ROSS STREET

Absolute colimits in enriched categories

Cahiers de topologie et géométrie différentielle catégoriques, tome 24, nº 4 (1983), p. 377-379

<http://www.numdam.org/item?id=CTGDC_1983__24_4_377_0>

CAHIERS DE TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE

Vol. XXIV-4 (1983)

ABSOLUTE COLIMITS IN ENRICHED CATEGORIES

by Ross STREET

What is it about an indexing type ϕ that ensures that every colimit indexed by ϕ is preserved by all functors? The present short note answers this question in the context of enriched categories. Appropriate references are listed at the end of the paper. The base for enrichment can be a bicategory W although the reader may take it to be a symmetric monoidal closed category should this be more commodious. We use the term *module* for what others have called *bimodule, profunctor, distributor.*

For each enriched category A, there are enriched categories PA, $P^{\dagger}A$ such that the category of enriched functors $C \to PA$ is naturally equivalent to the category of modules $C \to A$ and the category of enriched functors $C \to P^{\dagger}A$ is naturally equivalent to the dual of the category of modules $A \to C$. There are Yoneda embeddings

$$y: A \to PA, \quad y^{\dagger}: A \to P^{\dagger}A$$

which are fully faithful enriched functors. Also, recall that each enriched functor $f: A \to B$ gives rise to modules $f_*: A \to B$, $f^*: B \to A$ such that f^* is a right adjoint for f_* in the bicategory of modules.

For a module $\phi: U \to A$ and (enriched) functor $f: A \to X$, *the colimit of f weighted* (or «indexed») *by ϕ* is a functor $colim(\phi, f): U \to X$ together with a 2-cell

$$\begin{tikzcd} & X \arrow[dl, "colim(\phi{,}\, f)^*"'] \arrow[dr, "f^*"] \arrow[d, phantom, "\overset{\epsilon}{\Rightarrow}"] & \\ U \arrow[rr, "\phi"'] & & A \end{tikzcd}$$

which exhibits $colim(\phi, f)^*$ as a right lifting of f^* through ϕ in the bicategory of modules. A functor $g: X \to Y$ *preserves the colimit* when

$$g\, colim(\phi, f) \approx colim(\phi, gf);$$

that is, when $g^*: Y \to X$ is pasted onto ϵ the result still exhibits a right lifting through ϕ. The colimit is *absolute* when it is preserved by all functors out of X.

THEOREM. *Every colimit weighted by ϕ is absolute if and only if ϕ has a right adjoint in the bicategory of modules.*

PROOF. If ϕ has a right adjoint ψ, then $colim(\phi, f)$ is characterized by

$$colim(\phi, f)^* \approx \psi f^*.$$

But then

$$(g\, colim(\phi, f))^* \approx \psi (gf)^* ;$$

so

$$g\, colim(\phi, f) \approx colim(\phi, gf).$$

So $colim(\phi, f)$ is absolute.

Conversely, suppose all colimits weighted by ϕ are absolute. Let

$$t = colim(\phi, y^\dagger): U \to P^\dagger A.$$

Then we have a right lifting diagram

$$\begin{tikzcd} & P^\dagger A \arrow[dl, "t^*"'] \arrow[dr, "(y^\dagger)^*"] \arrow[d, phantom, "\Rightarrow"] & \\ U \arrow[rr, "\phi"'] & & A \end{tikzcd}$$

which is respected by all modules g^* where $g: P^\dagger A \to Y$. However, every module θ is isomorphic to a composite $g^* h_*$ for functors h, g; and left adjoints respect all right liftings. So the above lifting is respected by all modules into $P^\dagger A$. In particular, it is respected by $(y^\#)_*: A \to P^\dagger A$. Put $\psi = t^*(y^\dagger)_*$ and recall that $(y^\dagger)^*(y^\dagger)_* \approx 1$ (since $y^\dagger$ is fully faithful). So we have a right lifting

$$\begin{tikzcd} & A \arrow[dl, "\psi"'] \arrow[dr, "1"] \arrow[d, phantom, "\Rightarrow"] & \\ U \arrow[rr, "\phi"'] & & A \end{tikzcd}$$

which is respected by all modules into A. It follows that ψ is a right adjoint for ϕ . □

The above gives another characterization of the Cauchy completion of an enriched category as consisting of the weightings (indexing types) for absolute colimits. Hence a *category is Cauchy complete if and only if it admits all absolute colimits.*

As an example, since mapping cones and suspensions can be defined equationally for DG-categories (= categories enriched in complexes of abelian groups), Cauchy complete DG-categories admit suspensions and mapping cones (as well as the expected direct sums and splittings for idempotents).

REFERENCES.

1. R. BETTI, A. CARBONI, R.H. STREET & R.F.C. WALTERS, Variation through enrichment, *J. Pure Appl. Algebra* (to appear).
2. G.M. KELLY, *Basic concepts of Enriched Category Theory,* Cambridge Un. Press, 1982.
3. F.W. LAWVERE, Metric spaces, generalized logic, and closed categories, *Ren. Sem. Mat. Fis. Milano* 43 (1974), 135-166.
4. R.H. STREET, Enriched categories and cohomology, *Quaestiones Math.* 6 (1983), 265-283.
5. R.H. STREET, Cauchy characterization of enriched categories, *Rend. Sem. Mat. Fis. Milano* 51 (1981), 217-233.
6. R.H. STREET & R.F.C. WALTERS, Yoneda structures on 2-categories, *J. Algebra* 50 (1978), 350-379.

School of Mathematics and Physics
Macquarie University
NORTH RYDE, NSW 2113
AUSTRALIA