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ET GÉOMÉTRIE DIFFÉRENTIELLE

CONTINUOUS FAMILIES: CATEGORICAL ASPECTS

by David B. LEVER

CONTENTS.

0. INTRODUCTION.

Let *Top* denote the category of topological spaces and continuous functions, *Set* denote the category of sets and functions, and *SET* denote the *Top*-indexed category of sheaves of sets (see Section 1 for the definition of *SET* or see [16] for the theory of indexed categories). Although *SET* has an extensive history [8] its properties as an indexed category have been neglected until now. Accepting the view of the working mathematician [9, 11] that a sheaf of sets on a topological space X is a local homeomorphism whose fibers form a family of sets varying continuously over the space, the theory of indexed categories provides a language in which *SET* is *Set* suitably topologized, in that we have specified a «continuous function» $X \to SET$ to be a sheaf of sets on X. When *Set* is identified with the category of discrete topological spaces, we get $SET^1 = Set$. Guided by our view that *SET* is the «category of sets» of the *Top*-indexed world, this paper investigates some of the category theory of this *Top*-indexed category.

The importance of the category SET^X of sheaves of sets on a topological space X was emphasized by Grothendieck [2]. Later, Lawvere-

Tierney topos theory discovered the relationship between geometry and logic [18]. In the wake of the methods of topos theory, the last decade has seen considerable interest in categorical topology, a subject arising out of the older wish to have universal function spaces [17], while in [14] Niefield has characterized the admissibility of the exponent of a relative function space and she has shown local homeomorphisms fit into the picture. If *SET* is to be a convenient setting for mathematics, analogous to the discrete case, the category SET^{C} of continuous functors [13] on a topological category C (category object in *Top* [1, 6, 7]) should be a Grothendieck topos. This has been established in [12].

Below, we see *SET* is well-powered, cowell-powered, and has small homs. If C is a finite topological category, SET^{C} is shown to be equivalent to a presheaf category over *Set*. Our main result characterizes the *Top*-indexed functors $SET^{T} \to SET^{\mathrm{D}}$ in terms of the preservation of filtered colimits at *1* when T is a topological space and D is a topological category. In particular, if a «continuous functor» $Set \to Set$ is taken to be a *Top*-indexed functor $SET \to SET$, it is just an ordinary functor $Set \to Set$ preserving filtered colimits. Also, it follows the *Top*-indexed algebras of triples on *SET* are the same as the finitary algebras on *Set*.

1. CONTINUOUS FAMILIES.

The following is well known and easily established.

PROPOSITION 1.1. *(1) All homeomorphisms are local homeomorphisms.*

(2) If α and β are local homeomorphisms and composable, $\beta\alpha$ is a local homeomorphism.

(3) If α and β are local homeomorphisms and $\alpha = \beta\delta$, δ is a local homeomorphism.

(4) If $S \to X$ is a local homeomorphism, for all continuous functions $f: Y \to X$, the pullback $f^*S \to Y$ is a local homeomorphism.

(5) $S \to 1$ is a local homeomorphism iff S is a set (we identify Set with the category of discrete topological spaces).

(6) The image factorization of a local homeomorphism results in two local homeomorphisms.

(7) If

$$\begin{array}{ccc} S & \xrightarrow{a} & S' \\ \downarrow & & \downarrow \\ X & \xrightarrow{=} & X \end{array}$$

is a commutative diagram in Top with a a local homeomorphism, a is an isomorphism (monomorphism, epimorphism) iff for each point $x: 1 \to X$ *the fiber* $x^* a: x^* S \to x^* S'$ *of a over x is an isomorphism (monomorphism, epimorphism).* □

Benabou [3] has defined the notion of a calibration on a category in order to formalize the notion of relative smallness. The first four properties stated in 1.1 assert

PROPOSITION 1.2. *The class of local homeomorphisms calibrates Top.* □

DEFINITION. The *Top*-indexed category *SET* is given at $X \in Top$ by taking SET^X to be the comma category of local homeomorphisms with codomain X, i.e. sheaves on X, and by taking substitution along $f: Y \to X$ to be the pullback functor $f^*: SET^X \to SET^Y$.

PROPOSITION 1.3. *SET has stable monomorphisms, stable subobjects, stable epimorphisms, stable quotients, stable equivalence relations, stable finite limits and stable colimits.*

PROOF. By stable monomorphisms we mean: if $S' \rightarrowtail S$ is a monomorphism in SET^X and if $f: Y \to X$ is a continuous function then $f^* S' \to f^* S$ is a monomorphism. The other stability properties are defined in the same way [16]. These stability properties are well-known [2]. □

DEFINITION. The *Sierpinski two-point space* 2 has underlying set $\{1, 0\}$ and topology of $\{1\}$ open but not closed.

DEFINITION. For $X \in Top$, $2_X \to X$ is $proj: 2 \times X \to X$, and $t_X: X \to 2_X$ is $1 \times X: X \to 2 \times X$.

PROPOSITION 1.4. $t_X: X \to 2_X$ *is an open inclusion, so it is a subobject of 1 in* SET^X. □

PROPOSITION 1.5. *For a local homeomorphism* $S \to X$ *and continuous function* $f\colon Y \to X$, *there are bijections*

$$
\begin{array}{c}
\begin{array}{ccc}
S \underset{X}{\times} Y & \xrightarrow{\;<U>\;} & 2_X \\
\downarrow & & \downarrow \\
X & \xrightarrow{\;=\;} & X
\end{array} \\
\hline
\begin{array}{ccc}
<U>^{*} t_X & \xrightarrow{\;open\;} & S \underset{X}{\times} Y \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\;=\;} & Y
\end{array} \\
\hline
\text{subobject } U \rightarrowtail f^{*} S \quad \text{in } SET^{Y}.
\end{array}
$$

These bijections are natural in the variables $S \to X$ *and* $f\colon Y \to X$. □

Recall from [14] a continuous function $C \to X$ is cartesian if for each continuous function $Z \to X$ there is a continuous function $Z^C \to X$ and bijection

$$
\begin{array}{c}
\begin{array}{ccc}
Y & \longrightarrow & Z^C \\
\downarrow & & \downarrow \\
X & \xrightarrow{\;=\;} & X
\end{array} \\
\hline
\begin{array}{ccc}
Y \underset{X}{\times} C & \longrightarrow & Z \\
\downarrow & & \downarrow \\
X & \xrightarrow{\;=\;} & X
\end{array}
\end{array}
$$

which is natural in the variables $Y \to X$ and $Z \to X$. In [14], it is shown $C \to X$ is cartesian iff there is a continuous function $2_X^C \to X$ and bijection

$$
\begin{array}{c}
\begin{array}{ccc}
Y & \longrightarrow & 2_X^C \\
\downarrow & & \downarrow \\
X & \xrightarrow{\;=\;} & X
\end{array} \\
\hline
\begin{array}{ccc}
Y \underset{X}{\times} C & \longrightarrow & 2_X \\
\downarrow & & \downarrow \\
X & \xrightarrow{\;=\;} & X
\end{array}
\end{array}
$$

which is natural in the variable $Y \to X$.

Additionally, in [14] it is shown every local homeomorphism is cartesian.

Recall from [16] the *Top*-indexed category *SET* is well-powered if for each $X \in Top$ and $S \in SET^X$ there is a continuous function $sub^X S \to X$ and bijection

$$\frac{\begin{array}{ccc} Y & \xrightarrow{\langle U \rangle} & sub^X S \\ {\scriptstyle f}\downarrow & & \downarrow \\ X & \xrightarrow{=} & X \end{array}}{\textit{stable subobject } U \rightarrowtail f^* S \textit{ in } SET^Y}$$

which is natural in the variable $f\colon Y \to X$.

PROPOSITION 1.6. *SET is well-powered.*

PROOF. An object of SET^X is a local homeomorphism $S \to X$, which as we have noted is a cartesian function. Therefore if $S \in SET^X$, we have the natural bijection

$$\frac{\begin{array}{ccc} Y & \longrightarrow & 2^S_X \\ \downarrow & & \downarrow \\ X & \xrightarrow{=} & X \end{array}}{\begin{array}{ccc} Y \times_X S & \longrightarrow & 2_X \\ \downarrow & & \downarrow \\ X & \xrightarrow{=} & X \end{array}}$$

This in combination with Proposition 1.5 gives the natural bijection

$$\frac{\begin{array}{ccc} Y & \xrightarrow{\langle U \rangle} & 2^S_X \\ {\scriptstyle f}\downarrow & & \downarrow \\ X & \xrightarrow{=} & X \end{array}}{\textit{subobject } U \rightarrowtail f^* S \textit{ in } SET^Y}.$$

Therefore, *SET* is well-powered. □

Recall from [16] the definitions of cowell-poweredness and small homs.

PROPOSITION 1.7. *SET has small homs and is cowell-powered.*

PROOF. *SET* is well-powered by Proposition 1.6 and has finite stable limits by Proposition 1.3.

Let $\alpha : S \to S'$ be a morphism in SET^X, $X \in Top$. Let $M_\alpha \rightarrowtail X$ be the equalizer of

$$X \underset{<1_{S \times_{S'} S}>}{\overset{<(\alpha,\alpha)>}{\rightrightarrows}} sub^X(S \times_{S'} S).$$

Then $f: Y \to X$ factors through $M_\alpha \rightarrowtail X$ iff $f^*\alpha : f^*S \to f^*S'$ is a monomorphism. Let $I_\alpha \rightarrowtail M_\alpha$ be the equalizer of

$$M_\alpha \underset{<M_\alpha^* \alpha>}{\overset{<1_{M_\alpha^* S'}>}{\rightrightarrows}} sub^{M_\alpha}(M_\alpha^* S').$$

Then $f: Y \to X$ factors through $I_\alpha \rightarrowtail X$ iff $f^*\alpha : f^*S \to f^*S'$ is an isomorphism.

Now let S and S' be any two objects of SET^X, $X \in Top$. Let

$$P \rightarrowtail (sub^X S \times S')^* S \times S'$$

in $SET^{sub^X S \times S'}$ be the generic subobject of $S \times S'$, and let

$$p : P \longrightarrow (sub^X S \times S')^* S$$

be the projection. We have natural bijections

$$\begin{array}{ccc} Y & \xrightarrow{\quad <\alpha> \quad} & I_p \\ & & \downarrow \\ & & M_p \\ \Big\downarrow f & & \downarrow \\ & & sub^X(S \times S') \\ & & \downarrow \\ X & \xrightarrow{\quad = \quad} & X \end{array}$$

---

subobject $<\alpha>^*P \rightarrowtail f^*(S \times S') \xrightarrow{\cong} f^*S \times f^*S'$ in SET^Y such that $proj: <\alpha>^*P \to f^*S$ is an isomorphism

---

Morphism $\alpha : f^*S \to f^*S'$ in SET^Y.

Therefore, $I_p \to X$ serves as the object of morphisms $S \to S'$. Hence, the *Top*-indexed category *SET* has small homs.

For cowell-poweredness, let $S \in SET^X$ and $R \rightarrowtail (sub^X S \times S)^* S \times S$

be the generic subobject of $S \times S$. Let $(sub^X S \times S)^* S \twoheadrightarrow Q$ be the coequalizer of the pair of projections $R \rightrightarrows (sub^X S \times S)^* S$, and let $K \rightrightarrows (sub^X S \times S)^* S$ be the kernel pair of $(sub^X S \times S)^* S \twoheadrightarrow Q$. We have the comparison

$$\begin{array}{ccc} R & \rightrightarrows & (sub^X S \times S)^* S \\ \theta \downarrow & & \downarrow = \\ K & \rightrightarrows & (sub^X S \times S)^* S \end{array}$$

making the corresponding squares commute. As above, or as in [15], let $I_\theta \rightarrowtail sub^X S \times S$ be the subobject in Top such that $\langle U \rangle : Y \to sub^X S \times S$ factors through $I_\theta \rightarrowtail sub^X S \times S$ iff $\langle U \rangle^* \theta$ is an isomorphism. We have natural bijections

$$\begin{array}{ccc} Y & \xrightarrow{\langle R \rangle} & I_\theta \\ \downarrow & & \downarrow \\ X & \xrightarrow{=} & X \end{array} \quad \text{in } Top$$

$$\overline{\text{kernel pair } R \rightrightarrows f^* S \text{ in } SET^Y}$$

$$\overline{\text{quotient } f^* S \twoheadrightarrow (f^* S)/R \text{ in } SET^Y .}$$

Therefore, $I_\theta \to X$ will serve as $epi^X S \to X$. So SET is cowell-powered. □

Let $C \to X$ be cartesian. Taking a point $x: 1 \to X$, we get a bijection

$$\begin{array}{ccc} 1 & \xrightarrow{\langle U \rangle} & 2^C_X \\ x \downarrow & & \downarrow \\ X & \xrightarrow{=} & X \end{array}$$

$$\overline{\text{open subset } U \subset x^* C .}$$

Therefore, the underlying set of the fiber of $2^C_X \to X$ over the point x is isomorphic to $\{\text{open } U \subset x^* C\}$. In [14], a description of a topology on

$$\coprod_{x \in X} \{\text{ open } U \subset x^* C\}$$

making

$$proj: \coprod_{x \in X} \{\text{ open } U \subset x^* C\} \longrightarrow X$$

isomorphic to $2^C_X \to X$ in TOP^X is given (TOP denotes the indexing of *Top* by itself). When $X = 1$, this topology is called the Scott-topology; for general $C \to X$, we will call this topology the *Niefield-Scott topology,* signaling its appearance by the notation $\theta(C) \to X$. In [14] a subbasic open $H \subset \theta(C)$ is given by the requirement that it be saturated, binding, and have fup - saturated means if $x \in X$ then

$$U \in x^* H \text{ and } U \subset V \subset x^* C \text{ imply } V \in x^* H,$$

binding means if $U \subset C$ is open then $\{x \mid x^* U \in x^* H\}$ is an open subset of X, and fup means if $x \in X$ then

$(\bigcup_{a \in A} U_a) \in x^* H$ implies $(\bigcup_{a \in F} U_a) \in x^* H$ for some finite subset F of A .

Let $S \to X$ be a local homeomorphism. If

$$\begin{array}{ccc} W & \overset{s}{\rightarrowtail} & S \\ {\scriptstyle =}\uparrow & & \downarrow \\ W & \underset{open}{\rightarrowtail} & X \end{array}$$

is a local section of $S \to X$, let the subset $H_s \subset \theta(S)$ be defined at $x \in X$ by

$$x^* H_s = \begin{cases} \emptyset & \text{if } x \notin W \\ \{U \subset x^* S \mid s(x) \in U\} & \text{if } x \in W. \end{cases}$$

Clearly H_s is saturated and has fup. It is binding because if $U \subset S$ is open, then

$$\{x \mid x^* U \in x^* H_s\} = s^{-1} U.$$

Therefore, H_s is a subbasic open of $\theta(S)$ for each local section s of $S \to X$. Conversely, let $H \subset \theta(S)$ be saturated, binding, and have fup. If $U \in x^* H$ is non empty, then by the three given properties we can choose a finite set of local sections

$$\left\{ \begin{array}{ccc} W_1 & \overset{s_1}{\rightarrowtail} & S \\ {\scriptstyle =}\uparrow & & \downarrow \\ W & \underset{open}{\rightarrowtail} & X \end{array}, \quad \ldots, \quad \begin{array}{ccc} W_n & \overset{s_n}{\rightarrowtail} & S \\ {\scriptstyle =}\uparrow & & \downarrow \\ W & \underset{open}{\rightarrowtail} & X \end{array} \right\}$$

such that

$$s_i(x) \in U \text{ for each } i = 1, \ldots, n \text{ and } H_{s_1} \cap \ldots \cap H_{s_n} \subset H.$$

On the other hand, if $x \in X$ then the only neighborhoods of $\emptyset$ in $\theta(S)$ over x are $(\theta(S) \to X)^{-1} W$ for neighborhoods W of x. Therefore, together with the inverse images of opens of X along $\theta(S) \to X$, a subbasis of $\theta(S)$ may be taken to be subsets $H_s \subset \theta(S)$ defined by local sections s of $S \to X$.

THEOREM 1.1. *If $X \in Top$ and $S \in SET^X$, $sub^X S \to X$ is*

1. *an open function,*
2. *a closed function, and*
3. *a cartesian function.*

PROOF. Because $\theta(S) \to X$ and $sub^X S \to X$ are TOP^X homemorphic, we may work with the former of these.

1. To see $\theta(S) \to X$ is an open function, it is enough to see the image of a basic open is open. So let $H = H_1 \cap \ldots \cap H_n$ where $H_1, \ldots, H_n \subset \theta(S)$ are saturated, binding and have fup. Let $x \in X$ and $V_x \in x^* H$. Then V_x is an open subset of S_x and we can find an open subset V of S such that $x^* V = V_x$. By the binding property,

$$\begin{aligned} W &= \{ x' \mid x'^* V \in x'^* H \} \\ &= \{ x' \mid x'^* V \in x'^* H_1 \} \cap \ldots \cap \{ x' \mid x'^* V \in x'^* H_n \} \end{aligned}$$

is an open subset of X. But $x \in W$ and W is a subset of the image of $H \to X$. Therefore, $\theta(S) \to X$ is an open function.

2. Let $C \subset \theta(S)$ be a closed subset and let V be the complement of C. If $\emptyset \subset S$ is the empty subset of S then from the pullback

$$\begin{array}{ccccc} X & \xrightarrow{<\emptyset>} & sub^X S & \xrightarrow{\cong} & \theta(S) \\ \uparrow & & & & \uparrow \\ <\emptyset>^* V & & \longrightarrow & & V \end{array}$$

we see $x \in <\emptyset>^* V$ iff the fiber $x^* V$ contains $<\emptyset>(x)$ as an element. Now $x^* V$ is an open subset of $x^* \theta(S)$; therefore, all of $x^* \theta(S)$ if and only if $x \in <\emptyset>^* V$. Therefore, the image of $C \to X$ is the complement of the open subset $<\emptyset>^* V$ of X. Hence, $\theta(S) \to X$ is a closed function.

3. In [14], it is shown that a continuous function $Y \to X$ is cartesian if for each

$$x_0 \in X, \ \textit{open} \ U_{x_0} \subset x_0^* Y, \ \text{and} \ y \in U_{x_0}$$

there is $H \subset \theta(Y)$ such that $U_{x_0} \in H$ and H is saturated, binding, has fup, and $\cap H \subset Y$ is a (not necessarily open) neighborhood of y; here, $\cap H$ is defined at $x \in X$ by

$$x^* \cap H = \begin{cases} x^* Y & \text{if } x^* H = \emptyset \\ \bigcap_{V \in x^* H} V & \text{if } x^* H \neq \emptyset. \end{cases}$$

We will apply this to show $\theta(S) \to X$ is cartesian. For purposes of notation, let $Y \to X$ be $\theta(S) \to X$. Fix

$$x_0 \in X, \ \textit{open} \ U_{x_0} \subset x_0^* Y, \ \text{and} \ y \in U_{x_0}.$$

Now y is a subset of $x_0^* S$. First suppose y is the empty subset of $x_0^* S$. Then U_{x_0} is all of $x_0^* Y$. Define $H \subset \theta(Y)$ by $x^* H = \{x^* Y\}$, $x \in X$, so $x^* H$ is a singleton for each $x \in X$. Then $U_{x_0} \in x_0^* H$. Also $\cap H = Y$ is a neighborhood of y. H is saturated because $x^* H$ is the maximal subset of $x^* Y$ for each $x \in X$. H has fup because $x^* Y$ is compact for each $x \in X$. To see H is binding it is enough to see if W is a basic open of Y then $\{x \mid x^* W \in x^* H\}$ is open. But if $W = H_{s_1} \cap \ldots \cap H_{s_n}$ for local sections $s_1, \ldots, s_n$ of $S \to X$ then

$$\{x \mid x^* W \in x^* H\} = \{x \mid x^* W = x^* Y\} = \emptyset$$

because for each $x \in X$, $<\emptyset>(x) \in x^* Y$ while $<\emptyset>(x) \notin x^* W$, and if $W = (Y \to X)^{-1} V$ for some $V \subset X$ then

$$V = \{x \mid x^* W \in x^* H\}.$$

Therefore, H is binding. Now suppose y is not the empty subset of S. Then there are local sections

$$\begin{array}{ccc} V_1 & \xrightarrow{s_1} & S \\ {\scriptstyle =}\uparrow & & \downarrow \\ V_1 & \xrightarrow{\textit{open}} & X \end{array}, \quad \ldots, \quad \begin{array}{ccc} V_n & \xrightarrow{s_n} & S \\ {\scriptstyle =}\uparrow & & \downarrow \\ V_n & \xrightarrow{\textit{open}} & X \end{array}$$

such that

$$x_0 \in V_1 \cap \ldots \cap V_n, \quad \text{and} \quad y \in x_0^* (H_{s_1} \cap \ldots \cap H_{s_n}) \subset U_{x_0}.$$

Let $H \subset \theta(Y)$ be defined at each $x \in X$ by

$$H_x = \begin{cases} \emptyset \text{ if } x \notin V_1 \cap \ldots \cap V_n \\ \{ W_x \underset{open}{\subset} x^* Y \mid x^*(H_1 \cap \ldots \cap H_n) \subset W_x \} \text{ if } x \in V_1 \cap \ldots \cap V_n. \end{cases}$$

Then $H_{s_1} \cap \ldots \cap H_{s_n} \subset \cap H$ and since $y \in H_{s_1} \cap \ldots \cap H_{s_n}$, $\cap H$ is a neighborhood of y. Clearly H is saturated. For fup, note $x^*(H_{s_1} \cap \ldots \cap H_{s_n})$ is compact for each $x \in X$ because either it is empty or if $x \in V_1 \cap \ldots \cap V_n$, then $x^*(H_1 \cap \ldots \cap H_n)$ is homeomorphic to the Scott-space of

$$(x^* S) - \{ s_1(x), \ldots, s_n(x) \}.$$

For binding, let $W \subset Y$ be open. It is sufficient to assume W is a basic open. If $W = (Y \to X)^{-1} V$ for some open $V \subset X$ then $\{ x \mid x^* W \in x^* H \}$ is $V \cap V_1 \cap \ldots \cap V_n$ which is open, and if $W = H_{s'_1} \cap \ldots \cap H_{s'_m}$ for some local sections

$$\begin{array}{ccc} V'_1 & \xrightarrow{s'_1} & S \\ = \uparrow & & \downarrow \\ V'_1 & \xrightarrow{open} & X \end{array} \quad \cdots \quad \begin{array}{ccc} V'_m & \xrightarrow{s'_m} & S \\ = \uparrow & & \downarrow \\ V'_m & \xrightarrow{open} & X \end{array}$$

then

$$\{ x \mid x^* W \in x^* H \} = \{ x \mid x \in (V'_1 \cap \ldots \cap V'_m \cap V_1 \cap \ldots \cap V_n) \}$$

and

$$\{ s'_1(x), \ldots, s'_m(x) \} \subset \{ s_1(x), \ldots, s_n(x) \},$$

which is open. Therefore, H is binding. Hence, $\theta(S) \to X$ is cartesian. □

Recall from [4] a continuous function $P \to X$ is proper if for every continuous function $f: Y \to X$, $f^* P \to Y$ is a closed function; also, it is proper iff it is closed and has compact fibers.

COROLLARY. *For each $X \in Top$ and $S \in SET^X$, $sub^X S \to X$ is a proper function.*

PROOF. If $f: Y \to X$ is a continuous function then we have a pullback

$$\begin{array}{ccc} sub^Y f^* S & \longrightarrow & sub^X S \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X . \end{array}$$

The result follows from Theorem 1.1.2 applied to these pullbacks □

PROPOSITION 1.8. *If X is a topological space and $S \in SET^X$ then the direct image functor $SET^{sub^X S} \to SET^X$ preserves filtered colimits.*

PROOF. For notation, denote $sub^X S \to X$ by π. Let C be a small filtered category and $F: \mathrm{C} \to SET^{sub^X S}$ be a functor. In SET^X, we have a comparison $\epsilon: \varinjlim \pi_* \circ F \to \pi_* \varinjlim F$ coming from the universal property of $\varinjlim \pi_* \circ F$. We will show ϵ is both epi and mono, so iso.

Fix $x \in X$ and suppose that

$$\begin{array}{ccc} V & \overset{s}{\rightarrowtail} & \pi_* \varinjlim F \\ = \uparrow & & \downarrow \\ V & \overset{open}{\rightarrowtail} & X \end{array}$$

is a local section with $x \in V$. By adjunction, we get a local section

$$\begin{array}{ccc} \pi^{-1} V & \overset{\hat{s}}{\longrightarrow} & \varinjlim F \\ = \uparrow & & \downarrow \\ \pi^{-1} V & \rightarrowtail & sub^X S . \end{array}$$

Choose $c \in \mathrm{C}$ such that $\hat{s}(<\theta>(x))$ is in the image of $\rho_{F(c)}: F(c) \to \varinjlim F$. Let

$$\begin{array}{ccc} W & \overset{t}{\rightarrowtail} & F(c) \\ = \uparrow & & \downarrow \\ W & \overset{open}{\rightarrowtail} & sub^X S \end{array}$$

be a local section such that

$$\rho_{F(c)} \circ t(<\emptyset>(x)) = \hat{s}(<\emptyset>(x)).$$

Let $U = W \cap \pi^{-1} V$. Since $<\theta>(x) \in U$, we have $\pi^{-1} x \subset U$. Since π is proper, $\pi_* U$ is an open neighborhood of x. We have the commutative diagram

$$\begin{array}{ccc}
\varinjlim \pi_* \circ F & \xrightarrow{\ \epsilon\ } & \pi_* \varinjlim F \\
\uparrow & & \\
\pi_* F(c) & & \\
\pi_* t \Big\uparrow & & \Big\uparrow s \\
\pi_* W & & \\
\subset \Big\uparrow & & \\
\pi_* U & \xrightarrow{\ \subset\ } & V
\end{array}$$

Therefore, ϵ is an epimorphism. Next, with $x \in X$ fixed, suppose

$$\begin{array}{ccc}
V & \overset{s_1}{\underset{s_2}{\rightrightarrows}} & \varinjlim \pi_* \circ F \\
= \Big\uparrow & & \Big\downarrow \\
V & \xrightarrow{\ open\ } & X
\end{array}$$

with $x \in V$ such that $\epsilon \circ s_1 = \epsilon \circ s_2$. By the filteredness of C we can choose $c \in$ C and local sections

$$\begin{array}{ccc}
V' & \overset{s'_1}{\underset{s'_2}{\rightrightarrows}} & \pi_* F(c) \\
= \Big\uparrow & & \Big\downarrow \\
V' & \xrightarrow{\ open\ } & X
\end{array}$$

such that $x \in V' \subset V$ and the respective squares of

$$\begin{array}{ccc}
V & \overset{s_1}{\underset{s_2}{\rightrightarrows}} & \varinjlim \pi_* F \\
\subset \Big\uparrow & & \Big\uparrow \rho_{\pi_* F(c)} \\
V' & \overset{s'_1}{\underset{s'_2}{\rightrightarrows}} & \pi_* F(c)
\end{array}$$

commute. Because $\epsilon \circ s_1 = \epsilon \circ s_2$ and because

$$\begin{array}{ccc}
\varinjlim \pi_* F & \xrightarrow{\ \epsilon\ } & \pi_* \varinjlim F \\
\rho_{\pi_* F(c)} \Big\uparrow & \nearrow_{\pi_* \rho F(c)} & \\
\pi_* F(c) & &
\end{array}$$

commutes, by adjunction we get the commutative square

$$\begin{array}{ccc} \pi^{-1}V' & \xrightarrow{\hat{s}'_1} & F(c) \\ {\scriptstyle \hat{s}'_2}\downarrow & & \downarrow{\scriptstyle \rho_{F(c)}} \\ F(c) & \xrightarrow{\rho_{F(c)}} & \varinjlim F \end{array} .$$

Since

$$\rho_{F(c)} \circ \hat{s}'_1(\langle\emptyset\rangle(x)) = \rho_{F(c)} \circ \hat{s}'_2(\langle\emptyset\rangle(x))$$

there is a morphism $a : c \to d$ in C such that

$$F(a) \circ \hat{s}'_1(\langle\emptyset\rangle(x)) = F(a) \circ \hat{s}'_2(\langle\emptyset\rangle(x)).$$

Therefore there is an open neighborhood U of $\langle\emptyset\rangle(x)$ such that $U \subset \pi^{-1}V'$ and

$$\begin{array}{ccccc} U & \overset{\subset}{\rightarrowtail} & \pi^{-1}V' & \xrightarrow{\hat{s}'_1} & F(c) \\ {\scriptstyle \subset}\downarrow & & & & \\ \pi^{-1}V' & & & & \downarrow{\scriptstyle F(a)} \\ {\scriptstyle \hat{s}'_2}\downarrow & & & & \\ F(c) & & \xrightarrow{\quad F(a) \quad} & & F(d) \end{array}$$

commutes. Since $\pi^{-1}x \subset U$ and π is proper, $\pi_* U$ is an open neighborhood of x, and we have a commutative diagram

$$\begin{array}{ccc} V & \overset{s_1}{\underset{s_2}{\rightrightarrows}} & \varinjlim \pi_* F \\ {\scriptstyle \subset}\uparrow & & \uparrow{\scriptstyle \rho_{\pi_* F(d)}} \\ \pi_* U & \xrightarrow{\pi_* F(a) \circ s'_1 = \pi_* F(a) \circ s'_2} & \pi_* F(d) \end{array}$$

Therefore, locally at x, $s_1 = s_2$. Hence, ϵ is a monomorphism.

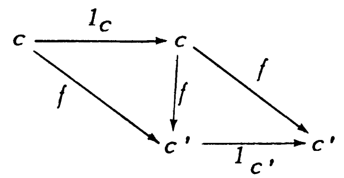
Therefore, π_* preserves filtered colimits. □

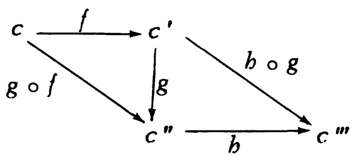
2. TOPOLOGICAL CATEGORIES.

DEFINITION. A *topological category* is a category object in *Top* ; that is, a topological category C has a topological space C_0 of objects, a topological space C_1 of morphisms, a continuous function $id: C_0 \to C_1$ which chooses an identity morphism $1_c = id(c)$ for each object $c \in C_0$, continuous functions $\partial_0 : C_1 \to C_0$ and $\partial_1 : C_1 \to C_0$ which assign to each morphism $f \in C_1$ its domain $\partial_0 f \in C_0$ and its codomain $\partial_1 f \in C_0$, and a continuous function $o : C_2 \to C_1$ of composition where

$$\begin{array}{ccc} C_2 & \xrightarrow{\pi_1} & C_1 \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \partial_0} \\ C_1 & \xrightarrow{\partial_1} & C_0 \end{array}$$

is a pullback in *Top*, and with $o(f, g)$ written $g \circ f$, satisfying the commutativity condition:

1.  $f \circ 1_c = f = 1_{c'} \circ f$;

3.  $(h \circ g) \circ f = h \circ (g \circ f)$.

In [1], 2.6, Adams has defined a topological category in the same way as we have above, but Ehresmann knew of them earlier [6, 7].

Topological groups, topological monoids, topological groupoids, and topological preorders provide examples of topological categories often arising in mathematics. If X is a topological space, we get a topological groupoid

$$X \times X \times X \; \begin{array}{c} \xrightarrow{Proj(1,2)} \\ \xrightarrow{Proj(1,3)} \\ \xrightarrow[Proj(2,3)]{} \end{array} \; X \times X \; \begin{array}{c} \xrightarrow{Proj(1)} \\ \xleftarrow{Diag} \\ \xrightarrow[Proj(2)]{} \end{array} \; X$$

and if we take from this the subspace of $X \times X$ consisting of pairs (x, x') such that x is in the closure of x' as the morphisms of a subspace-subcategory, we get a topological preorder which we will denote by $\leq_X$. *Top* inverse limits of finite set categories, the profinite categories, provide another important class of topological categories [10].

DEFINITION. A *continuous functor* $(S, \mathcal{S})$ from a topological category C to SET is a local homeomorphism $S \to C_0$ and a continuous function

$$\begin{array}{ccc} \partial_0^* S & \xrightarrow{\;\&\;} & \partial_1^* S \\ \downarrow & & \downarrow \\ C_1 & \xrightarrow{\;=\;} & C_1 \end{array}$$

(necessarily a local homeomorphism as well) such that for each object c of C,

$$\&(1_c) = \&_{1_c} = 1_{c*S}$$

and for each composable pair of morphisms $(f, g) \in C$,

$$\&(g \circ f) = \&_{(g \circ f)} = \&_g \circ \&_f .$$

In the language of [16], a continuous functor from a topological category C to SET is an internal functor from C to SET. Continuous functors on a topological category are used in [13] to construct vector bundles.

DEFINITION. A *continuous natural transformation* between two continuous functors $(S, \&)$ to $(S', \&')$ on a topological category C is a morphism $\eta : S \to S'$ in SET^{C_0} such that the diagram

$$\begin{array}{ccc} \partial_0^* S & \xrightarrow{\;\partial_0^* \eta\;} & \partial_0^* S' \\ \& \downarrow & & \downarrow \&' \\ \partial_1^* S & \xrightarrow{\;\partial_1^* \eta\;} & \partial_1^* S' \end{array}$$

commutes in SET^{C_1}.

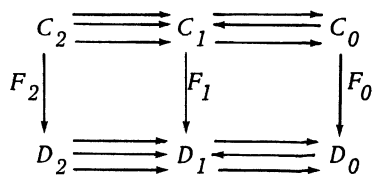
Continuous natural transformations are composable and each continuous functor has an identity continuous natural transformation. If C is a topological category, we will let SET^{C} denote the category of continuous functors from C to SET and continuous natural transformations of such. This definition of SET^{C} is guided by the ideas of [16]. From [12], we have :

THEOREM 2.1. *If* C *is a topological category,* SET^{C} *is a Grothendieck topos; if*

$$b = max(\aleph_0, card \coprod_{U \text{ open } C_0} \{P \rightarrowtail \partial_0^* U\})$$

then the set of continuous functors $(S, \mathcal{E})$ *such that* S *can be covered by a set of local sections whose cardinality does not exceed* b *is a set of generators of* SET^{C}. □

DEFINITION. If C and D are two topological categories, *a topological functor* $F: \mathrm{C} \to \mathrm{D}$ is a triple of continuous functions (F_0, F_1, F_2) making the corresponding squares commute in the diagram



If $F: \mathrm{C} \to \mathrm{D}$ is a topological functor, a continuous functor $(S, \mathcal{E})$ in SET^{D} may be pulled back along F to a continuous functor

$$F^*(S, \mathcal{E}) = (F_0^* S, F_1^* \mathcal{E}) \in SET^{\mathrm{C}}.$$

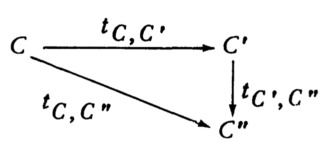
We get a geometric morphism $(F_*, F^*): SET^{\mathrm{C}} \to SET^{\mathrm{D}}$; the existence of F_* uses the special adjoint functor Theorem, which is necessary by its use in the particular case $C_o \to \mathrm{C}$ arising in the proof that the category SET^{C} is a Grothendieck topos.

DEFINITION. A *finite topological category* is a topological category such that its space of objects and its space of morphisms are finite.

THEOREM 2.2. *If* C *is a finite topological category then* SET^{C} *is equivalent to a presheaf category.*

PROOF. Define a diagram C^+ in *Set* by taking C_o^+ to be the underlying set of C_o and by taking C_1^+ to be the underlying set of C_1 together with new morphisms $t_{C,C'}: C \to C'$, $C, C' \in C_o^+$, whenever C is in the closure of C' in C_o, and subject to $t_{C,C} = 1_C$ for each $C \in C_o^+$ and the commutativity conditions :

1. all compositions defined in C ;
2.



$$\begin{array}{ccc} \partial_0 f & \xrightarrow{\;t_{\partial_0 f,\partial_0 g}\;} & \partial_0 g \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \partial_1 f & \xrightarrow{\;t_{\partial_1 f,\partial_1 g}\;} & \partial_1 g \end{array}$$

whenever f and g are morphisms of C and f is in the closure of g. Let Set^{C^+} be the topos of all diagrams $C^+ \to Set$ which respect any compositions defined for C^+. Note, by freely generating a small category from C^+ and dividing out by the proper relations, we get a small category $\hat{C}^+$ such that Set^{C^+} is equivalent to $Set^{\hat{C}^+}$. We will show SET^C is equivalent to $Set^{\hat{C}^+}$ by showing it is equivalent to Set^{C^+}.

First, we define a functor $SET^C \to Set^{C^+}$. Let $(S, \mathcal{E}) \in Set^C$. For $C \in C_o$, let S_C be the fiber of S over C and for $f \in C_1$, let

$$\mathcal{E}_f : S_{\partial_0 f} \to S_{\partial_1 f}$$

be the fiber of $\mathcal{E}$ over f. For $C \in C_o$, let $S^+(C) = S_C$. If C is in the closure of C' in C_o then for any local section

$$\begin{array}{ccc} U & \xrightarrow{\;\;s\;\;} & S \\ \big\uparrow & & \big\downarrow \\ U & \xrightarrow{\;open\;} & C_o \end{array}$$

with $C \in U$ then $C' \in U$, and for any other local section

$$\begin{array}{ccc} U' & \xrightarrow{\;\;s'\;\;} & S \\ {\scriptstyle =}\big\uparrow & & \big\downarrow \\ U' & \xrightarrow{\;open\;} & C_o \end{array}$$

with $C \in U'$ and $s(C) = s'(C)$ then $s(C') = s'(C')$. Therefore, if C is in the closure of C', we have a function $\mathcal{E}^+(t_{C,C'}) : S^+(C) \to S^+(C')$ induced from the restriction of local sections at C to local sections at C'. Clearly, $\mathcal{E}^+(t_{C,C}) = 1_{S^+(C)}$ for each $C \in C_o^+$, and

$$\mathcal{E}^+(t_{C',C''}) \circ \mathcal{E}^+(t_{C,C'}) = \mathcal{E}^+(t_{C,C''})$$

for each C in the closure of C' and C' in the closure of C'' in C_o. Suppose f is in the closure of g in C_1. Then it must be that $\partial_0 f$ is in the closure of $\partial_0 g$ and $\partial_1 f$ is in the closure of $\partial_1 g$. If U_1 is an open neighborhood of f and

$$\begin{array}{ccc} U_1 & \xrightarrow{\;s_1\;} & \partial_0^* S \\ {\scriptstyle =}\uparrow & & \downarrow \\ U_1 & \xrightarrow{\;open\;} & C_1 \end{array}$$

is a local section then s_1 is equivalent to ∂_0^* of a local section

$$\begin{array}{ccc} U_0 & \xrightarrow{\;s_0\;} & S \\ {\scriptstyle =}\uparrow & & \downarrow \\ U_0 & \xrightarrow{\;open\;} & C_o \end{array}$$

at $\partial_0 f \in U_0$; that is

$$s_1 \,|\, U_1 \cap \partial_0^* U_0 = \partial_0^* s_0 | U_1 \cap \partial_0^* U_0 .$$

Therefore, since the points of $(\partial_0^* S)_f$ and equivalence classes of local sections of $S \to C_o$ at $\partial_0 f$ are the same thing where the sheaf $\partial_0^* S \to C_1$ is concerned, the function

$$\partial_0^* \mathcal{E}^+(t_{\partial_0 f, \partial_0 g}) : (\partial_0^* S)_f \to (\partial_0^* S)_g ,$$

induced by the restriction of local sections of $\partial_0^* S \to C_1$ at f, makes the diagram

$$\begin{array}{ccc} (\partial_0^* S)_f & \xrightarrow{\;\partial_0^* \mathcal{E}^+(t_{\partial_0 f, \partial_0 g})\;} & (\partial_0^* S)_g \\ {\scriptstyle (\partial_0')_{f, \partial_0 f}}\downarrow \approx & & \approx \downarrow {\scriptstyle (\partial_0')_{g, \partial_0 g}} \\ S^+(\partial_0 f) & \xrightarrow{\;\mathcal{E}^+(t_{\partial_0 f, \partial_0 g})\;} & S^+(\partial_0 g) \end{array}$$

commute, where ∂_0' is defined by the pullback

$$\begin{array}{ccc} \partial_0^* S & \xrightarrow{\;\partial_0'\;} & S \\ \downarrow & & \downarrow \\ C_1 & \xrightarrow{\;\partial_0\;} & C_o . \end{array}$$

A similar diagram exists with respect to $\partial_1^* S \to C_1$ at f and g. Define $\mathcal{E}^+(f)$ by $(\partial_1')_{f,\partial_1 f} \circ \mathcal{E}_f \circ (\partial_0')^{-1}_{f,\partial_0 f}$, and define $\mathcal{E}^+(g)$ by

$$(\partial_1')_{g,\partial_1 g} \circ \mathcal{E}_g \circ (\partial_0')^{-1}_{g,\partial_0 g}.$$

By the continuity of $\mathcal{E}$, if $f \in U$ and

$$\begin{array}{ccc} U & \xrightarrow{\quad s \quad} & \partial_0^* S \\ {\scriptstyle =}\uparrow & & \downarrow \\ U & \xrightarrow{\quad open \quad} & C_1 \end{array}$$

is a local section, we get a local section

$$\begin{array}{ccc} U & \xrightarrow{\quad \mathcal{E} \circ s \quad} & \partial_1^* S \\ \uparrow & & \downarrow \\ U & \xrightarrow{\qquad} & C_1 \end{array} .$$

Since $g \in U$ this implies the commutativity of

$$\begin{array}{ccc} (\partial_1^* S)_f & \xrightarrow{\quad \partial_1^* \mathcal{E}^+(t_{\partial_1 f, \partial_1 g}) \quad} & (\partial_1^* S)_g \\ {\scriptstyle \mathcal{E}_f}\uparrow & & \uparrow{\scriptstyle \mathcal{E}_g} \\ (\partial_0^* S)_f & \xrightarrow{\quad \partial_0^* \mathcal{E}^+(t_{\partial_0 f, \partial_0 g}) \quad} & (\partial_0^* S)_g \end{array} .$$

Therefore, we have

$$\mathcal{E}^+(g) \circ \mathcal{E}^+(t_{\partial_0 f, \partial_0 g}) = (\partial_1')_{g,\partial_1 g} \circ \mathcal{E}_g \circ (\partial_0')^{-1}_{g,\partial_0 g} \circ \mathcal{E}^+(t_{\partial_0 f, \partial_0 g})$$

$$= (\partial_1')_{g,\partial_1 g} \circ \mathcal{E}_g \circ \partial_0^* \mathcal{E}^+(t_{\partial_0 f, \partial_0 g}) \circ (\partial_0')^{-1}_{f,\partial_0 f}$$

$$= (\partial_1')_{g,\partial_1 g} \circ \partial_1^* \mathcal{E}^+(t_{\partial_1 f, \partial_1 g}) \circ \mathcal{E}_f \circ (\partial_0')^{-1}_{f,\partial_0 f} =$$

$$= \mathcal{E}^+(t_{\partial_1 f, \partial_1 g}) \circ \mathcal{E}^+(f).$$

If f and g are composable (but not necessarily f in the closure of g) with $\partial_0 g = \partial_1 f$, we have

$$(\partial_1')^{-1}_{g,\partial_0 g} = (\partial_1')^{-1}_{gf,\partial_1 gf}, \quad (\partial_0')_{g,\partial_0 g} = (\partial_1')_{f,\partial_1 f},$$

$$(\partial_0')_{f,\partial_0 f} = (\partial_0')_{gf,\partial_0 gf}, \qquad \text{and}$$

$$\mathcal{E}^+(g)\circ\mathcal{E}^+(f) = (\partial_1')_{g,\partial_1 g}\circ\mathcal{E}_g\circ(\partial_0')^{-1}_{g,\partial_0 g}\circ(\partial_1')_{f,\partial_1 f}\circ\mathcal{E}_f\circ(\partial_0')_{f,\partial_0 f}$$
$$= (\partial_1')_{gf,\partial_1 gf}\circ\mathcal{E}_{gf}\circ(\partial_0)_{gf,\partial_0 gf} = \mathcal{E}^+(gf).$$

Therefore, $(S^+, \mathcal{E}^+)\in Set^{\mathbf{C}^+}$. This gives us the functor $SET^{\mathbf{C}}\to Set^{\mathbf{C}^+}$.

In the other direction, let $(S^+, \mathcal{E}^+)$ be an object of $Set^{\mathbf{C}^+}$. Take $S\to C_0$ to be defined on the Set level as

$$proj : \Big(\coprod_{C\in C_0} S^+(C)\Big) \to C_0 .$$

For $C\in C_0$ let U_C denote the smallest neighborhood of C. Similarly, if $f\in C_1$, let U_f denote the smallest neighborhood of f. For $x\in U_C$, we have $t_{C,x}: C\to x$ in C^+. For $a\in S^+(C)$, define a local section

$$\begin{array}{ccc} U_C & \xrightarrow{\;s_{C,a}\;} & S \\ \uparrow & & \downarrow \\ U_C & \longrightarrow & C_0 \end{array}$$

at $x\in U_C$ by $s_{C,a}(x) = \mathcal{E}^+(t_{C,x})(a)$. Suppose $s_{C,a}(x) = s_{C',a'}(x)$. Then $x\in U_C$ and $x\in U_{C'}$, and for each $y\in U_x$, we have the C^+ commutative diagram

$$\begin{array}{ccccc} C & \xrightarrow{\;t_{C,x}\;} & x & \xleftarrow{\;t_{C',x}\;} & C' \\ {\scriptstyle t_{C,y}}\downarrow & & \downarrow{\scriptstyle t_{x,y}} & & \downarrow{\scriptstyle t_{C',y}} \\ y & \xrightarrow{\;=\;} & y & \xleftarrow{\;=\;} & y \end{array}$$

Therefore, for any $y\in U_x$, we have

$$s_{C,a}(y) = \mathcal{E}^+(t_{C,y})(a) = \mathcal{E}^+(t_{x,y}\circ t_{C,x})(a) =$$
$$= \mathcal{E}^+(t_{x,y})(\mathcal{E}^+(t_{C,x})(a)) = \mathcal{E}^+(t_{x,y})(s_{C,a}(x)) =$$
$$= \mathcal{E}^+(t_{x,y})(s_{C',a'}(x)) = \mathcal{E}^+(t_{x,y})(\mathcal{E}^+(t_{C',x})(a')) =$$
$$= \mathcal{E}^+(t_{x,y}\circ t_{C',x})(a') = \mathcal{E}^+(t_{C',y})(a') = s_{C',a'}(y).$$

Therefore, collectively, the local sections $s_{C,a}$ make $S\to C_0$ into a local homeomorphism. Now for each $f\in C_1$, identify the fiber $(\partial^* S)_f$ with $S_{\partial_0 f}$ and the fiber $(\partial_1^* S)_f$ with $S_{\partial_1 f}$; this saves us from the involvment of the isomorphisms $(\partial_0')_{f,\partial_0 f}$ and $(\partial_1')_{f,\partial_1 f}$. Define a function

$$\mathcal{E} : \partial_0^* S \to \partial_1^* S \ \text{ by } \ \mathcal{E}_f = \mathcal{E}^+(f) \ \text{ for each } f \in C_1 .$$

We will show $\mathcal{E}$ is continuous. Let $f \in C_1$.

$$U_f \subset (\partial_0^* U_{\partial_0 f}) \cap (\partial^* U_{\partial_1 f}).$$

A section

$$\begin{array}{ccc} U_f & \xrightarrow{s_{f,a}} & \partial^* S \\ {\scriptstyle =}\uparrow & & \downarrow \\ U_f & \longrightarrow & C_1 \end{array}$$

comes from a section

$$\begin{array}{ccc} U_{\partial_0 f} & \xrightarrow{s_{\partial_0 f, a}} & S \\ {\scriptstyle =}\uparrow & & \downarrow \\ U_{\partial_0 f} & \longrightarrow & X \end{array}$$

by $s_{f,a} = \partial_0^* s_{\partial_0 f, a} | U_f$. For any $g \in U_f$, we have

$$(\mathcal{E} \circ s_{f,a})(g) = (\mathcal{E} \circ \partial_0^* s_{\partial_0 f, a})(g) = \mathcal{E}^+(g) \circ \mathcal{E}^+(t_{\partial_0 f, \partial_0 g})(a) =$$

$$= \mathcal{E}^+(g \circ t_{\partial_0 f, \partial_0 g})(a) = \mathcal{E}^+(t_{\partial_1 f, \partial_1 g} \circ f)(a) = \mathcal{E}^+(t_{\partial_1 f, \partial_1 g}) \circ \mathcal{E}^+(f)(a)$$

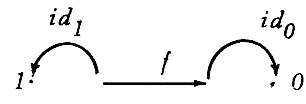
$$= s_{\partial_1 f, \mathcal{E}^+(f)(a)}(\partial_1 g) = (\partial_1^* s_{\partial_1 f, \mathcal{E}^+(f)(a)})(g).$$

Therefore, we have a commutative diagram

$$\begin{array}{ccccc} U_f & \xrightarrow{\subset} & \partial_0^* U_{\partial_0 f} & \xrightarrow{\partial_0^* s_{f,a}} & \partial_0^* S \\ {\scriptstyle \subset}\downarrow & & & & \downarrow{\scriptstyle \mathcal{E}} \\ \partial_1^* U_{\partial_1 f} & & \xrightarrow{\partial_1^* s_{\partial_1 f, \mathcal{E}^+(f)(a)}} & & \partial_1^* f \end{array}$$

Therefore, $\mathcal{E}$ is continuous. So now we have the functor $Set^{C^+} \to SET^{C}$, inverse to $SET^{C} \to Set^{C^+}$. Therefore, SET^{C} is equivalent to $Set^{\hat{C}+}$. □

EXAMPLES. (1) Let C be the topological category



with topologies defined by taking the opens of $\{1, 0\}$ to be

$$\{\emptyset, \{1\}, \{1, 0\}\}$$

(so C_0 is homeomorphic to the Sierpinski space) and by taking the opens of $\{id_1, f, id_0\}$ to be

$$\{\emptyset, \{id_1\}, \{f\}, \{id_1, id_0\}, \{id_1, f\}, \{id_1, f, id_0\}\}.$$

The functions ∂_0 and ∂_1 are continuous because

$$\partial_0^{-1}\{1\} = \{id_1, f\} \quad \text{and} \quad \partial_1^{-1}\{1\} = \{id_1\}.$$

The function $id: C_0 \to C_1$ is continuous because it is a subspace inclusion, and $\circ: C_2 \to C_1$ is continuous because C_2 is a subspace of $C_1 \times C_1$,

$$\circ^{-1}\{id_1\} = \{(id_1, id_1)\}.$$

$$\circ^{-1}\{f\} = \{(id_1, f), (f, id_0)\} = \{f\} \times C_1 \cup C_1 \times \{f\} \cap C_2,$$

and

$$\circ^{-1}\{id_1, id_0\} = \{(id_1, id_1), (id_0, id_0)\} = \{id_1, id_0\} \times \{id_1, id_0\} \cap C_2.$$

The topos $SET^{\mathbf{C}}$ is equivalent to the category of presheaves on the category freely generated by the graph

$$1 \underset{t_{0,1}}{\overset{f}{\rightleftarrows}} 0$$

The absence of relations between f and the topologically induced arrow $t_{0,1}$ arises because $\{f\}$ is both open and closed. The Beck condition

$$\begin{array}{ccc} SET^{C_1} & \xrightarrow{\partial_{0*}} & SET^{C_0} \\ {\scriptstyle \pi_1^*}\downarrow & & \downarrow{\scriptstyle \partial_1^*} \\ SET^{C_2} & \xrightarrow{\pi_{0*}} & SET^{C_1} \end{array}$$

does not hold for this topological category, for if $S \in SET^{C_1}$ then $(\partial_1^* \partial_{0*} S) \approx (\partial_{0*} S)_0 \approx S_f \times S_0$ while $(\pi_{0*} \pi_1^* S) \approx (\pi_1^* S)_{(f, id_0)} \approx S_0$.

(2) Take $\mathbf{C}$ as in Example (1), but with topology on arrows given as

$$\{\emptyset, \{id_1\}, \{id_1, f\}, \{id_1, f, id_0\}\}.$$

∂_0, ∂_1 and id are continuous for the same reasons as before. Composition is continuous because $o^{-1}\{id_1\} = \{(id_1, id_1)\}$ and

$$o^{-1}\{id_1, f\} = \{(id_1, id_1), (id_1, f), (f, id_0)\} = \{id_1, f\} \times C_1 \cap C_2 .$$

In C^+ we must have the commutative diagram

$$\begin{array}{ccccc} 1 & \xleftarrow{t_{0,1}} & 0 & \xleftarrow{t_{0,0}} & 0 \\ {\scriptstyle id_1}\uparrow & & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle id_0} \\ 1 & \xleftarrow{t_{1,1}} & 1 & \xleftarrow{t_{0,1}} & 0 \end{array}$$

because id_0 is in the closure of f and f is in the closure of id_1. It follows that $f \in C_1^+$ is forced to be a left and right inverse to the topologically induced arrow $t_{0,1}$. Therefore, SET^{C} is equivalent to Set. For use later we note

$$\mathrm{C}^{op} \approx \leqslant_2 \quad \text{and} \quad SET^{\mathrm{C}^{op}} \simeq SET^2 .$$

The Beck condition $\partial_1^* \partial_{0*} \xrightarrow{\simeq} \pi_{0*}\pi_1^*$ holds for this topological category, because global sections of $S \in SET^{C_1}$ correspond by restriction to the elements of the fiber of S at id_0.

(3) C is as in Example (1), except the topology on C_1 is given by

$$\{\emptyset, \{id_1\}, \{f\}; \{id_1, f\}, \{id_1, f, id_0\}\}.$$

All the maps ∂_0, ∂_1, id, and o are continuous for the reasons given in Example (1). In C^+ we must have the commutative diagram

$$\begin{array}{ccccc} 1 & \xleftarrow{t_{0,1}} & 0 & \xrightarrow{t_{0,0}} & 0 \\ {\scriptstyle id_1}\uparrow & & \uparrow{\scriptstyle id_0} & & \uparrow{\scriptstyle f} \\ 1 & \xleftarrow{t_{0,1}} & 0 & \xrightarrow{t_{0,1}} & 1 \end{array}$$

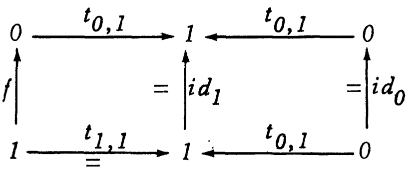
because id_0 is in the closures of id_1 and f. Therefore, we see that if we first apply the topologically induced arrow $t_{0,1}$, then f, we must have id_0. This means $f \in C_1^+$ is a split epimorphism. However, there are no further conditions.

As in Example (2), the Beck condition $\partial_1^* \partial_{0*} \xrightarrow{\simeq} \pi_{0*}\pi_1^*$ holds for this topological category.

(4) C is as in Example (1), except the topology on C_1 is given by

$$\{\emptyset, \{id_1\}, \{id_1, f\}, \{id_1, id_0\}, \{id_1, f, id_0\}\}.$$

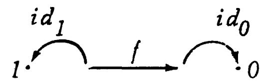
All the maps ∂_0, ∂_1, id, and o are continuous for the reasons given in Example (2). In C^+ we have the commutative diagram



Therefore, f is a retract in C^+ with $t_{0,1}$ providing the retraction.

As in Example (1), the Beck condition $\partial_1^* \partial_{0*} \xrightarrow{\cong} \pi_{0*} \pi_1^*$ fails for this topological category.

PROPOSITION 2.1. *When $\{1, 0\}$ is given the topology $\{\emptyset, \{1\}, \{1, 0\}\}$, there are only four topologies on $\{id_1, f, id_0\}$ making*



into a topological category.

PROOF.

$$\partial_0^{-1}\{1\} = \{id_1, f\}, \quad \partial_1^{-1}\{1\} = \{id_1\}, \quad \partial_0^{-1}\{0\} = \{id_0\},$$

$$\text{and} \quad \partial_1^{-1}\{0\} = \{f, id_0\},$$

so $\{id_1, f\}$ and $\{id_1\}$ are open while $\{id_0\}$ and $\{f, id_0\}$ are closed in C_1. $\{id_0\}$ and $\{f, id_0\}$ cannot be open because id is a continuous function and $\{0\}$ is not open in C_0. Therefore, the only possible neighborhood of id_0 other then $\{id_1, f, id_0\}$ is $\{id_1, id_0\}$ as in Example (1) and Example (4), and the only possible neighborhood of f other than $\{id_1, f\}$ and $\{id_1, f, id_0\}$ is $\{f\}$ as in Examples (1) and (3). □

3. ASSOCIATED SHEAF FUNCTOR.

DEFINITION. A topological category C is *filtered* if

(1) it is nonempty,

(2) it is pseudofiltered; that is, for any two objects c, c' of C,

there is an object c'' of C and morphisms $c \to c''$ and $c' \to c''$, and if

$$c \overset{f}{\underset{g}{\rightrightarrows}} c'$$

is a parallel pair of morphisms in C, there is a morphism $h : c' \to c''$ of C such that $h \circ f = h \circ g$.

This definition may be abstracted to define a filtered category object in a category with finite limits [5], in which case the condition of nonemptiness becomes the requirement that $C_0 \to 1$ be the coequalizer of ∂_0 and ∂_1. If C is a category object of TOP^X, the comma category of spaces over $X \in Top$, for C to be filtered it is not sufficient for each fiber $x^*\mathrm{C}$, $x \in X$, to be filtered because the comparison function

$$coeq(\partial_0, \partial_1) \to X$$

may not be an isomorphism even though it will be both a monomorphism and an epimorphism; as an example, take C but not X to have discrete topology.

PROPOSITION 3.1. *If* C *is a category object of* SET^X, *it is filtered iff each fiber* $x^*\mathrm{C} \in Set$, $x \in X$, *is filtered.*

PROOF. Filteredness is preserved by substitution, so if $\mathrm{C} \in SET^X$ is filtered so is $x^*\mathrm{C}$ for each $x \in X$. Alternatively, suppose each fiber of C is filtered. The only question is if $coeq(\partial_0, \partial_1) \to X$ is an isomorphism. But each fiber of this comparison is an isomorphism by hypothesis, so the result follows from Proposition 1.1 (7). □

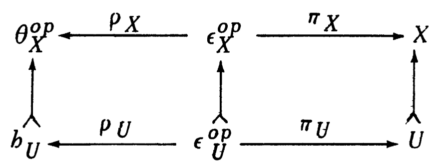
Let $X \in Top$ and let θ_X be the partially ordered set of open subsets of X. Let $ev : \theta_X^{op} \times X \to 2$ be the evaluation morphism, and define $(\rho_X, \pi_X) : \epsilon_X^{op} \rightarrowtail \theta_X^{op} \times X$ by the pullback

$$\begin{array}{ccc} \theta_X^{op} \times X & \xrightarrow{ev} & 2 \\ {\scriptstyle (\rho_X, \pi_X)} \uparrow & & \uparrow {\scriptstyle open} \\ \epsilon_X^{op} & \longrightarrow & 1 \end{array}$$

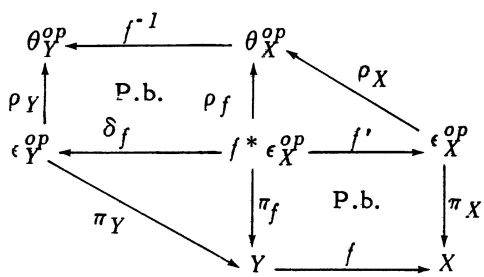
PROPOSITION 3.2. *For a topological space* X, *the projection* $\rho_X : \epsilon_X^{op} \to X$ *presents* ϵ_X^{op} *as a filtered category object of* SET^X.

PROOF. The object part of ϵ_X^{op} is homeomorphic to the Top coproduct $\coprod_{U \epsilon \theta_X} U$, and π_X is the obvious projection, so π_X makes ϵ_X^{op} into a category object of SET^X. For $x \epsilon X$, $x^* \epsilon_X^{op}$ is equivalent to the partially ordered set of open neighborhoods of x, so $x^* \epsilon_U^{op}$ is filtered. The result follows from Proposition 3.1. □

Let $X \epsilon Top$ and $U \epsilon \theta_X$. The inclusion $\epsilon_X^{op} \to \epsilon_X^{op}$ defines a subobject of 1 in $SET^{\epsilon_X^{op}}$, and if $h_U \to \theta_X^{op}$ is the representable corresponding to U, we have $h_U \approx \theta_U^{op}$ and a diagram of pullbacks



Also, if $f: Y \to X$ is a continuous function, we have a commutative diagram



providing us with some notation.

Let $X \epsilon Top$, and $F: \mathrm{D} \to \mathrm{C}$ be an internal functor between two category objects in SET^X. Recall [5] that $F^*: SET^{\mathrm{C}} \to SET^{\mathrm{D}}$ has a left adjoint $\underrightarrow{lim}_F$, the left Kan extension along F. If $f: Y \to X$ is a continuous function then f^* preserves $\underrightarrow{lim}_F$; that is,

$$f^* \circ \underrightarrow{lim}_{F} \xrightarrow{\approx} \underrightarrow{lim}_{f^* F} .$$

Also, if $\mathrm{C} = X$ and $(S, \mathcal{E}) \epsilon SET^{\mathrm{D}}$, then

$$\underrightarrow{lim}_F (S, \mathcal{E}) \to X$$

comes from

$$\begin{array}{ccccc} \partial_0^* S \overset{proj}{\longrightarrow} & & S & \overset{coeq}{\longrightarrow} & \underset{\vec{F}}{lim}(S, \&) \\ \underset{\&}{\longrightarrow} \partial_1^* S \underset{proj}{\longrightarrow} & & \downarrow & & \downarrow \\ & & D_0 \overset{F_0}{\longrightarrow} C_0 & \overset{=}{\longrightarrow} & X \end{array}$$

Still if $C = X$ and if F presents D as a filtered category object of SET^X then $\underset{\vec{F}}{lim}$ is left exact and $\underset{\vec{F}}{lim}\, F^* \xrightarrow{\cong} 1_{SET^X}$.

PROPOSITION 3.3. *For $X \in Top$, $\underset{\vec{\pi}_X}{lim} : SET^{\epsilon_X^{op}} \to SET^X$ is the inverse part of a geometric morphism.*

PROOF. π_X presents ϵ_X^{op} as a filtered category object of SET^X, so $\underset{\vec{\pi}_X}{lim}$ is left exact. Since $\pi_X^* : SET^X \to SET^{\epsilon_X^{op}}$ is the right adjoint to $\underset{\vec{\pi}_X}{lim}$, it follows $\underset{\vec{\pi}_X}{lim}$ is the inverse part of a geometric morphism. □

Let X be a topological space and $S : \theta_X^{op} \to Set$ a presheaf. This is the same thing as an object $S \in SET^{\theta_X^{op}}$; $S \to \theta_X^{op}$ is gotten by taking the coproduct of the values of the presheaf and projecting to θ_X^{op}. Recall [9], page 17, there is an associated sheaf $a_X S \to X$.

THEOREM 3.1. *If $S : \theta_X^{op} \to S$ is a presheaf then $a_X S \to X$ is equivalent to $\underset{\vec{\pi}_X}{lim}\, \rho_X^* S \to X$.*

PROOF. Let $S' \to X$ be a local homeomorphism and $U \in \theta_X$. We have a diagram of pullbacks

$$\begin{array}{ccccc} U & \xleftarrow{\pi_U} & \epsilon_U^{op} & \longrightarrow & h_U \\ \downarrow & & \downarrow & & \downarrow \\ X & \xleftarrow{\pi_X} & \epsilon_X^{op} & \xrightarrow{\rho_X} & \theta_X^{op} \end{array}$$

and

$$U \xrightarrow{\cong} \underset{\vec{\pi}_X}{lim}\,(\epsilon_U^{op} \rightarrowtail \epsilon_X^{op}) \quad \text{in } SET^X .$$

Therefore, we have natural bijections

$$\frac{\frac{\frac{\frac{h_U \to \rho_{X*}\pi_X^* S' \quad \text{in } SET^{\theta_X^{op}}}{\rho_X^* h_U \to \pi_X^* S' \quad \text{in } SET^{\epsilon_U^{op}}}}{\epsilon_U^{op} \to \pi_X^* S' \quad \text{in } SET^{\epsilon_U^{op}}}}{\varinjlim_{\pi_X}(\epsilon_U^{op}) \to S' \quad \text{in } SET^X}}{U \to S' \quad \text{in } SET^X .}$$

Therefore, $\rho_{X*}\pi_X^* : SET^X \to SET^{\theta_X^{op}}$ is the local section functor. But the associated sheaf functor a_X and the functor $\varinjlim_{\pi_X} \rho_X^*$ are both left adjoints to $\rho_{X*}\pi_X^*$. Therefore, $\varinjlim_{\pi_X} \rho_X^*$ is equivalent to the associated sheaf functor. □

This theorem computes, within the category theory of *Top*, the associated sheaf functor. This seems more natural than first constructing the underlying set of an associated sheaf and then forcing on the topology which makes things work.

It follows from the above theorem that, if $S \to X$ is a sheaf on $X \in Top$ then $S \xrightarrow{\simeq} \varinjlim_{\pi_X} \rho_X^* \rho_{X*} \pi_X^* S$. Also, if $S \to \theta_X^{op}$ is a presheaf of sets, it is a complete presheaf iff $S \xrightarrow{\simeq} \rho_{X*}\pi_X^* \varinjlim_{\pi_X} \rho_X^* S$. For these reasons and as usual today, we may write a sheaf $S \to X$ as a presheaf $S \to \theta_X^{op}$ (or $S : \theta_X^{op} \to Set$) whenever it suits our needs; if there is a possibility of of confusion, we will use Γ_X for $\rho_{X*}\pi_X^*$ and a_X for $\varinjlim_{\pi_X} \rho_X^*$.

COROLLARY. *If $S \to X$ is a sheaf of sets on $X \in Top$ and $f : Y \to X$ is a continuous function then $f^* S \approx \varinjlim_{\pi_X} \rho_f^* S$.*

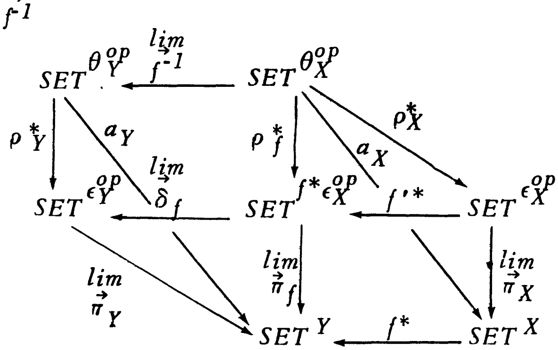
PROOF. $\pi_f = f^*\pi_X$. $f^* : SET^X \to SET^Y$ preserves internal colimits because it is a left adjoint. Therefore,

$$f_X^* S \approx f^* a_X S \approx f^* \varinjlim_{\pi_X} \rho_X^* S \approx \varinjlim_{\pi_f} f'^* \rho_X^* S \approx \varinjlim_{\pi_f} \rho_f^* S. \quad \square$$

In a like manner, because

$$\begin{array}{ccc} f^* \epsilon_X^{op} & \xrightarrow{\rho_f} & \theta_X^{op} \\ {\scriptstyle \delta_f}\downarrow & & \downarrow{\scriptstyle f^{-1}} \\ \epsilon_Y^{op} & \xrightarrow{\rho_Y} & \theta_Y^{op} \end{array}$$

is a pullback with f^{-1} a local homeomorphism on each component, we have $\varinjlim_{\delta_f} \rho_f^* S \approx \rho_Y^* \varinjlim_{f^{-1}} S$. In summary, the following diagram commutes:



DEFINITION. For $T \in Top$, SET^T is Top-indexed by taking $(SET^T)^X$ to be $SET^{T\times X}$ and by taking substitution along $f: Y \to X$ to be $(T\times f)^*$.

This definition gives the indexing of SET^T as suggested by the general theory of indexed categories. For $X \in Top$ let

$$T\times a_X = \varinjlim_{T\times\pi_X} (T\times\rho_X)^* \quad\text{and}\quad T\times\Gamma_X = (T\times\rho_X)_*(T\times\pi_X)^*.$$

With the same kind of reasoning as before we conclude the well-known

PROPOSITION 3.4. *If T and X are topological spaces, the functor $T\times a_X: SET^{T\times\theta_X^{op}} \to SET^{T\times X}$ is the associated sheaf functor to the inclusion $T\times\Gamma_X: SET^{T\times X} \to SET^{T\times\theta_X^{op}}$ of Grothendieck toposes.* $\square$

This allows us to view an object $S \in SET^{T\times X}$ as an object of $SET^{T\times\theta_X^{op}}$, something we do at our convenience. Thus if $f: Y \to X$ is a continuous function, we may take the inverse image of S along $T\times f$ to be $(T \times a_Y) \varinjlim_{T \times f^{-1}} S$.

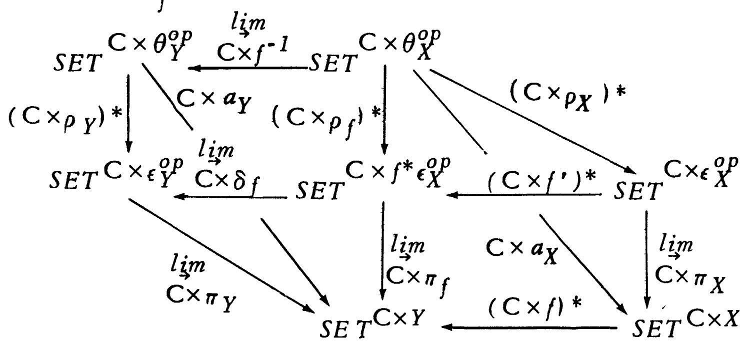
If C is a topological category, SET^{C} is indexed by $(SET^{\mathrm{C}})^X = SET^{\mathrm{C} \times X}$. We have $(SET^{\mathrm{C}})^X \simeq (SET^X)^{\mathrm{C}}$. If $h: \mathrm{C} \to \mathrm{D}$ is a topological functor then the pullback functor $h^*: SET^{\mathrm{D}} \to SET^{\mathrm{C}}$ is *Top*-indexed by taking $(h^*)^X : (SET^{\mathrm{D}})^X \to (SET^{\mathrm{C}})^X$ to be

$$(h \times X)^* : SET^{\mathrm{D} \times X} \to SET^{\mathrm{C} \times X} ;$$

this applies to the functors ρ_X^*, f^*, ρ_f^*, and ρ_Y^* above. Additionally, if $h: \mathrm{C} \to \mathrm{D}$ is a local homeomorphism on C_0, C_1 and C_2 then the functor $\varinjlim_h : SET^{\mathrm{C}} \to SET^{\mathrm{D}}$ is *Top*-indexed by taking

$$(\varinjlim_h)^X : (SET^{\mathrm{C}})^X \to (SET^{\mathrm{D}})^X \text{ to be } \varinjlim_{h \times X} : SET^{\mathrm{C} \times X} \to SET^{\mathrm{D} \times X};$$

this applies to the functors $\varinjlim_{\pi_X}$, $\varinjlim_{\pi_f}$, $\varinjlim_{\delta_f}$, and $\varinjlim_{f^1}$ above. Therefore, letting $\mathrm{C} \times a_X = \varinjlim_{\mathrm{C} \times \pi_f} . (\mathrm{C} \times \rho_X)^*$, we get the commutative diagram



for each topological category C and continuous function $f: Y \to X$.

4. MAIN RESULT.

Recall from above the definition of the topological preorder $\leq_2^+$. It is equivalent to $0 . \longrightarrow . 1$; that is, the forgetful functor $SET^{\leq_2} \to SET^2$ is an equivalence of categories (see Example (2), Section 2).

PROPOSITION 4.1. *For each $X \in Top$, the forgetful functor $SET^{\leq X} \to SET^X$ is a Top-indexed equivalence.*

PROOF. If $T \in Top$ and $(S, \&) \in SET^{T \times \leq_2}$, substitution along

$$t \times \leq_2 : \leq_2 \to T \times \leq_2$$

for each $t: 1 \to T$ shows $\&$ is completely determined by S, $(T \times \partial_0)^* S$, and $(T \times \partial_1)^* S$. On the other hand, let $S \to T \times 2$ be a local homeomorphism. Substitution along $t \times 2 : 2 \to T \times 2$ for each $t: 1 \to T$ gives us the fibers of a function $\& : (T \times \partial_0)^* S \to (T \times \partial_1)^* S$ which is easily shown to be continuous by a routine examination of local sections, and such that $(S, \&)$ is an object of $SET^{T \times \leq_2}$. Therefore, the forgetful functor $SET^{T \times \leq_2} \to SET^{T \times 2}$ is an equivalence of categories. Even more, as T ranges over the objects of Top these forgetful functors define a Top-indexed equivalence of $SET^{\leq_2}$ and SET^2. For any topological space X and elements $x_0 \leq x_1$ of $\leq_X$, we have a topological functor $\leq_{x_0, x_1} : \leq_2 \to \leq_X$, coming from the continuous function

$$2 \to X \quad (0 \mapsto x_0,\ 1 \mapsto x_1).$$

Therefore, for each $T \in Top$, we have commutative diagrams

$$\begin{array}{ccc} SET^{T \times \leq_X} & \xrightarrow{forget} & SET^{T \times X} \\ {\scriptstyle (T \times \leq_{x_0, x_1})^*} \downarrow & & \downarrow {\scriptstyle (T \times (\leq_{x_0, x_1})_0)^*} \\ SET^{T \times \leq_2} & \underset{\approx}{\xrightarrow{forget}} & SET^{T \times 2} \end{array}$$

providing us with a prescription for reconstructing (up to isomorphism) an internal functor $(S, \&) \in SET^{T \times \leq_X}$ given the object part $S \in SET^{T \times X}$. These same diagrams give us a function $\& : (T \times \partial_0)^* S \to (T \times \partial_1)^* S$ in $SET^{(T \times \leq_X)_1}$ given $S \in SET^{T \times X}$, and we will show $\&$ is continuous. Let $t \in T$ and f be a morphism of $\leq_X$. $\partial_0 f$ is in the closure of $\partial_1 f$, so any neighborhood of $\partial_0 f$ contains $\partial_1 f$. Let

$$\begin{array}{ccc} U & \xrightarrow{s_1} & (T \times \partial_0)^* S \\ \uparrow & & \downarrow \\ U & \rightarrowtail & (T \times \leq_X)_1 \end{array}$$

be a small enough local section with $(t, f) \in U$ and so that there is an open rectangle $V \times W$ in $T \times X$ and a local section

$$\begin{array}{ccc} V\times W & \xrightarrow{\quad s_0 \quad} & S \\ {\scriptstyle =}\Big\uparrow & & \Big\downarrow \\ V\times W & \rightarrowtail & T\times X \end{array}$$

such that $(T\times\partial_0)^* s_0|_U = s_1$. We will show $\mathcal{E} \circ s_1$ is continuous. Let $(t', f') \in U$. $\partial_0 f'$ is in the closure of $\partial_1 f'$, so $\partial_0 f' \in W$ implies $\partial_1 f' \in W$. $t' \in V$. Therefore, pulling back along $(t', f') : \leq_2 \to T \times \leq_X$ shows that $\mathcal{E}_{(t',f')}(s_1(t', f'))$ is $s_0(t', \partial_1 f)$ when $(t', f')^*(T\times\partial_1)^* S$ is identified with $(t', \partial_1 f')^* S$. Therefore $\mathcal{E} \circ s_1 = (T\times\partial_1)^* s_0|_U$ (the restriction exists because $\partial_0^{-1} W \subset \partial_1^{-1} W$). Therefore, the forgetful functor $SET^{T\times\leq X} \to SET^{T\times X}$ is an equivalence of categories. As T varies in *Top* this defines a *Top*-indexed equivalence $SET^{\leq X} \to SET^{X}$. □

THEOREM 4.1. *If $F : SET^{\mathbf{C}} \to SET^{\mathbf{D}}$ is a Top-indexed functor then F^1 preserves filtered colimits.*

PROOF. Let P be a directed set. Define the topological space X by taking the underlying set of X to be P_0 and an additional point ∞, and by taking a basis of the topology of X to be the subsets of the form

$$X_p = \{\infty\} \cup \{q \mid p \leq q\}, \quad p \in P_0 \,;$$

the directnedness of P ensures this is a basis, and that the point ∞ is not isolated. If T is a topological space and $S \in SET^{T\times X}$, then

$$[(T\times p_X)_* (T\times\pi_X)^* S](x_p) \approx (T\times x_p)^* S$$

for each $p \in P_0$ by Proposition 3.4, because X_p is a minimal neighborhood of the point $x_p \in X$ corresponding to p. Therefore, $SET^{T\times X} \xrightarrow{\approx} SET^{T\times P}$. This equivalence decomposes as

$$SET^{T\times X} \xrightarrow[\approx]{\ forget^{-1}\ } SET^{T\times\leq X} \xrightarrow{\ (T\times i_p)^*\ } SET^{T\times P}$$

where i_p is the topological functor $P \to \leq_X$ mapping p to x_p. We have already mentioned $i_p^* : SET^{\leq X} \to SET^{P}$ is *Top*-indexed by $(i_p^*)T = (T\times i_p)^*$, while the forgetful functor $SET^{\leq X} \to SET^{X}$ is *Top*-indexed by

Proposition 4.1. Therefore,

$$(SET^{C})^{P} = SET^{C\times P} \approx SET^{C\times X} = (SET^{C})^{X}$$

for any topological category C ; also, $(SET^{C})^{P}$ is the category of functors $P \to SET^{C}$ because P has discrete topology. Therefore, if $F: SET^{C} \to SET^{D}$ is a *Top*-indexed functor we have a diagram

$$\begin{array}{ccccccc}
(SET^{C})^{P} = SET^{C\times P} & \xrightarrow{(C\times i_p)^*} & SET^{C\times \leq_X} & \xrightarrow{forget} & SET^{C\times X} = (SET^{C})^{X} \\
\Big\downarrow F^{P} & & & & \Big\downarrow F^{X} \\
(SET^{D})^{P} = SET^{D\times P} & \xrightarrow{(D\times ip)^*} & SET^{D\times \leq_X} & \xrightarrow{forget} & SET^{D\times X} = (SET^{D})^{X}
\end{array}$$

which in combination with

$$\begin{array}{ccc}
(SET^{C})^{X} & \xrightarrow{\infty^*} & SET^{C} \\
\Big\downarrow F^{X} & & \Big\downarrow F^{1} \\
(SET^{D})^{X} & \xrightarrow{\infty^*} & SET^{D}
\end{array}$$

gives the commutative diagram

$$\begin{array}{ccccc}
(SET^{C})^{P} & \xrightarrow{\approx} & (SET^{C})^{X} & \xrightarrow{\infty^*} & SET^{C} \\
\Big\downarrow F^{P} & & & & \Big\downarrow F^{1} \\
(SET^{D})^{P} & \xrightarrow{\approx} & (SET^{D})^{X} & \xrightarrow{\infty^*} & SET^{D}
\end{array}$$

By the corollary to Theorem 3.1 the vlaue of

$$(SET)^{P} \xrightarrow{\approx} (SET^{C})^{X} \xrightarrow{\infty^*} SET^{D}$$

at $S: P \to SET^{C}$ is $\varinjlim_{p \in P} S(p)$ (similarly, with C replaced by D). For any $p \in P$ we have a commutative diagram

$$\begin{array}{ccc}
(SET^{C})^{P} & \xrightarrow{p^*} & SET^{C} \\
\Big\downarrow F^{P} & & \Big\downarrow F^{1} \\
(SET^{D})^{P} & \xrightarrow{p^*} & SET^{D}
\end{array} \quad ;$$

that is, if $S \in (SET^{C})^{P}$, we have

$$\begin{array}{ccc} P & \xrightarrow{\quad S \quad} & SET^{\mathrm{C}} \\ & \searrow^{F^P S} & \downarrow F^1 \\ & & SET^{\mathrm{D}} \end{array}$$

Combining this with the above gives us

$$F^1 \varinjlim_{p \in P} S(p) = F^1 \infty^* S = \infty^* F^P S = \varinjlim_{p \in P} (F^1 \circ S)(p) = \varinjlim_{p \in P} F^1(S(p))$$

for $S \in (SET^{\mathrm{C}})^P$. Therefore, F^1 preserves directed colimits. But it is well known [2], I.1.6, a functor preserves filtered colimits iff it preserves directed colimits. Therefore, F^1 preserves filtered colimits. □

COROLLARY. *If T is a topological space, D is a topological category, and $F: SET^T \to SET^{\mathrm{D}}$ is a Top-indexed functor then for each $X \in Top$, $F^X \approx (\mathrm{D} \times a_X) F^{\theta_X^{op}}$ where the right hand side of this equivalence views an object of $(SET^T)^X$ as an object of $(SET^T)^{\theta_X^{op}}$.*

PROOF. We have $\mathrm{D} \times a_X = \varinjlim_{\mathrm{D} \times \pi_X} (\mathrm{D} \times \rho_X)^*$, so for $S \in (SET^T)^X$ we have a comparison

$$\begin{array}{rcl} (\mathrm{D} \times a_X) F^{\theta_X^{op}} S = \varinjlim_{\mathrm{D} \times \pi_X} (\mathrm{D} \times \rho_X)^* F^{\theta_X^{op}} S & \approx & \varinjlim_{\mathrm{D} \times \pi_X} F^{\epsilon_X^{op}} (T \times \rho_X)^* S \\ & & \downarrow a \\ & & F^X S = F^X \varinjlim_{T \times \pi_X} (T \times \rho_X)^* S \end{array}$$

Let $x: 1 \to X$ be a point of X. We have a commutative triangle

$$\begin{array}{l} (\mathrm{D} \times x)^* (\mathrm{D} \times a_X) F^{\theta_X^{op}} S \qquad \searrow^{(\mathrm{D} \times x)^* a} \\ \downarrow \approx \\ \varinjlim_{x \in U} (F^{\theta_X^{op}} S)(U) \\ \downarrow \approx \\ \varinjlim_{x \in U} \{U\}^* F^{\theta_X^{op}} S \\ \downarrow \approx \\ \varinjlim_{x \in U} F^1 \{U\}^* S \\ \downarrow \approx \\ \varinjlim_{x \in U} F^1 S(U) \xrightarrow[\approx]{\text{by 4.1}} F^1 \varinjlim_{x \in U} S(U) \xrightarrow{\approx} F^1 (T \times x)^* S \xrightarrow{\approx} (\mathrm{D} \times x)^* F^X S \end{array}$$

Therefore, the fiber $(\mathrm{D}\times x)^*\alpha$ of α over each $x \in X$ is an isomorphism. Hence, α is an isomorphism. □

THEOREM 4.2. *If* $G: SET^T \to SET^{\mathrm{D}}$ *is a functor preserving filtered co-limits then there is a unique (up to isomorphism) Top-indexed functor* $F: SET^T \to SET^{\mathrm{D}}$ *such that* $F^1 = G$.

PROOF. By the corollary to Theorem 4.1, uniqueness will follow from existence. For $Z \in Top$, define

$$F^{\theta_Z^{op}}: (SET^T)^{\theta_Z^{op}} \longrightarrow (SET^{\mathrm{D}})^{\theta_Z^{op}} \quad \text{by}$$

$$(F^{\theta_Z^{op}} S) = (\theta_Z^{op} \xrightarrow{S} SET^T \xrightarrow{G} SET^{\mathrm{D}}) = G \circ S.$$

Let $f: Y \to X$ be a continuous function. For each $u \in \theta_Y$,

$$\{ v \in \theta_X \mid f^{-1}v \supset u \}$$

is directed. Therefore,

$$\begin{array}{ccc} (SET^T)^{\theta_X^{op}} & \xrightarrow{\varinjlim_{T\times f^{-1}}} & (SET^T)^{\theta_Y^{op}} \\ {\scriptstyle F^{\theta_X^{op}}}\downarrow & & \downarrow{\scriptstyle F^{\theta_Y^{op}}} \\ (SET^{\mathrm{D}})^{\theta_X^{op}} & \xrightarrow{\varinjlim_{\mathrm{D}\times f^{-1}}} & (SET^{\mathrm{D}})^{\theta_Y^{op}} \end{array}$$

commutes. Also, for $S \in (SET^T)^X$, viewing $S \in (SET^T)^{\theta_X^{op}}$, and $(T\times f)^* S \in (SET^T)^{\theta_Y^{op}}$ gives a comparison functor

$$\eta: \varinjlim_{T\times f^{-1}} S \longrightarrow (T\times f)^* S$$

because $(T\times f)^* S$ is the sheafication of $\varinjlim_{T\times f^{-1}} S$. This gives us a comparison functor

$$(\mathrm{D}\times a_Y) F^{\theta_Y^{op}} \eta : (\mathrm{D}\times a_Y) F^{\theta_Y^{op}} \varinjlim_{T\times f^{-1}} S \to (\mathrm{D}\times a_Y) F^{\theta_Y^{op}} (T\times f)^* S.$$

For each $y \in Y$ we have a commutative triangle

$$
\begin{array}{l}
(\mathrm{D}\times y)^*(\mathrm{D}\times a_Y)F^{\theta_Y^{op}} \varinjlim_{T\times f^{-1}} S \\
\quad \downarrow \approx \\
(\mathrm{D}\times y)^*(\mathrm{D}\times a_Y) \varinjlim_{\mathrm{D}\times f^{-1}} F^{\theta_X^{op}} S \\
\quad \downarrow \approx \\
\varinjlim_{y\in f^{-1}y} (F^{\theta_X^{op}} S)(v) \\
\quad \downarrow \approx \\
\varinjlim_{y\in f^{-1}v} G(S(v)) \\
\quad \downarrow \approx \\
G \varinjlim_{y\in f^{-1}v} S(v) \\
\quad \downarrow \approx \\
G \varinjlim_{y\in f^{-1}v} [(T\times f)^*S)](f^{-1}v) \approx \varinjlim_{y\in f^{-1}v} [F^{\theta_Y^{op}}(T\times f)^*S](f^{-1}v) \xrightarrow{\approx} (\mathrm{D}\times y)^*(\mathrm{D}\times a_Y)F^{\theta_Y^{op}}(T\times f)^*S
\end{array}
$$

$$(\mathrm{D}\times y)^*(\mathrm{D}\times a_Y)F^{\theta_Y^{op}} \varinjlim_{T\times f^{-1}} S \xrightarrow{(\mathrm{D}\times y)^*(\mathrm{D}\times a_Y)F^{\theta_Y^{op}}\eta} (\mathrm{D}\times y)^*(\mathrm{D}\times a_Y)F^{\theta_Y^{op}}(T\times f)^*S$$

Therefore, $(\mathrm{D}\times a_Y)F^{\theta_Y^{op}}\eta$ is an isomorphism.

Now for each $Z \in Top$ define $F^Z : (SET^T)^Z \to (SET^{\mathrm{D}})^Z$, at $S \in (SET^T)^Z$, by $F^Z S = (\mathrm{D}\times a_Z)F^{\theta_Z^{op}} S$, where the right hand side of the equality views S as an object of $(SET^T)^{\theta_Z^{op}}$. Therefore, for $S \in (SET^T)^X$, we have

$$
\begin{aligned}
f^* F^X S &\xrightarrow{\approx} (\mathrm{D}\times f)^* F^X S \xrightarrow{\approx} (\mathrm{D}\times f)^*(\mathrm{D}\times a_X)F^{\theta_X^{op}} S \\
&\xrightarrow{\approx} (\mathrm{D}\times a_Y) \varinjlim_{T\times f^{-1}} F^{\theta_X^{op}} S \xrightarrow{\approx} (\mathrm{D}\times a_Y)F^{\theta_Y^{op}} \varinjlim_{T\times f^{-1}} S \\
&\xrightarrow{(\mathrm{D}\times a_Y)F^{\theta_Y^{op}}\eta} (\mathrm{D}\times a_Y)F^{\theta_Y^{op}}(T\times f)^* S \xrightarrow{\approx} F^Y f^* S
\end{aligned}
$$

Therefore, we have a Top-indexed functor $F: SET^T \to SET^{\mathrm{D}}$ such that $F^1 = G$. □

COROLLARY. *For $X \in Top$ and $S \in SET^X$, the direct image functor $SET^{sub^X S} \to SET^X$ along $sub^X S \to X$ may be Top-indexed.*

PROOF. By Proposition 1.8, the direct image functor $SET^{sub^X S} \to SET^X$ preserves filtered colimits. Therefore, with $T = sub^X S$ and $\mathrm{C} = X$ the theorem says the direct image functor along $sub^X S \to X$ has a unique (up to isomorphism) *Top*-indexing. □

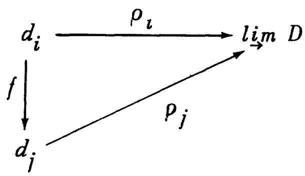
THEOREM 4.3. *Up to isomorphism the Top-indexed functors* $F: SET \to SET$ *are in bijective correspondeuce with the filtered colimit preserving functors* $F^1: Set \to Set$.

PROOF. This is just Theorems 4.1 and 4.2 combined with $T = 1$ and $C = 1$. □

Let *Fin* denote the category of finite sets and functions.

PROPOSITION 4.2. *A functor* $F: Fin \to Set$ *preserves all the filtered colimits existing in* *Fin*.

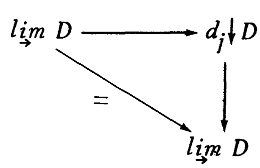
PROOF. Let D be a filtered diagram in *Fin* with $\varinjlim D \in Fin$. By the filteredness of D and because the objects of D are objects of *Fin*, there is a diagram



representing a section of the colimiting cone of $\varinjlim D$ with ρ_i onto $\varinjlim D$ and

$$Image(f) \rightarrowtail d_j \xrightarrow{\rho_j} \varinjlim D$$

an isomorphism. Therefore, we have a commutative diagram



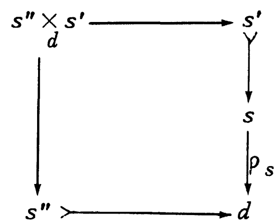
Since $d_j \downarrow D \rightarrowtail D$ is final, $\varinjlim(d_j \downarrow D) \xrightarrow{\cong} \varinjlim D$. Therefore, $\rho: D \to \varinjlim D$ is an absolute colimiting cone. Therefore, every functor on *Fin* preserves the colimit of D. □

A functor $Fin \overset{\wedge}{\to} Set$ has an extension $\hat{F}: Set \to Set$ defined at $S \in Set$ by $\hat{F}(S) = \varinjlim(F(Fin(S)))$, where $Fin(S)$ is the partially ordered set of finite subsets of S. Referring to the next diagram, for any filtered diagram D in Set with $d = \varinjlim D$, we have

$$\varinjlim_{s \in D} \hat{F}(s) \approx \varinjlim_{s \in D} \Big(\varinjlim_{d' \in Fin(s)} (F(s')) \Big)$$

$$\approx \varinjlim_{s \in D} \Big(\varinjlim_{s' \in Fin(s)} \big(li \varinjlim_{s'' \in Fin(d)} (F(s'' \times_d s')) \big) \Big)$$

$$\approx \varinjlim_{s'' \in Fin(d)} \varinjlim_{s \in D} \Big(\varinjlim_{s' \in Fin(s)} (F(s'' \times_d s')) \Big) \approx \hat{F}(d);$$



Therefore, the extension F of F preserves filtered colimits.

THEOREM 4.4. *The category of isomorphism classes of Top-indexed functors $SET \to SET$ is equivalent to Set^{Fin}.*

PROOF. Since every set is the canonical colimit of its finite subsets, any functor $Set \to Set$ preserving colimits is determined up to isomorphism by what it does to finite sets. Therefore, the filtered colimit preserving functors $Set \to Set$ correspond to functors $Fin \to Set$. The result now follows from Theorem 4.3. □

5. SHEAVES OF FINITARY ALGEBRAS.

THEOREM 5.1. *The algebraic theories whose algebras can be Top-indexed as the algebras of a Top-indexed triple $SET \to SET$ are precisely the finitary algebraic theories.*

PROOF. It is well-known that the finitary algebraic theories arise exactly from triples $Set \to Set$ preserving filtered colimits. The result follows from this and Theorem 4.3. □

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