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ROSS STREET

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**ABSOLUTE COLIMITS IN ENRICHED CATEGORIES**

by Ross STREET

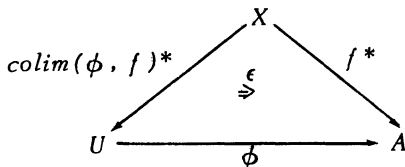
What is it about an indexing type  $\phi$  that ensures that every colimit indexed by  $\phi$  is preserved by all functors? The present short note answers this question in the context of enriched categories. Appropriate references are listed at the end of the paper. The base for enrichment can be a bicategory  $W$  although the reader may take it to be a symmetric monoidal closed category should this be more commodious. We use the term *module* for what others have called *bimodule*, *profunctor*, *distributor*.

For each enriched category  $A$ , there are enriched categories  $PA$ ,  $P^\dagger A$  such that the category of enriched functors  $C \rightarrow PA$  is naturally equivalent to the category of modules  $C \rightarrow A$  and the category of enriched functors  $C \rightarrow P^\dagger A$  is naturally equivalent to the dual of the category of modules  $A \rightarrow C$ . There are Yoneda embeddings

$$y: A \rightarrow PA, \quad y^\dagger: A \rightarrow P^\dagger A$$

which are fully faithful enriched functors. Also, recall that each enriched functor  $f: A \rightarrow B$  gives rise to modules  $f_*: A \rightarrow B$ ,  $f^*: B \rightarrow A$  such that  $f^*$  is a right adjoint for  $f_*$  in the bicategory of modules.

For a module  $\phi: U \rightarrow A$  and (enriched) functor  $f: A \rightarrow X$ , the colimit of  $f$  weighted (or «indexed») by  $\phi$  is a functor  $colim(\phi, f): U \rightarrow X$  together with a 2-cell



which exhibits  $colim(\phi, f)^*$  as a right lifting of  $f^*$  through  $\phi$  in the bicategory of modules. A functor  $g: X \rightarrow Y$  preserves the colimit when

$$g \, colim(\phi, f) \approx colim(\phi, gf);$$

that is, when  $g^*: Y \rightarrow X$  is pasted onto  $\epsilon$  the result still exhibits a right lifting through  $\phi$ . The colimit is *absolute* when it is preserved by all functors out of  $X$ .

**THEOREM.** *Every colimit weighted by  $\phi$  is absolute if and only if  $\phi$  has a right adjoint in the bicategory of modules.*

**PROOF.** If  $\phi$  has a right adjoint  $\psi$ , then  $\text{colim}(\phi, f)$  is characterized by

$$\text{colim}(\phi, f)^* \approx \psi f^*.$$

But then

$$(g \text{colim}(\phi, f))^* \approx \psi(gf)^* ;$$

so

$$g \text{colim}(\phi, f) \approx \text{colim}(\phi, gf).$$

So  $\text{colim}(\phi, f)$  is absolute.

Conversely, suppose all colimits weighted by  $\phi$  are absolute. Let

$$t = \text{colim}(\phi, y^\dagger): U \rightarrow P^\dagger A.$$

Then we have a right lifting diagram

$$\begin{array}{ccc} & P^\dagger A & \\ t^* \swarrow & \Rightarrow & \searrow (y^\dagger)^* \\ U & \xrightarrow{\phi} & A \end{array}$$

which is respected by all modules  $g^*$  where  $g: P^\dagger A \rightarrow Y$ . However, every module  $\theta$  is isomorphic to a composite  $g^* b_*$  for functors  $b, g$ ; and left adjoints respect all right liftings. So the above lifting is respected by all modules into  $P^\dagger A$ . In particular, it is respected by  $(y^\#)_*: A \rightarrow P^\dagger A$ . Put  $\psi = t^*(y^\dagger)_*$  and recall that  $(y^\dagger)^*(y^\dagger)_* \approx 1$  (since  $y^\dagger$  is fully faithful). So we have a right lifting

$$\begin{array}{ccc} & A & \\ \psi \swarrow & \Rightarrow & \searrow 1 \\ U & \xrightarrow{\phi} & A \end{array}$$

which is respected by all modules into  $A$ . It follows that  $\psi$  is a right ad-

joint for  $\phi$ .  $\square$

The above gives another characterization of the Cauchy completion of an enriched category as consisting of the weightings (indexing types) for absolute colimits. Hence a *category is Cauchy complete if and only if it admits all absolute colimits.*

As an example, since mapping cones and suspensions can be defined equationally for DG-categories (= categories enriched in complexes of abelian groups), Cauchy complete DG-categories admit suspensions and mapping cones (as well as the expected direct sums and splittings for idempotents).

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School of Mathematics and Physics  
 Macquarie University  
 NORTH RYDE, NSW 2113  
 AUSTRALIA