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ON THE CATEGORY OF TOPOLOGICAL TOPOLOGIES

by Maria Cristina PEDICCHIO

SUMMARY.

We study the main properties of the category $\mathcal{I}$ of all pairs (Y, Y^*) with Y a topological space and Y^* a topological topology on the lattice of open sets of Y. By $\mathcal{I}$, it is possible to classify monoidal closed and monoidal biclosed structures on *Top*.

INTRODUCTION.

Isbell in [8] gave the following classification for adjoint endofunctors of the category *Top* of topological spaces. A pair $F \dashv G: Top \to Top$ is completely determined by a pair (Y, Y^*) of topological spaces where

$$Y = F(\{*\}) \text{ and } Y^* = G(2),$$

with 2 a Sierpinski space; furthermore the set underlying Y^* is, up to isomorphism, just the lattice of open sets of Y, and the topology of Y^* is a *topological topology*, that is, it makes finite intersection and arbitrary union continuous operations.

It is natural then to define a category $\mathcal{I}$ whose objects are all pairs (Y, Y^*), and whose morphisms are all continuous maps $f: X \to Y$, such that $f^{-1}: Y^* \to X^*$ is continuous too.

Our aim is to study the main properties of such a category $\mathcal{I}$.

Section 1 is completely devoted to recall definitions and properties about topological functors and initially complete categories drawing mainly on Herrlich [6, 7].

In Section 2, first we prove that $\mathcal{I}$ is a small-fibred, initially complete category on *Top*, in order to apply the previous results; then we construct a monoidal closed (also biclosed) structure on $\mathcal{I}$.

Section 3 is devoted to the problem of classification of monoidal closed structures on the category of topological spaces [2, 3]). As a generalization of our results about structures induced by adjoining systems of filters [10], we give a complete description both of all monoidal closed structures and of all monoidal biclosed structures on *Top*, relating them to opportune functors from $\mathfrak{I}$ to *Top*. The author is greatly indebted to G.M. Kelly for many helpful conversations and advices.

1. For a concrete category over a base category X, we mean a pair (A, U), where $U: A \to X$ is a forgetful functor, that is, a faithful, amnestic and transportable functor.

A X-morphism $c: X \to Y$ is said to be a constant morphism provided that, for each Z in X and for all $u, v \in X(Z, X)$, $c.u = c.v$. If X is *Set* or *Top* this definition is just the classical one.

DEFINITION 1.1. A concrete category (A, U) over X is called a *topological category* iff the following hold:

(i) $U: A \to X$ is a topological functor in the sense of [6];

(ii) A is small-fibred;

(iii) every constant morphism $U(A) \to U(B)$ underlies some A-morphism $A \to B$.

If the base category X is *Set* or *Top*, then condition (iii) is equivalent to the following

(iv) For any X with cardinality one, the fibre $U^{-1}(X)$ is a singleton.

We shall say that a concrete category is a *small-fibred, initially complete category* iff it verifies (i) and (ii) of Definition 1.1 (see [1, 11]).

If (A, U) and (B, V) are concrete categories over X, for a *functor F over* X, $F: (A, U) \to (B, V)$, we mean any functor $F: A \to B$ with $V.F \simeq U$.

DEFINITION 1.2. A functor F over X, $F: (A, U) \to (B, V)$, is called an *extension* (of (A, U)) if F is full and faithful.

DEFINITION 1.3. An extension $E: (A, U) \to (B, V)$ is called a *small-*

fibred, initial completion (of (A , U)) if (B , V) is small-fibred and initially complete.

The following list exhibits some properties of small-fibred, initially complete categories over the category *Top* of topological spaces ; the same results are well known for topological categories over *Set* [7].

PROPOSITION 1.4. *If* (A, U) *is a small-fibred, initially complete category over Top, then the following hold:*

(1) A *is finally complete ;*

(2) A *is complete and cocomplete and* U : A → *Top preserves limits and colimits ;*

(3) An A-*morphism is a monomorphism (epimorphism) iff it is injective (surjective);*

(4) A *is wellpowered and cowellpowered;*

(5) U : A → Top *has a full and faithful left adjoint* D *(= discrete structure);*

(6) U : A → *Top has a full and faithful right adjoint* K *(= indiscrete structure);*

(7) For any A-*morphism* f, f *is an embedding iff* f *is an extremal monomorphism iff* f *is a regular monomorphism;* f *is a quotient map iff* f *is an extremal epimorphism iff* f *is a regular epimorphism ;*

(8) For any topological space X, *the fibre* $U^{-1}(X)$ *of* X *is a small complete lattice ;*

(9) Any A-*object* A, *with* $UA \neq \emptyset$ *and* $A = DUA$, *is a separator;*

(10) An A-*object* A *is a coseparator iff there exists an embedding of an indiscrete object with two points into* A *;*

(11) An A-*object* A *is projective iff* $A = DUA$ *;*

(12) An A-*object* A *is injective iff* $UA \neq \emptyset$ *and* $A = KUA$.

PROOF. (1), (2) and (3) follow from [6], Sections 5 and 6.

(4) is true for *Top* is well- and cowellpowered.

(5) and (6) follow by Theorem 7.1 of [6].

(7) follows easily by the definition of regular and extremal morphism.

(8) For any pair A^x, $\bar{A}^x$ of objects in $U^{-1}(X)$, let us put

$A^x \leq \bar{A}^x$ iff there is a A-morphism $g: A^x \to \bar{A}^x$ with $U(g) = 1_X$;
with this order relation, $U^{-1}(X)$ forms a small complete lattice and, for any family $(A_i^x)_{i \in I}$ in the fibre, $inf(A_i^x)$ is the U-initial lifting of the source $(X^i, 1_X: X \to U(A_i^x)_{i \in I})$.

(9), (10), (11) and (12) follow by properties of discrete and indiscrete structures. □

2. Let us recall the following definition from [8].

A topology T on the lattice OY of the open sets of a topological space Y is a *topological topology* iff it makes finite intersection and arbitrary union continuous maps (we use the symbol Y^* for the set OY topologized by T). Isbell [8, 12] has proved that any pair of adjoint endofunctors $F \dashv G$ of *Top* is determined (up to functorial isomorphism) by a pair (Y, Y^*), where $Y = F(\{*\})$ and $Y^* = G(2)$, with 2 a Sierpinski space; furthermore $G(2)$ has, as underlying set, just OY (up to bijection) and Y^* is a topological topology on this lattice.

We define a category $\mathcal{I}$ (= Isbell category) as follows:
Objects of $\mathcal{I}$ are all pairs (X, X^*), with $X \in Top$ and X^* a topological topology on OX. An $\mathcal{I}$-morphism $f: (X, X^*) \to (Y, Y^*)$ is a continuous map $f: X \to Y$ such that $f^{-1}: OY \to OX$ is continuous from Y^* to X^* (we shall write f^* for f^{-1}). We call V the forgetful functor $V: \mathcal{I} \to Top^o$ defined by

$$V(X, X^*) = X^* \quad \text{and} \quad V(f) = f^*,$$

and we call U the forgetful functor $U: \mathcal{I} \to Top$ defined by

$$U(X, X^*) = X \quad \text{and} \quad U(f) = f.$$

Then, if *Adj* denotes the category whose objects are the pairs $F \dashv G$ of adjoint endofunctors of *Top*, and whose morphisms are the pairs of natural transformations

$$(\alpha, \beta): (F, G) \to (F', G'), \quad \text{with } \alpha: F \to F' \text{ and } \beta: G' \to G$$

Isbell's result implies the following theorem:

THEOREM 2.1. *The categories Adj and $\mathcal{I}$ are equivalent.*

PROPOSITION 2.2. *$(\mathcal{J}, U)$ is a small-fibred, initially complete category over Top.*

PROOF. If $(X_i, X_i^*)_{i \in I}$ is a family of $\mathcal{J}$-objects and

$$(X, f_i : (X_i, X_i^*) \to X)_{i \in I}$$

is a sink in *Top*, we denote by X^* the set OX with the initial topology with respect to $(f_i^*)_{i \in I}$. To show that X^* is a topological topology, it suffices to consider the following commutative diagram

$$\begin{array}{ccc} X^* \times X^* & \xrightarrow{\ \cap\ } & X^* \\ {\scriptstyle f_i^* \times f_i^*} \downarrow & & \downarrow {\scriptstyle f_i^*} \\ X_i^* \times X_i^* & \xrightarrow[\ \cap_i\ ]{} & X_i^* \end{array}$$

and to apply properties of the initial topology to the continuous composite $\cap_i . (f_i^* \times f_i^*)$ (the same applies to the union map).. Since, for any

$$h : \ U(X, X^*) = X \longrightarrow \ Y = U(Y, Y^*),$$

$h^* : Y^* \to X^*$ is continuous iff $f_i^* . h^*$ is continuous for any i, then h is an $\mathcal{J}$-morphism iff $h . f_i$ is an $\mathcal{J}$-morphism (for any i). This shows that (X, X^*) is the $\mathcal{J}$-final structure on X with respect to $(f_i)_{i \in I}$.

Any finally complete category is initially complete; if

$$(X, g_i : X \to U(X_i, X_i^*))_{i \in I}$$

is a source in *Top* and Φ denotes the following set:

$$\Phi = \{ X^* \mid X^* \text{ is a topological topology on } OX \text{, and } g_i : (X, X^*) \to (X_i, X_i^*) \text{ is an } \mathcal{J}\text{-morphism for any } i \}$$

then the $\mathcal{J}$-initial structure on X with respect to $(g_i)_{i \in I}$ is the $\mathcal{J}$-final structure on X with respect to all the functions $1 : (X, X^*) \to X$, $X^* \in \Phi$.

□

REMARK 2.3. If S is an open set of a topological topology X^* and r, s, are objects of X^* such that $r \in S$ and $s \supseteq r$, then $s \in S$. Consider the continuous map $\bigcup_{n \in N} : \prod (X^*)_n \to X^*$; since $\bigcup^{-1}(S)$ is open in the topological product $\prod_{n \in N} (X^*)_n$ and $(r_n)_{n \in N} = (r, r, \dots, r) \in \bigcup^{-1}(S)$, then there exists a neighborhood $U_r \subseteq \cup^{-1}(S)$ of (r_n) with

$$U_r = \prod_{n \in N} T_n = T_{l_1} \times T_{l_2} \times \ldots \times T_{l_h} \times OX \times \ldots \times OX.$$

and T_{l_j}, $1 \leq j \leq h$ open sets of X^* containing r. The sequence $(t_n)_{n \in N}$, with $t_n = r$ if $n = l_1, l_2, \ldots, l_h$ and $t_n = s$ otherwise is in U_r, so s is in S.

Note that a construction of $\mathcal{I}$-initial structures by *Top*-final structures generally fails.

As an example, consider a constant map $k: 2 \to U(X, X^*)$, with $(X, X^*) \in \mathcal{I}$, and denote by $3 = \{0, 1, 2\}$ the ordered set $O2$. Since the set $\{0, 2\}$ is open in the final topology 2^*_f on 3 with respect to the map $k^*: X^* \to 3$, it follows, by Remark 2.3, that 2^*_f is not a topological topology, so, if $(2, 2^*)$ is the $\mathcal{I}$-initial structure with respect to k, $2^* \leq 2^*_f$.

By Theorem 2.2, Proposition 1.4 can be applied, hence completeness and cocompleteness of $\mathcal{I}$ follow, where limits and colimits are obtained from limits and colimits in *Top* by $\mathcal{I}$-initial and $\mathcal{I}$-final structures respectively, with respect to the maps of the limit (or colimit) cone of *Top*. Again by Proposition 1.4, $U: \mathcal{I} \to Top$ has a left adjoint D and a right adjoint K with the following properties:

$$\mathcal{I}(D(X), (Y, Y^*)) \approx Top(X, U(Y, Y^*) = Y),$$
$$Top(U(Y, Y^*) = Y, X) \approx \mathcal{I}((Y, Y^*), K(X))$$

for any $X \in Top$ and $(Y, Y^*) \in \mathcal{I}$. If we denote by (X, X^*_D) the discrete structure $D(X)$ and by (X, X^*_K) the indiscrete structure $K(X)$, then X^*_D and X^*_K determine, resp., the coarsest and the finest topological topology definable on OX. For $X = \{*\}$, the fibre

$$U^{-1}(\{*\}) = D(\{*\}) = (\{*\}, 2), \quad K(\{*\}) = (\{*\}, 2).$$

(where 2 denotes an indiscrete space with two objects), hence $\mathcal{I}$ is not topological. Since *Top* is topological over *Set* with the forgetful functor $\bar{U}: Top \to Set$, then $\mathcal{I}$ is small-fibred, initially complete over *Set* with the forgetful functor $W = \bar{U} . U$. We write $\bar{D}$ and $\bar{K}$ for the left and right adjoints to $\bar{U}$ and $\tilde{D}$ and $\tilde{K}$ for the composite functors $D . \bar{D}$ and $K . \bar{K}$.

For $X, Z \in Top$ and $(Y, Y^*) \in \mathcal{J}$, the symbol $X \otimes Y$ denotes the set $\bar{U}(X) \times \bar{U}(Y)$ with the topology $\mathrm{O}(X \otimes Y) = Top(X, Y^*)$, and the symbol $[Y, Z]$ denotes the set $Top(Y, Z)$ with the initial topology with respect to the maps

$$\{\lambda_r : Top(Y, Z) \to Y^*\}_{r \in \mathrm{O}Z} \text{ defined by } \lambda_r(f) = f^{-1}(r),\ f: Y \to Z$$

[8, 10]. Furthermore, from now on, we shall use letters A, B, C to denote $\mathcal{J}$-objects

$$A = (X, X^*), \quad B = (Y, Y^*), \quad C = (Z, Z^*) ;$$

J will be the $\mathcal{J}$-object $D(\{*\}) = (\{*\}, 2)$ and I the $\mathcal{J}$-object

$$K(\{*\}) = (\{*\}, 2).$$

THEOREM 2.4. *The category $\mathcal{J}$ admits a structure of monoidal closed category* [4] $(\mathcal{J}, \circ, I, \{-, -\})$ *with*

$$A \circ B = (X \otimes Y, (X \otimes Y)^* = [X, Y^*]), \quad \{B, C\} = ([Y, Z], [Y, Z]^*)$$

where $[Y, Z^*]$ *is the set* $\mathrm{O}[Y, Z]$ *with the final topological topology with respect to the maps* $\{p_S : Z^* \to \mathrm{O}[Y, Z]\}_{S \in \mathrm{O}Y^*}$, *defined by*

$$p_S(r) = \lambda_r^{-1}(S) \quad \text{with} \quad r \in Z^* \text{ and } \lambda_r : [Y, Z] \to Y^*.$$

PROOF. Since the initial topology on $Top(X, Y^*)$ with respect to all the maps $\{\lambda_S\}_{S \in \mathrm{O}Y^*}$ is a topological topology (see Theorem 2.2), then the tensor product object is in $\mathcal{J}$. For $\mathcal{J}$-morphisms $f: A \to A'$ and $g: B \to B'$ we define

$$f \circ g : (X \otimes Y, [X, Y^*]) \to (X' \otimes Y', [X', Y'^*])$$

as the product $f \times g$. If $r: X' \to Y'^*$ is an open set of $X' \otimes Y'$, then

$$(f \times g)^{-1}(r) = g^* . r . f : X \to Y^*,$$

hence $f \times g : X \otimes Y \to X' \otimes Y'$ is continuous; to prove the continuity of $(f \times g)^* : [X', Y'^*] \to [X, Y^*]$, it suffices to apply properties of the initial topology. $I = (\{*\}, 2)$, so

$$A \circ I \simeq A \text{ and } I \circ B \simeq B, \quad \text{for } A, B \in \mathcal{J}.$$

The associativity of $\circ$ follows from Proposition 2.3 of [10], for

$$(X \otimes Y)^* = [X, Y^*].$$

As for the internal hom we get the final topological topology $[Y,Z]^*$ as in Theorem 2.3 and the $\mathcal{J}$-morphism $\{f,g\}:\{B,C\}\to\{B',C'\}$ is $hom(f,g)$, for any $f: B'\to B$ and $g: C\to C'$. If $B=(Y,Y^*)\in\mathcal{J}$, it is known by [8], that

$$Top(X\otimes Y,Z)\underset{\phi}{\xrightarrow{\simeq}} Top(X,[Y,Z]).$$

Let us denote by $k: X\to[Y,Z]$ the map $\phi(h)$, for any $h: X\otimes Y\to Z$. h^* is continuous iff the following composite maps t_S

$$\begin{array}{ccc} Z^* & \xrightarrow{\quad t_S \quad} & X^* \\ & {\scriptstyle h^*}\searrow \quad \nearrow{\scriptstyle \lambda_S} & \\ & [X,Y^*] & \end{array}$$

are continuous, for any $S\in \mathrm{O}\,Y^*$, but, for the commutative diagram

$$\begin{array}{ccc} Z^* & \xrightarrow{\quad t_S \quad} & X^* \\ & {\scriptstyle p_S}\searrow \quad \nearrow{\scriptstyle k^*} & \\ & [Y,Z]^* & \end{array}$$

and, by properties of the final topological topology, t_S is continuous (for any S) iff k^* is continuous; hence

$$\phi: \mathcal{J}(A\circ B,C) = \mathcal{J}((X\otimes Y,[X,Y^*]),(Z,Z^*)) \xrightarrow{\simeq}$$
$$\to \mathcal{J}(A,\{B,C\}) = \mathcal{J}((X,X^*),([Y,Z],[Y,Z]^*))$$

and $-\circ B \dashv \{B,-\}$ for any $B\in\mathcal{J}$. □

We end this section with two further comments on Theorem 2.4. The tensor $\circ$ is not symmetric, for there exist non symmetric monoidal closed structures over *Top* (for example Greve structures [5]), induced by $(\mathcal{J},-\circ-,I,\{-,-\})$ (see Section 3). $A\circ -$ preserves coproducts and coequalizers for any $A\in\mathcal{J}$, then by Freyd's Special Adjoint Functor Theorem, it has a right adjoint, hence the $\mathcal{J}$-structure is biclosed.

3. Consider *Top* as a concrete category over *Top* itself:

$$(Top,\ 1: Top\to Top).$$

PROPOSITION 3.1 (see also [10]). *A monoidal closed structure on Top,*

$$(Top, -\Box-, \{*\}, <-,->)$$

is equivalent to a functor over Top, $F: (Top, 1) \to (\mathcal{I}, U)$ *such that*

(i) $F(\{*\}) = I$,

(ii) $F(X \otimes UFY) \approx F(X) \circ F(Y)$.

PROOF. By [8] (or [12]),

$$F(Y) = <Y, 2>, \quad <Y, Z> \approx [Y, VF(Z)], \quad X \Box Y = X \otimes UF(Y)$$

for $X, Y, Z \in Top$. From Proposition 2.3 of [10], (i) and (ii) are equivalent to monoidality of $(Top, -\Box-, \{*\})$. □

Then, for any monoidal closed structure

$$(Top, -\Box-, \{*\}, <-,->),$$

the associated functor F is strict monoidal and the pair $(\mathcal{I}, F: Top \to \mathcal{I})$ is a small-fibred, initial completion of $(Top, 1)$. Further the monoidal structure $(\mathcal{I}, -\circ-, I)$ restricted to the full subcategory $F(Top) \approx Top$, is just $(Top, -\Box-, \{*\})$.

EXAMPLES. The canonical symmetric monoidal closed structure on *Top* (separate continuity and pointwise convergence) is determined by $F(Y) = (Y, Y^*)$ where the topology of Y^* is generated by the family of all principal ultrafilters of open sets. Similarly the $\mathfrak{A}$-structures of Booth and Tillotson [2] and Greve [5] are obtained associating to any $Y \in Top$ the topological topology generated by the family

$$\{<h(X)>,\ h \in Top(X, Y),\ X \in \mathfrak{A}\}$$

where $<h(X)>$ denotes the set of all open sets of Y containing $h(X)$.

PROPOSITION 3.2. *A monoidal biclosed structure over Top is equivalent to a pair of functors over Top,* $(F, G): Top \to \mathcal{I}$, *such that F satisfies (i) and (ii) of Proposition 3.1 and* $V.F \dashv V^\circ.G^\circ : Top \to Top^\circ$.

PROOF. If

$$(Top, -\Box-, \{*\}, <-,->, <<-,->>)$$

is a monoidal biclosed structure, F and G are defined by

$$F(X) = (X, <X, 2>) \quad \text{and} \quad G(X) = (X, <<X, 2>>).$$

Since, for X, Y, $Z \in Top$,

$$Top^{o}(<X, Z>, Y) \approx Top(X, <<Y, Z>>),$$

then if $Z = 2$,

$$Top^{o}(VF(X), Y) \approx Top(X, V^{o}G^{o}(Y)),$$

so $V.F \dashv V^{o}.G^{o}$. Conversely, F determines a monoidal closed structure

$$(Top, -\Box-, \{*\}, <-,->) \quad \text{with } <X, 2> = VF(X)$$

and G determines an adjunction $-\bar{\Box}Y \dashv <<Y, \cdot>>$ with

$$<<-,->> \cdot\ Top^{o} \times Top \to Top, \quad <<Y, 2>> = VG(Y).$$

$V.F \dashv V^{o}.G^{o}$ implies

$$Top(Y\Box X, 2) \approx Top(Y, <X, 2>) \approx Top(X, <<Y, 2>>) \approx$$
$$\approx Top(X\bar{\Box}Y, 2).$$

Since

$$Top(Y\Box X, 2) \approx Top(X\bar{\Box}Y, 2)$$

and $(2, 2)$ is a strong cogenerator of Top, then $Y\bar{\Box}X \approx Y\Box X$, for any X, hence $Y\Box- \dashv <<Y, \cdot>>$, for any Y. $\Box$

Observe that a symmetric monoidal closed structure over Top is determined by a pair (F, G) with $G = F$.

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