{∞}-manifold in the usual sense (with charts) and is Hausdorff.*

Proof. The mapping

$$exp^{M} = exp = Geo^{M}(\ )(1) : TM \to M$$

is smooth by 7.1. Fix $x \in M$ and consider the mapping $exp_x : T_xM \to M$.

$$T_0(exp_x) : \{0\} \times T_xM \to T_xM$$

is the identity, since

$$\begin{aligned} T_0(exp_x).v_x &= T_0(exp_x)(\tfrac{d}{dt}|_0(t.v_x)) = T(exp_x).\delta_0(.\ v_x) \\ &= \delta_0(exp_x(.\ v_x)) = \delta_0\, Geo^{M}(v_x) = v_x . \end{aligned}$$

Now consider the mapping

$$\varepsilon \circ \exp_x : T_xM \to S(M, \mathrm{R})'.$$

This is smooth between C^∞-complete bornological lcs, by 8.6, so it is C^∞ by 8.5. Since M is regular,

$$d(\varepsilon \circ \exp_x)(0) = T_0(\varepsilon \circ \exp_x) = T_0\, \varepsilon . T\,(\exp_x) = T_x \varepsilon : T_xM \to S(M, \mathrm{R})'$$

is injective. Let $v_1,\dots, v_n$ be a basis of T_xM, then the elements

$$w_i := d(\varepsilon \circ \exp_x)(0)(v_i)$$

span an n-dimensional linear subspace of $S(M, \mathrm{R})$. Choose $f_1 ,\dots, f_n$ in $S(M, \mathrm{R})$ such that

$$<f_i , w_j> = \delta_{ij}$$

(see Schaeffer, IV,1.1). Put

$$F = (f_1,\dots, f_n) \in S(M, \mathrm{R}^n);$$

then we have a C^∞-mapping

$$F \circ \exp_x = (ev_{f_1} \circ \varepsilon \circ \exp_x, \dots, ev_{f_n} \circ \varepsilon \circ \exp_x) : T_xM \to \mathrm{R}^n$$

such that $d(F \circ \exp_x)(0)$ is invertible (in fact the identity if T_xM has the basis (v_i)). So by the usual inverse function theorem $F \circ \exp_x$ is a diffeomorphism from a convex neighborhood of zero V_x in T_xM onto an open neighborhood of $F(x)$ in R^n. So in particular

$$\exp_x | V_x : V_x \to M$$

is injective.

Claim : Let $c \in S(\mathrm{R}, M)$ with $c(0) = x$. Then there is a piece of a C^∞-curve c_1 in V_x such that $c(t) = \exp_x c_1(t)$ for small t.

In particular, $\exp_x(V_x)$ is open in M in the natural topology (2.2), i.e. the C^∞-curve final topology.

Given $c \in S(\mathrm{R}, M)$ with $c(0) = x$ consider the mapping

$$\varphi := \exp_x \circ Pt^{TM}(c, \ , \ ) : \mathrm{R} \times T_xM \to TM \to M.$$

We have

$$\varphi | \{0\} \times V_x = \exp_x ,$$

so $\varepsilon \circ \varphi$ has at least rank n at $\{0\} \times V_x \subset \mathrm{R} \times T_xM$. Repeating the argument involving w_i and f_i from above we see that $\varepsilon \circ \varphi$ has rank $\geq n$ in a convex neighborhood U of $\{0\} \times V_x \subset \mathrm{R} \times T_xM$. We claim that $\varepsilon \circ \varphi$ has rank n in U. Suppose not, then

$$d(\varepsilon \circ \varphi)(r, v) : \mathrm{R} \times T_xM \to S(M, \mathrm{R})'$$

spans a $(n+1)$-dimensional subspace of $S(M, \mathbb{R})'$; as above choose

$$g_1, \ldots, g_{n+1} \in S(M, \mathbb{R})$$

such that
s

$$(ev_{g_1}, \ldots, ev_{g_{n+1}}) \circ d(\varepsilon \circ \varphi)(r, v)$$

is a linear isomorphism from $\mathbb{R} \times T_xM$ onto $\mathbb{R}^{n+1}$. Then for

$$G = (g_1, \ldots, g_{n+1}) \in S(M, \mathbb{R}^{n+1})$$

we have

$$G \circ \varphi : \mathbb{R} \times T_x M \to \mathbb{R}^{n+1}$$

a C^∞-mapping which is a diffeomorphism at (r, v), so it is a diffeomorphism in a neighborhood W of (r, v) in $\mathbb{R} \times T_xM$. Choose C^∞-curves

$$c_1, \ldots, c_{n+1} : \mathbb{R} \to W \quad \text{with} \quad c_i(0) = (r, v)$$

and such that

$$\{ (G \circ \varepsilon \circ c_i)'(0) : \; i = 1, \ldots, n+1 \}$$

is linear independent in $\mathbb{R}^{n+1}$. Then the curves

$$\bar{c_i} = \varphi \circ c_i = exp \circ Pt^{TM}(c, \; , \;) \circ c_i$$

are in $S(\mathbb{R}, M)$ and

$$\delta \, {}_0\bar{c}_i = T_{(r,v)} \varphi .c_i'(0)$$

are linear independent in $T_{\varphi(r,v)}M$ by the choice of c_i. So

$$dim \; T_{\varphi(r,v)} \; M \geqq n+1 \; .$$

But

$$t \mapsto exp \; Pt^{TM} (c, \; tr, \; tv)$$

is a smooth curve $\tilde{c}$ with

$$\tilde{c}(0) = x \quad \text{and} \quad \tilde{c}(1) = (r, v),$$

so

$$Pt^{TM}(\tilde{c}, 1) : T_xM \to T_{\varphi(r,v)} M$$

is a linear isomorphism, so

$$dim \; T_{\varphi(r,v)} \; M = dim \; T_xM = n.$$

Contradiction. Thus $\varepsilon \circ \varphi$ has rank n in U. Now remember the mapping $F \in S(M, \mathbb{R})$ from above, let $f \in S(M, \mathbb{R})$ be arbitrary and consider $(f, F) \in S(M, \mathbb{R}^{n+1})$. Then $(f, F) \circ \varphi$ has rank n in U , so $(f, F) \varphi(U)$

is an n-dimensional submanifold of $\mathbb{R}^{n+1}$, containing $(f, F) \circ exp_x (V_x)$, which is again an n-dimensional submanifold of $\mathbb{R}^{n+1}$ by the choice of F. So

$$(f, F) \circ \varphi(U) = (f, F) \circ exp_x (V_x)$$

near 0, so

$$(f, F)(c(t)) = (f, F) \circ \varphi(t, 0) = (f, F) \circ exp_x (c_f(t))$$

for some $c_f(t) \in V_x$. Since

$$F(c(t)) = F(exp_x (c_f(t)))$$

and F is injective on $exp_x (V_x)$ we see that $c_f(t)$ does not depend on the choice of f, so

$$c_f(t) = c_1(t) \quad \text{for all} \quad f \in S(M, \mathbb{R}).$$

So finally

$$f(c(t)) = f \circ exp_x \circ c_1(t),$$

i.e.

$$< f, \varepsilon \circ c(t)> = <f, \varepsilon \circ exp_x \circ c_1(t) >$$

for all f, so

$$\varepsilon \circ c(t) = \varepsilon \circ exp_x c_1(t),$$

so

$$c(t) = exp_x c_1(t) \quad \text{for small} \quad t.$$

c_1 is C^∞ since

$$c_1(t) = (F \circ exp_x V_x)^{-1} \circ (F \circ c)(t)$$

So the claim follows.

Now we have constructed the following data : for each $x \in M$ a convex neighborhood of zero V_x in T_xM and a mapping $F_x \in S(M, \mathbb{R}^{n(x)})$ such that

$$F_x \circ exp_x : V_x \to \mathbb{R}^{n(x)}$$

is a diffeomorphism onto its image.

Claim : The mappings

$$(exp_x |V_x)^{-1} : exp_x (V_x) \to V_x \subset T_xM$$

for $x \in M$ generate a C^∞-atlas on M.

Let $x \neq y$ be such that

$$exp_x (V_x) \cap exp_y(V_y) =: U \neq \emptyset \text{ in } M.$$

Then

$$dim\ T_xM = dim\ T_yM$$

for we can join x and y by a smooth curve. Put

$$U_x := (\exp_x | V_x)^{-1}(U), \qquad U_y := (\exp_y | V_y)^{-1}(U).$$

We have to show that

$$(\exp_y |V_y)^{-1} \circ \exp_x : U_x \to U_y$$

is C^∞ . For that consider the mapping

$$(F_x, F_y) \in S(M, \mathrm{R}^{2n}).$$

Then clearly

$$(F_x, F_y) \circ \exp_x : V_x \to \mathrm{R}^{2n} \quad \text{and} \quad (F_x, F_y) \circ \exp_y : V_y \to \mathrm{R}^{2n}$$

have both rank n , are injective, fit together nicely, so they parametrize a submanifold of R^{2n} , the chart change of which is clearly C^∞ and coincides with

$$(\exp_y | V_y)^{-1} \circ \exp_x : U_x \to U_y .$$

Now any mapping in $S(M, \mathrm{R})$ is a smooth function on M with this C^∞ -atlas by construction, so the C^∞-functions separate points, so M is Hausdorff. Any curve in $S(\mathrm{R}, M)$ is a C^∞-curve in this new atlas by the claim above, and conversely by the construction of the charts. It is clear that the identity gives a diffeomorphism between the new M and the old M. Finally note that M is paracompact since it admits a connection. QED

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