KYUNG CHAN MIN

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AN EXPONENTIAL LAW FOR REGULAR ORDERED BANACH SPACES

by KYUNG CHAN MIN

0. INTRODUCTION.

The notion of a category upholding an exponential law

$$[E \otimes F, G] \approx [E, [F, G]]$$

(more precisely, a symmetric monoidal closed category), pioneered by S. Eilenberg and G.M. Kelly [4] and others, provides a setting in which elegant functorial techniques become available. These techniques, which exploit the presence of a large number of canonical morphisms, become even more powerful if the category also admits all the usual limit and colimit constructions (technically, if it is complete and cocomplete). Thus, a «well-equipped» category is one which upholds an exponential law and is complete and cocomplete.

An important example is

Ban = (Banach spaces, linear maps with norm at most *1*).

The «well-equipped» feature of this category has been put to effective use by a number of authors so as to bring interesting new developments into the theory of Banach spaces. See for example J. Cigler, V. Losert and P. Michor [2], C. Herz and J. Wick Pelletier [10], L.D. Nel [15] and J. Wick Pelletier [18]. Numerous further papers could be cited to illustrate the effectiveness of other «well-equipped» categories in Analysis and (topological) Algebra.

In the realm of *ordered* Banach spaces, however, despite a vast literature involving numerous special classes of spaces, no corresponding «well-equipped» category has so far emerged. The main purpose of this paper is to bring to light such a category.

To be specific, we will show that the class of regular ordered Banach spaces (previously studied from a different point of view [3,7]) can be structured into a «well-equipped» category *ROBan* by taking as morphisms all positive linear maps with norm at most *1* and by introducing appropriate internal Hom-objects $[E, F]$ (operator spaces), so as to obtain an exponential law, in fact a symmetric monoidal closed structure.

ROBan seems to be the appropriate «ordered version» of *Ban*. While it is of course a subcategory of *Ban*, it is remarkable how little structure it inherits from *Ban*. As will be shown below, the internal Hom-objects $[E, F]$, the categorical tensor products $E \boxtimes F$ and even the equalizers (roughly, the regular ordered Banach subspaces) of *ROBan* carry norms that are in general different from the norms of the corresponding «parent» objects in *Ban*. Products and coproducts in *ROBan*, though, are formed with the same Banach space structure as in *Ban*. Coequalizers (roughly, quotient spaces) in *ROBan* are something else again: We have so far been unable to obtain their explicit form and we prove only their existence by indirect categorical arguments.

It is fortunate that *ROBan* includes virtually all interesting ordered Banach spaces; in particular, it includes all Banach lattices. It is worth pointing out that the spaces arising from the study of an axiomatic foundation of quantum mechanics are regular ordered Banach spaces and the crucial operators are morphisms in *ROBan* (cf. [7]). Thus, because of its excellent theoretical attributes and its relevance to real world problems, the category *ROBan* seems destined to play an important role in Analysis.

For general categorical background we refer to H. Herrlich and G.E. Strecker [9] and for closed categories to S. Eilenberg and G. M. Kelly [4].

This paper is based on a chapter in my doctoral dissertation [13]. It is a pleasure to thank my supervisor, Professor L. D. Nel, for suggesting this investigation and for his stimulating guidance and encouragement durring the research and preparation of this work.

1. A REGULAR ORDERED BANACH SPACE AND ITS POSITIVE UNIT BALL

For ordered vector space theory we generally follow the terminology of Y.-C. Wong and K.-F. Ng [22].

An ordered normed space $(E, C, \| \, \|)$ is called *regular* [3] when the positive cone C of E is closed and $\| \ \|$ is a Riesz norm; i.e., $\| \ \|$ satisfies the following two conditions:

(R1) if $-y \leq x \leq y$, then $\|x\| \leq \|y\|$ (absolute-monotone),

(R2) for any $x \in E$ with $\|x\| < 1$, there exists $y \geq 0$ with $\|y\| < 1$ such that $-y \leq x \leq y$.

The condition (R2) implies that C is generating: indeed,

$$x = 2^{-1}(y+x) - 2^{-1}(y-x).$$

Furthermore, (R2) is equivalent to the following statement: If $x \in E$ and $\epsilon > 0$, then there is $y \geq 0$ with $\|y\| \leq \|x\| + \epsilon$ such that $-y \leq x \leq y$.

It is well known and easily seen that every Banach lattice is a regular ordered Banach space and an ordered normed space $(E, \| \ \|)$ is a locally solid space iff it possesses an equivalent Riesz norm $\| \ \|_1$, where

$$\|x\|_1 = \inf\{ \|x\| : -y \leq x \leq y \} \quad \text{for all } x \in E.$$

A regular ordered Banach space E is fully characterized by its positive unit ball as is a Banach space by its unit ball. By the positive unit ball of E is meant the set

$$UE = \{ x \in E : \|x\| \leq 1, \ x \geq 0 \}.$$

1.1. The boundedness of a positive linear map between regular ordered normed spaces is determined by the positive unit ball of the domain space.

PROPOSITION. *Let E and F be regular ordered normed spaces and $f: E \to F$ a bounded positive linear map. Then the sup norm satisfies*

$$\|f\| = \sup\{ \|f(x)\| : \|x\| \leq 1, \ x \in C \}.$$

PROOF. Let $x \in E$ and $\|x\| < 1$. Then there is $y \in C$ with $\|y\| < 1$ such that $-y \leq x \leq y$. Thus

$$-f(y) \leq f(x) \leq f(y)$$

and therefore $\|f(x)\| \leq \|f(y)\|$. Hence

$$\begin{aligned} \|f\| &= \sup\{ \|f(x)\| : \|x\| < 1 \} \\ &\leq \sup\{ \|f(y)\| : \|y\| < 1, \ y \in C \} \leq \|f\|. \quad / \end{aligned}$$

We note that every positive linear map between regular ordered Banach spaces is bounded (cf. II.2.16 [16]).

Consider the function $U: ROBan \to Set$ defined by:

UE = the positive unit ball of E on objects, and

$U(f) = f \mid UE$ for every $f: E \to F$ in $ROBan$.

Then U is a faithful functor $(E = C - C)$. This positive unit ball functor U is closely related to the unit ball functor $U_1: Ban \to Set$

1.2. PROPOSITION. *The functor $U: ROBan \to Set$ has a left adjoint, namely $l_1(\cdot, \mathrm{R})$.*

PROOF. It is well known (cf. I.1.11 [2]) that $l_1(\cdot, \mathrm{R})$ is left adjoint to $U_1: Ban \to Set$ via the natural isomorphism

$$\psi_{SE}: Ban(l_1(S, \mathrm{R}), E) \to Set(S, U_1E), \quad \psi(t)(s) = t(\eta_S(s)),$$

where $\eta_S(s)$ is the characteristic function of $\{s\}$. Since $l_1(S, \mathrm{R})$ is a Banach lattice (cf. II.4.12 [16]), hence already lies in $ROBan$ and since $ROBan$ is a subcategory of Ban, it is enough to check that for every E in $ROBan$, ψ carries the subset $ROBan(l_1(S, \mathrm{R}), E)$ onto $Set(S, UE)$. This is routine. /

1.3. PROPOSITION. *The mono-sources in ROBan are precisely the points separating-sources.*

PROOF. Let $\{m_i: E \to E_i\}_I$ be a mono-source in $ROBan$, where I is an index class. Then since U is a right adjoint functor $\{U(m_i): UE \to UE_i\}_I$ is a mono-source in Set, which separates points of UE. Suppose $x \neq 0$ in E and $x = x_1 - x_2$ with $x_1, x_2 \in C$. Then

$$\|x_1 + x_1\|^{-1} x_1 \neq \|x_1 + x_2\|^{-1} x_2 \quad \text{and} \quad \|x_1 + x_2\|^{-1} x_i \in UE$$

$(i = 1, 2)$. Thus $m_j(x) \neq 0$ for some $j \in I$. /

1.4. The functor U shares with algebraic forgetful functor (e.g. the functor $Group \to Set$) the following useful property.

PROPOSITION. *Every mono-source in ROBan is U-initial* [8].

PROOF. Let $\{m_i: E \to E_i\}_I$ be a mono-source in $ROBan$, where I is an index class. For a source $\{g_i: F \to E_i\}_I$ in *ROBan* and a function

$$f: UF \to UE \quad \text{such that} \quad U(m_i)\circ f = U(g_i) \text{ for each } i \in I,$$

we can define a function $\bar{f}: F \to E$ by

$$\bar{f}(x) = \|x_1\| f(\|x_1\|^{-1}x_1) - \|x_2\| f(\|x_2\|^{-1}x_2),$$

where $x = x_1 - x_2$ with $x_1, x_2 \in CF$, subject to the convention $\|0\|^{-1}0 = 0$. Indeed, $\bar{f}$ is well-defined: For each $i \in I$,

$$m_i(\|x_1\| f(\|x_1\|^{-1}x_1) - \|x_2\| f(\|x_2\|^{-1}x_2)) =$$
$$= \|x_1\| m_i(f(\|x_1\|^{-1}x_1)) - \|x_2\| m_i(f(\|x_2\|^{-1}x_2))$$
$$= \|x_1\| g_i(\|x_1\|^{-1}x_1) - \|x_2\| g_i(\|x_2\|^{-1}x_2) = g_i(x).$$

Hence, $\bar{f}(x)$ is independent on the choice of x_1, x_2, because $\{m_i\}$ separates points of E by Proposition 1.3. For $x, y \in F$,

$$m_i(\bar{f}(x+y)) = g_i(x+y) = g_i(x) + g_i(y)$$
$$= m_i(\bar{f}(x)) + m_i(\bar{f}(y)) = m_i(\bar{f}(x) + \bar{f}(y))$$

for each $i \in I$, and therefore $\bar{f}(x+y) = \bar{f}(x) + \bar{f}(y)$. In a similar way, for $a \in \mathrm{R}$ and $x \in F$, $\bar{f}(ax) = a\bar{f}(x)$. Obviously, $\bar{f}$ is positive. Moreover,

$$\|\bar{f}\| = sup\{\|\bar{f}(x)\| : x \in UF\}$$
$$= sup\{\|x\| f(\|x\|^{-1}x) : x \in UF\} \leq 1.$$

$U(\bar{f}) = f$: For $x \in UF$,

$$m_i(\bar{f}(x)) = g_i(x) = m_i(f(x)) \quad \text{for each } i \in I,$$

and therefore $\bar{f}(x) = f(x)$. The uniqueness of such a map $\bar{f}$ follows immediately, since the functor U is faithful. /

2. INTERNAL HOM-FUNCTOR FOR *ROBan*.

We now embark on the derivation of an exponential law for *ROBan*. An obvious starting point in the quest for an appropriate internal Hom-object $[E, F]$ is the vector space of all bounded linear maps $E \to F$ which can be expressed as a difference of two positive linear maps. However, the choice of norm is not obvious, since the usual *sup* norm turns out to fail in general.

2.1. For $E, F \in ROBan$, let $[E, F]$ be the set of all linear maps from E to F which can be expressed as the difference of two (bounded) positive linear maps from E to F. Then $[E, F]$ is an ordered vector space with a natural generating positive cone C (= the set of all positive linear maps from E to F). Consider the function $\| \; \|_1 : [E, F] \to R^+$ defined by:

$$\|f\|_1 = \inf\{ \|g\| : -g \leq f \leq g, \; g \in [E, F] \},$$

where $\|g\|$ is the *sup* norm of g. Then it is easy to check that $\| \; \|_1$ is a semi-norm on $[E, F]$.

2.2. It is known (cf. IV.1 [17]) that for Banach lattices E and F, then $([E, F], \| \; \|_1)$ is an ordered Banach space with a normal B-cone. Here, we generalize this result to regular ordered Banach spaces. As a matter of fact, for $E, F \in ROBan$, $([E, F], C, \| \; \|_1)$ will be seen to be again a regular ordered Banach space.

LEMMA 1. *Let E and F be regular ordered Banach spaces. Then for every $f \in [E, F]$, $\|f\| \leq \|f\|_1$. Further, if f is positive, then $\|f\| = \|f\|_1$.*

PROOF. Let $g \in [E, F]$ such that $-g \leq f \leq g$. Take $x \in E$ with $\|x\| < 1$, and $y \in CE$ such that $\|y\| < 1$ and $-y \leq x \leq y$. Then

$$-g(y+x) \leq f(y+x) \leq g(y+x) \quad \text{and} \quad -g(y-x) \leq f(y-x) \leq g(y-x)$$

which implies

$$-g(y-x) \leq f(x-y) \leq g(y-x).$$

By adding the first and the last inequalities, we have

$$-g(y) \leq f(x) \leq g(y), \quad \text{and therefore} \quad \|f(x)\| \leq \|g(y)\|.$$

Thus

$$\|f\| = \sup\{ \|f(x)\| : \|x\| < 1 \}$$
$$\leq \sup\{ \|g(y)\| : \|y\| < 1, \; y \in CE \} = \|g\|.$$

Hence $\|f\| \leq \|f\|_1$. If f is positive, then $\|f\|_1 \leq \|f\|$ by the definition of $\| \; \|_1$, and hence $\|f\| = \|f\|_1$. /

LEMMA 2 (*Jameson, 3.5.11* [11]). *Let E be a metrizable topological vector space. If it is open decomposable and each increasing Cauchy sequence in CE has a limit, then E is complete.*

THEOREM. *Let E and F be regular ordered Banach spaces. Then the space $[E, F]$ equipped with the pointwise order and with the norm $\| \|_1$ is a regular ordered Banach space.* (Henceforth, $[E, F]$ will be supposed always to carry this structure.)

PROOF. By the definition of $\| \|_1$ and the above Lemma 1, it is easy to see that $\| \|_1$ is a Riesz norm. Observe that the positive cone C of $[E, F]$ is closed with respect to the *sup* norm $\| \|$. Hence the positive cone C is closed with respect to the stronger topology of $\| \|_1$. Therefore, $([E, F], C, \| \|_1)$ is a regular ordered normed space.

To show the completeness of $([E, F], \| \|_1)$, let $\{f_n\}$ be an increasing Cauchy sequence in C (with respect to $\| \|_1$). Then $\{f_n\}$ is an increasing Cauchy sequence with respect to the weaker topology of $\| \|$ and hence converges to some bounded linear map f. Furthermore, since the sequence $\{f_n\}$ is increasing and C is closed with respect to $\| \|$, $f = \sup_{N} f_n$ and therefore $f \in C$. As a matter of fact, f is a limit of $\{f_n\}$ in $([E, F], \| \|_1)$, because

$$\|f - f_n\|_1 = \|f - f_n\| \quad \text{for all } n \in \mathrm{N}$$

by Lemma 1. Thus $([E, F], \| \|_1)$ is complete by Lemma 2. /

2.3. It is known (IV.1.4 [17]) that $\| \|_1 > \| \|$ in general. Here we remark on some relationships between $[E, F]$ and the ordered Banach space $L(E, F)$ of bounded linear maps from E to F, with respect to the *sup* norm and the pointwise order.

(1) A. J. Ellis (1 [5]) showed that if E is a base normed space and F is an order-unit normed space, then $L(E, F)$ is an order-unit normed space. It is known (cf. IV.1.5 [17]) that for Banach lattices E and F if

(a) F is an order complete AM-space with unit (a largest element in the unit ball),

or (b) E is an AL-space and there exists a positive contractive projection $P: F'' \to F$ (by means of evaluation, F is considered as a subspace of F''),

then $L(E, F)$ is a Banach lattice.

Thus, in these cases, it is easy to see that $[E, F] = L(E, F)$ in *ROBan*.

(2) A.W. Wickstead [19] showed that for $E, F \in ROBan$ each of the following cases implies that $L(E, F)$ is a locally solid space :

(a) Let Ω be a *stonean* space, i. e. a compact Hausdorff space such that the closure of every open subset is open, and $C(\Omega)$ (= the set of all real-valued continuous functions on Ω) the Banach lattice with respect to the *sup* norm and the pointwise order. Let $F = C(\Omega)$.

(b) Let E or F be finite dimensional.

Thus, in these cases, $[E, F] = L(E, F)$, as ordered vector spaces, and $\| \|_1$ and the *sup* norm $\| \|$ are equivalent.

2.4. We conclude this section by showing that the category *ROBan* is closed.

The regular ordered Banach space $[E, F]$ defines a functor

$$[-,-] : ROBan^* \times ROBan \to ROBan$$

on objects; its definition on morphisms proceeds in the obvious way. Indeed,

$$\begin{aligned} \|[f, g]\| &= sup\{ \|g \circ h \circ f\|_1 : h \in U[E, F]\} \\ &= sup\{ \|g \circ h \circ f\| : h \in U[E, F]\} \end{aligned}$$

by Lemma 2.2.1 ≤ 1.

Moreover, each of the following maps in *ROBan* induces a natural transformation :

$i = i_E : E \to [\mathrm{R}, E]$, defined by $i(x) = f_x$ with $f_x(1) = x$,

an isomorphism,

$j = j_E : \mathrm{R} \to [E, E]$, defined by $j(1) = 1_E$,

$c = c_{EFG} : [F, G] \to [[E, F], [E, G]]$, defined by $c(f)(g) = f \circ g$.

Thus,

$$ROBan = (ROBan, U, [-,-], \mathrm{R}, i, j, c)$$

is a closed category in the sense of [4].

3. THE PROJECTIVE TENSOR PRODUCT OF REGULAR ORDERED BANACH SPACES.

The appropriate tensor product for *ROBan* turns out to have been studied already recently form a different point of view. Adopting the projective tensor product of regular ordered normed spaces due to G. Wittstock [20], we obtain a bifunctor ⊠ on *ROBan* which will turn out to be adjoint to $[-,-]$.

3.1. We recall some results of [20] concerning tensor products of regular ordered normed spaces.

Let E and F be regular ordered normed spaces and let

$$C_p = \{ \sum_{i=1}^{n} x_i \otimes y_i : x_i \in CE,\ y_i \in CF,\ n \in \mathrm{N} \}.$$

Then C_p is a generating cone for the vector space $E \otimes F$. For each $v \in C_p$, let

$$\| v \|_p = sup \{ \Psi(v) : \Psi \in B_b(E,F)_+,\ \|\Psi\| \leq 1 \},$$

where

$B_b(E,F)_+$ = the cone of all bounded positive bilinear functionals on $E \times F$.

Then, the functional

$$\| u \|_p = inf \{ \|v\|_p : -v \leq u \leq v \}$$

is a norm on $E \otimes F$. Indeed, $E \otimes_p F = (E \otimes F, \bar{C}_p, \| \ \|_p)$ is a regular ordered normed space such that

$$\| x \otimes y \|_p \leq \|x\| \, \|y\| \quad \text{for all } x \in CE \text{ and } y \in CF$$

and $\| f \otimes g \| \leq \|f\| \, \|g\|$ for all $f \in CE'$ and $g \in CF'$.

Moreover, the canonical bilinear map $\Phi_{EF} : E \times F \to E \otimes_p F$, which is positive and bounded, has the following universal property:

If $\Psi : E \times F \to G$ is a bounded positive bilinear map into a regular ordered normed space G, then the induced linear map $\tilde{\Psi} : E \otimes_p F \to G$ is bounded, positive and $\|\tilde{\Psi}\| = \|\Psi\|$:

REMARK [21]. In fact, for each $u \in E \otimes F$,

$$\|u\|_p = inf\{\sum_{i=1}^{n} \|x_i\| \|y_i\| : -\sum_{i=1}^{n} x_i \otimes y_i \leq u \leq \sum_{i=1}^{n} x_i \otimes y_i,$$
$$x_i \in CE,\ y_i \in CF,\ n \in \mathrm{N}\}.$$

3.2. THEOREM. *There exists a functor* $\boxtimes : ROBan \times ROBan \to ROBan$ *and for any* E, F *in ROBan, there exists a universal bilinear map* $\Theta_{EF}: E \times F \to E \boxtimes F$ *for ROBan, i. e. for every positive bilinear map* $\Psi : E \times F \to G$ *with norm at most* 1, *there is precisely one map* $\bar{\Psi} : E \boxtimes F \to G$ *in ROBan such that* $\bar{\Psi} \circ \Theta_{EF} = \Psi$.

(We call $\boxtimes$ the *projective tensor product* for the category *ROBan*.)

PROOF. Let $E \boxtimes F$ be the completion of $E \underset{p}{\otimes} F$. Then $E \boxtimes F$ is a regular ordered Banach space with respect to the order generated by the closure of C_p in $E \boxtimes F$ (cf. 2.4 [20]). Let $ci: E \underset{p}{\otimes} F \to E \boxtimes F$ be the canonical injection and

$$\Theta_{EF} = ci \circ \Phi_{EF}: E \times F \to E \boxtimes F,$$

where $\Phi_{EF}: E \times F \to E \underset{p}{\otimes} F$ is the canonical bilinear map. Then the positive bilinear map Θ_{EF} is universal for *ROBan* (note that $\|\Theta_{EF}\| = 1$) : Let $\Psi : E \times F \to G$ be a positive bilinear map with norm at most 1. Then, by 3.1, there is a unique bounded positive linear map

$$\tilde{\Psi} : E \underset{p}{\otimes} F \to G \quad \text{such that} \quad \tilde{\Psi} \circ \Phi_{EF} = \Psi \quad \text{and} \quad \|\tilde{\Psi}\| = \|\Psi\|.$$

Let $\bar{\Psi} : E \boxtimes F \to G$ be the unique extension of $\tilde{\Psi}$. Then $\|\bar{\Psi}\| = \|\tilde{\Psi}\|$ and $\bar{\Psi}$ is positive, since CG is closed. Thus $\bar{\Psi}$ is a map in *ROBan*. Moreover

$$\bar{\Psi} \circ \Theta_{EF} = \bar{\Psi} \circ ci \circ \Phi_{EF} = \tilde{\Psi} \circ \Phi_{EF} = \Psi.$$

Therefore a functor

$$\boxtimes : ROBan \times ROBan \to ROBan$$

is determined by the universal positive bilinear maps Θ_{EF} for *ROBan* : Indeed, $\Theta_{EF}: E \times F \to E \boxtimes F$ is a natural transformation, and $f \boxtimes g: E \boxtimes F \to G \boxtimes H$ is given by the factorization

$$\Theta_{GH} \circ (f \times g) = (f \boxtimes g) \circ \Theta_{EF}. \quad /$$

REMARK. D.H. Fremlin (1E [6]) showed that for Banach lattices E and F, $E \boxtimes F$ is again a Banach lattice.

3.3. We conclude this section by showing that every bounded linear map from a finite combination of the projective tensor product $\boxtimes$ in *ROBan* to a regular ordered Banach space is determined by the values on certain positive elements in the domain space.

PROPOSITION. *Let E, F, G and H be regular ordered Banach spaces and $f, g: (E \boxtimes F) \boxtimes G \to H$ bounded linear maps. If for all $x \in UE$, $y \in UF$ and $z \in UG$, $f((x \otimes y) \otimes z) = g((x \otimes y) \otimes z)$, then $f = g$:*

PROOF. By the definition of $\boxtimes$, it is enough to show that

for all $u \in UE \boxtimes F$ and $z \in UG$, $f(u \otimes z) = g(u \otimes z)$:

Indeed,

$$f(u \otimes z) = f((\lim_n \sum_{i=1}^{k(n)} (x_{in} \otimes y_{in})) \otimes z),$$

where $x_{in} \in CE$ and $y_{in} \in CF$ for all i, n,

$$= f(\lim_n ((\sum_{i=1}^{k(n)} (x_{in} \otimes y_{in})) \otimes z)),$$

since $\Phi_{E \boxtimes F\, G}$ is bounded,

$$= \lim_n \sum_{i=1}^{k(n)} f((x_{in} \otimes y_{in}) \otimes z)$$

$$= \lim_n \sum_{i=1}^{k(n)} \|x_{in}\| \, \|y_{in}\| f((\|x_{in}\|^{-1} x_{in} \otimes \|y_{in}\|^{-1} y_{in}) \otimes z$$

subject to the convention : $\|0\|^{-1} 0 = 0$,

$$= g(u \otimes z), \quad \text{by assumption.} \quad /$$

4. A SYMMETRIC MONOIDAL CLOSED STRUCTURE OF *ROBan*.

We first show that *ROBan* upholds an exponential law.

4.1. *Exponential law for ROBan: There exists a natural isomorphism*

$$\alpha_{EFG}: [E \boxtimes F, G] \to [E, [F, G]]$$

such that $\alpha(f)(x)(y) = f(x \otimes y)$ for all $f \in [E \boxtimes F, G]$, $x \in E$ and $y \in F$.

PROOF. For a positive linear map $g: E \boxtimes F \to G$, define a function

$$\bar{g}: E \to [F, G] \quad \text{by} \quad \bar{g}(x)(y) = g \circ \Theta_{EF}(x, y).$$

Then, by routine verification, $\bar{g}$ is a positive linear map and $\|\bar{g}\| = \|g\|$. Now, define a function

$$a_{EFG} : [E \boxtimes F, G] \to [E, [F, G]] \text{ by } a(f) = \bar{f}_1 \cdot \bar{f}_2,$$

where $f = f_1 \cdot f_2$, f_1, f_2 are positive. Then, using the universal property of Θ_{EF}, we can check without difficulty that a is a bijective linear map. Obviously, a is positive and norm preserving. /

REMARK. This result can also be obtained via U-bimorphisms, using results of B. Banaschewski & E. Nelson [1] and Proposition 1.4 (see [13]).

4.2. LEMMA. *There is a natural isomorphism*

$$ic : [E, [F, G]] \to [F, [E, G]]$$

such that $ic(f)(y)(x) = f(x)(y)$ *for all* $f \in [E, [F, G]]$, $x \in E$ *and* $y \in F$.

PROOF. For a positive linear map $g : E \to [F, G]$, define a function

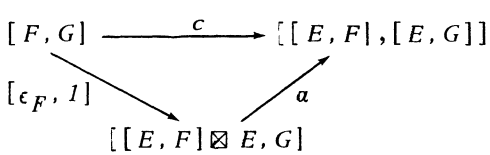
$$\bar{g} : F \to [E, G] \text{ by } \bar{g}(y)(x) = g(x)(y).$$

Then $\bar{g}$ is a positive linear map and $\|\bar{g}\| = \|g\|$. In fact, $ic(f) = \bar{f}_1 \cdot \bar{f}_2$, where $f = f_1 \cdot f_2$, f_1, f_2 are positive. /

4.3. Now we can obtain the main result in this section, which incorporates the exponential law just obtained.

THEOREM. *ROBan is a symmetric monoidal closed category.*

PROOF. To obtain a monoidal closed structure on *ROBan*, it is enough to show (2.4.1 [4]) that the following diagram commutes



where ϵ is the counit of the adjunction $- \boxtimes E \dashv [E, -]$. Indeed, for all $g \in [F, G]$, $f \in [E, F]$ and $x \in E$,

$$(a \circ [\epsilon_F, 1])(g)(f)(x) = a(g \circ \epsilon_F)(f)(x) =$$

$$= g(\epsilon_F(f \otimes x)) = g(f(x)) = c(g)(f)(x).$$

Hence $a \circ [\epsilon_F, 1] = c$. Thus,

$$ROBan = (ROBan, \boxtimes, \mathrm{R}, r, l, a)$$

is a monoidal closed category, where a, r and l are the natural isomorphisms determined by α and i, that give coherence.

NOTE 1. $a_{EFG}((x \otimes y) \otimes z) = x \otimes (y \otimes z)$ for all $x \in E$, $y \in F$ and $z \in G$.

For symmetry, apply Lemma 4.2 to obtain a natural isomorphism

$$U(ic \circ \alpha) : (F \boxtimes E, G) \to (E, [F, G])$$

such that

$$U(ic \circ \alpha)(f)(x)(y) = f(y \otimes x) \quad \text{for all } x \in E \text{ and } y \in F.$$

It follows that there exists exactly one natural isomorphism $s : \cdot \boxtimes F \to F \boxtimes \cdot$ making the diagram

$$\begin{array}{ccccc}
(F \boxtimes E, G) & \xrightarrow{U(\alpha)} & (F, [E, G]) & \xrightarrow{U(ic)} & (E, [F, G]) \\
\downarrow{\scriptstyle (s,1)} & & & & \downarrow{\scriptstyle (1,1)} \\
(E \boxtimes F, G) & & \xrightarrow{\quad U(\alpha) \quad} & & (E, [F, G])
\end{array}$$

commutative (cf. 4, [12]).

NOTE 2. $s(x \otimes y) = y \otimes x$ for all $x \in E$ and $y \in F$.

Moreover, since $\boxtimes$ is a bifunctor, $s : E \boxtimes F \to F \boxtimes E$ is a natural isomorphism. The coherence properties follow by the above Note 1 and Note 2, Theorem 3.2 and Proposition 3.3. /

REMARK. $r(x \otimes \alpha) = l(\alpha \otimes x) = \alpha x$ for all $x \in E$ and $\alpha \in \mathrm{R}$.

5. COMPLETENESS AND COCOMPLETENESS OF *ROBan*.

Neither the proof of completeness nor of cocompleteness is a routine matter. While the obvious forgetful functor $ROBan \to Ban$ preserves products and coproducts, it does not preserve equalizers. The general form of coequalizers turns out to be rather elusive.

By establishing existence of products and equalizers, we show that the category *ROBan* is complete.

5.1. PROPOSITION. *ROBan has products, in fact, the underlying functor from ROBan to Ban preserves and reflects products.*

PROOF. Recall (cf. [2]) that in *Ban* the product of spaces E_i $(i \in I)$ is formed by the space

$$l_\infty(I, \{E_i\}_I) = (\{(x_i) \in \prod_I E_i : \sup_I \|x_i\| < \infty\}, \|(x_i)\|_\infty = \sup_I \|x_i\|)$$

with projection functions

$$pr_i : l_\infty(I, \{E_i\}_I) \to E_i \quad (i \in I).$$

In view of Proposition 1.4 this will give also a product in *ROBan* provided that $l_\infty(I, \{E_i\}_I)$ equipped with the pointwise order lies in *ROBan* and the projections are positive. But both of these follow by routine verification. /

5.2. Hereafter, CE denotes the positive cone of E.

PROPOSITION. *ROBan has equalizers.*

PROOF. For a pair of maps $f, g : E \to F$ in *ROBan*, let

$$D = \{x \in E : f(x) = g(x)\} \quad \text{and} \quad D_I = (D \cap CE) - (D \cap CE).$$

Then D_I is an ordered vector space with a generating positive cone $D \cap CE$. For each $x \in D_I$, let

$$\|x\|_1 = \inf\{\|y\| : -y \leq x \leq y,\ y \in D\},$$

where $\| \ \|$ is the Riesz norm on E. Observe that for every $x \in D_I$, $\|x\| \leq \|x\|_1$, and moreover if $x \in D \cap CE$, then $\|x\|_1 \leq \|x\|$, and hence $\|x\|_1 = \|x\|$.

Thus, by routine work, $\| \ \|_1$ is shown to be a norm on D_I. Indeed, $\| \ \|_1$ is a Riesz norm on D_I by the definition of $\| \ \|_1$ and the above observation. Furthermore, since $D \cap CE$ is closed with respect to $\| \ \|$, the positive cone $D \cap CE$ of D_I is closed with respect to the stronger topology of $\| \ \|_1$. Thus $(D_I, D \cap CE, \| \ \|_1)$ is a regular ordered normed space.

To show the completeness of $(D_I, \| \ \|_1)$, let $\{x_n\}$ be an increasing Cauchy sequence in $D \cap CE$ with respect to $\| \ \|_1$. Then $\{x_n\}$ is an increasing Cauchy sequence in $(E, \| \ \|)$ and therefore converges to an x in E. Indeed, $x \in D$, because D is closed in $(E, \| \ \|)$. Moreover, since CE is closed with respect to $\| \ \|$ and the sequence is increasing, $x = \sup_{\mathrm{N}} x_n$. Thus $x \in D \cap CE$. As a matter of fact, x is a limit of the sequence $\{x_n\}$ in $(D_I, \| \ \|_1)$, because

$$\|x - x_n\|_1 = \|x - x_n\| \quad \text{for all} \quad n \in \mathrm{N}.$$

Hence by Lemma 2.2.2 D_1 is complete with respect to $\| \|_1$.

From now on, D_1 means the regular ordered Banach space: $(D_1, D \cap CE, \| \|_1)$. Let $e: D_1 \to E$ be the canonical injection. Then e is an order-isomorphic linear map and $\|e\| = 1$. Indeed, (D_1, e) is an equalizer in *ROBan* of f and g: Obviously, $f \circ e = g \circ e$. For each map $h: H \to E$ in *ROBan* with $f \circ h = g \circ h$, there is a function $\bar{h}: H \to D_1$ defined by $\bar{h}(u) = h(u)$. By Proposition 1.4, $\bar{h}$ is a morphism in *ROBan* and the uniqueness of such $\bar{h}$ is obvious.

5.3. In general, D_1 need not coincide with D, nor need $\| \|_1$ coincide with $\| \|$.

COUNTEREXAMPLES. Let $C[0, 1]$ be the Banach lattice of real-valued continuous functions on $[0, 1]$, with respect to the *sup* norm and the pointwise order.

(1) *A case in which* $D_1 \neq D$: Consider the two real-valued functions

$$[0] \text{ (constant map) and } I(f) = \int_0^1 f(x)\,dx$$

on $C[0, 1]$. Then $[0]$ and I are positive linear maps with norm ≤ 1. Let

$$D = \{ f \in C[0, 1] : I(f) = 0 \}.$$

Then it is easy to see that $D_1 = D \cap C - D \cap C = \{0\}$.

(2) *A case in which* $\| \|_1 \neq \| \|$: Consider a real-valued function $J(f) = 2^{-1} f(0)$ on $C[0, 1]$. Then J is a positive linear map with norm ≤ 1. Let

$$D = \{ f \in C[0. 1] : I(f) = J(f) \} \text{ and } D_1 = D \cap C - D \cap C.$$

Define a function f on $[0, 1]$ by

$$f(x) = 2^{-1} \text{ on } [0, 2^{-1}] \text{ and } -2x + 3/2 \text{ on } [2^{-1}, 1].$$

Then $f \in D$. Moreover, $f \in D_1$: Consider the two functions f_1 and f_2 on $C[0, 1]$ defined by

$$f_1(x) = -2^{-1}x + 3/4 \text{ on } [0, 1/2], \quad -2x + 3/2 \text{ on } [1/2, 3/4], \text{ and } 0 \text{ on } [3/4, 1],$$

$$f_2(x) = -2^{-1}x + 1/4 \text{ on } [0, 1/2], \quad 0 \text{ on } [1/2, 3/4] \text{ and } 2x - 3/2 \text{ on } [3/4, 1].$$

Then $f_1, f_2 \in D \cap C$ and $f = f_1 - f_2$. Note that

$$\|f\|_1 = inf\{\|g+h\| : f = g-h,\ g, h \in D \cap C\}.$$

Let $f = g-h$, where $g, h \in D \cap C$. Then

$$g(0)/2 = \int_0^1 g(x)dx > 1/4 + 1/16 \quad \text{and} \quad h(0)/2 = \int_0^1 h(x)dx > 1/16$$

since

$$f \vee 0 \leq g \quad \text{and} \quad (-f) \vee 0 \leq h.$$

Therefore $g(0)+h(0) > 3/4$. Thus $\|f\|_1 \geq 3/4$, while $\|f\| = 1/2$.

COROLLARY. *The obvious forgetful functor from ROBan to Ban does not preserve equalizers and therfore does not have a left adjoint.*

5.4. THEOREM. *ROBan is complete.*

PROOF. It is immediate from Proposition 2.1 and Proposition 2.2. /

5.5. Usually, cocompleteness is obtained by showing the existence of coproducts and coequalizers. However, for the category *ROBan* it is troublesome to detect coequalizers, while easy to exhibit coproducts. Thus, instead, we use indirect categorical results to show cocompleteness.

LEMMA 1. *ROBan is well-powered.*

PROOF. For $F \in ROBan$, let $\{[(E, CE, \|\ \|_E), m]\}$ be a representative class of subobjects of F. Then there is a function

$$w: \{[(E, CE, \|\ \|_E), m]\} \to \mathcal{P}(F) \times \mathcal{P}(CF) \times \mathcal{P}(\mathrm{R}^{+F})$$

defined by

$$w([(E, CE, \|\ \|_E), m]) = (m(E), m(CE), \{f \in \mathrm{R}^{+F} : f \circ m = \|\ \|_E\})$$

where $\mathcal{P}(F)$, $\mathcal{P}(CF)$ and $\mathcal{P}(\mathrm{R}^{+F})$ are the power-sets of F, CF and the set R^{+F} of all functions from F to R^+, respectively. Indeed, w is injective (cf. Proposition 1.3). /

LEMMA 2. R *is a coseparator for the category ROBan.*

PROOF. Let $f, g: E \to F$ be distinct maps in *ROBan*. Take $x \in E$ with $f(x) \neq g(x)$. Then we have $h \in C'$ such that $h(f(x)) \neq h(g(x))$ by observing that the dual F' of F separates points of F and $F' = C' - C'$, where C' is the set of all positive linear functionals on F (cf. 6.7 [12]). Thus, we have a map $\|h\|^{-1} h : F \to \mathrm{R}$ in *ROBan* such that

$$\|h\|^{-1} h \circ f \neq \|h\|^{-1} h \circ g. \quad /$$

THEOREM. *ROBan is co(well-powered) and cocomplete.*

PROOF. Theorem 2.4 and the above Lemma 1 and Lemma 2 imply the result (cf. 23.14 [9]). /

5.6 Coproducts in *ROBan* are formed, as in *Ban*, as follows.

PROPOSITION. *For a family* $\{E_i\}_I$ *in ROBan, where* I *is an index set, the coproduct* $\coprod_I E_i$ *in ROBan is the space of all elements*

$$x = (x_i)_I, \quad x_i \in E_i \quad \textit{such that} \quad \sum_I \|x_i\| < \infty,$$

with a norm $\|(x_i)\|_1 = \sum_I \|x_i\|$ *and the pointwise order.*

PROOF. Let

$$l_1(I, \{E_i\}_I) = (\{(x_i) \in \prod_I E_i : \sum_I \|x_i\| < \infty\}, \| \|_1).$$

This is the coproduct of the spaces E_i $(i \in I)$ in *Ban*. One verifies without difficulty that $l_1(I, \{E_i\}_I)$ equipped with the pointwise order lies in *ROBan* whenever all E_i are; moreover, the canonical injections

$$e_j : E_j \to l_1(I, \{E_i\}_I) \quad (j \in I)$$

are positive. The remaining arguments are routine. /

5.7. REMARKS. Consider a new category $ROBan_\infty$ of regular ordered Banach spaces and all (bounded) positive linear maps. The relations between *ROBan* and $ROBan_\infty$ are quite analogous to those between *Ban* and

$$Ban_\infty = (\text{Banach spaces, bounded linear maps}).$$

Similarly as in *ROBan*, we can show, in the obvious way, that $ROBan_\infty$ is also a symmetric monoidal closed category. In fact, the adjunction $[E \boxtimes F, G] \approx [E, [F, G]]$ in *ROBan* implies immediately

$$[E \boxtimes F, G] \approx [E, [F, G]] \quad \text{in} \quad ROBan_\infty.$$

However, this category $ROBan_\infty$ has, like Ban_∞, rather bad properties with respect to limits and colimits. In fact, it can be shown that infinite products and infinite coproducts do not exist by the same reasoning as in Ban_∞.

We note that if the index set I is finite, then the products $\prod_I E_i$, and coproducts $\coprod_I E_i$ in $ROBan$ are also products and coproducts in $ROBan_\infty$ respectively, and are isomorphic in $ROBan_\infty$. In the category $ROBan$. $\prod_I E_i$ and $\coprod_I E_i$ are of course not isomorphic in general, because they carry different norms.

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Department of Mathematics
You Sei University
Seodaemoon-Ku
SEOUL, KOREA 120-00