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BOUNDED FUNCTORS, FINITE LIMITS AND AN APPLICATION OF INJECTIVE TOPOI

by Robert ROSEBRUGH

1. INTRODUCTION.

Let $\underline{S}$ be a topos with natural numbers object (NNO) $1 \xrightarrow{0} N \xrightarrow{s} N$. Using bounded functors to provide solutions to recursion problems and showing that inverse images of endomorphisms of internal presheaf topoi are bounded provided a proof that $\underline{S}/N$ is the natural numbers object in $\underline{\underline{DTOP}}/\underline{S}$, the 2-category of presheaf topoi over $\underline{S}$ [8]. The main object of this paper is to broaden this to $\underline{\underline{BTOP}}/\underline{S}$, the topoi bounded over $\underline{S}$, and so recover the result of Johnstone and Wraith [4]. The first step is to study the relationship between flat and left exact indexed functors on a finitely complete category object in $\underline{S}$. This is the key to showing that a topos of presheaves on a finitely complete internal category is injective. We complete the Giraud Theorem for bounded topoi [1] by showing that a bounded topos is embedded in presheaves on a *finitely complete* internal category. We combine these results with a transfer of solutions to recursion problems to complete the main result. In the remainder of this section are definitions, the transfer lemma just mentioned, and some results on comparison of bounded endofunctors.

I would like to thank Bob Paré for helpful discussions and for suggesting Lemma 2.9, and Chris Mikkelsen for asking if K is bounded.

The reader is assumed to be somewhat familiar with the Paré-Schumacher theory of indexed categories [6]. We recall a little notation which will be useful. If $\underset{\sim}{A}$ is an $\underline{S}$-indexed category, the underlying ordinary category $\underline{A}^1$ is denoted $\underline{A}$; if C is an internal category in $\underline{S}$, the $\underline{S}$-indexed externalisation of C [6, II.1.2] is denoted [C] and in particular, the dis-

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crete category on the object I in $\underline{S}$ is $[I]$. The bijective correspondence between objects of $\underline{A}^I$ and (indexed) functors $[I] \to \underset{\sim}{A}$ should be recalled as should the notion of «stable», i. e. preserved by substitution. We denote the 2-category of internal category objects in $\underline{S}$ by $\underline{\underline{cat}}(\underline{S})$.

The following definitions are from [7].

1.1. DEFINITIONS. 1. A *recursion problem in* $\underset{\sim}{A}$ is a pair (A_0, F) where A_0 is in A and $F: \underset{\sim}{A} \to \underset{\sim}{A}$, i. e.

$$[1] \xrightarrow{A_0} \underset{\sim}{A} \xrightarrow{F} \underset{\sim}{A}.$$

A *solution to* (A_0, F) is A in $\underline{A}^N$ such that

$$A_0 = o^* A \quad \text{and} \quad F^N A = s^* A, \quad \text{i. e.} \quad [N] \xrightarrow{A} \underset{\sim}{A}$$

with a diagram of $\underline{S}$-categories commuting [7].

2. An *e-functor* is a functor $E: \underset{\sim}{A} \to \underset{\sim}{S}$ with small fibres (as an $\underline{S}$-indexed functor [7, 1.3].

3. $F: \underset{\sim}{A} \to \underset{\sim}{A}$ is called *mono bounded* (resp. *epi-mono bounded*) relative to E if for each A in $\underline{A}$ there is a B in $\underline{S}$ such that

i) $E(A) \rightarrowtail B$ (resp. $E(A) \twoheadleftarrow . \rightarrowtail B$),

ii) for all A' in $\underline{A}^I$ if $E^I A' \rightarrowtail I^* B$ then $E^I F^I A' \rightarrowtail I^* B$ (resp. $E^I A' \twoheadleftarrow . \rightarrowtail I^* B \Rightarrow E^I F^I A' \twoheadleftarrow . \rightarrowtail I^* B$).

There are two results which are important for the sequel. The first is that if $F: \underset{\sim}{A} \to \underset{\sim}{A}$ is epi-mono bounded (or mono bounded) then every recursion problem (A_0, F) has an essentially unique solution [7]. The second concerns the inverse image $f^*: \underline{S}^{\mathbf{C}^{op}} \to \underline{S}^{\mathbf{C}^{op}}$ of a geometric endomorphism f (over $\underline{S}$) on an internal presheaf topos: such f^* are epi-mono bounded (and hence all recursion problems (X, f^*) have a solution [8]). Until further notice (after 2.2) $\underline{S}$ need only be assumed to be a category with finite limits and a NNO.

1.2. PROPOSITION. *Let* $F: \underset{\sim}{A} \to \underset{\sim}{A}$, $G: \underset{\sim}{B} \to \underset{\sim}{B}$ *and* $H: \underset{\sim}{A} \to \underset{\sim}{B}$ *be* $\underline{S}$-*indexed with* $HF = GH$, *and* B_0 *in* $\underline{B}$. *If there is* A_0 *in* $\underline{A}$ *such that* $HA_0 = B_0$ *and* (A_0, F) *has a solution, then* (B_0, G) *has a solution.*

PROOF. Just consider the following diagram in $\underline{S}\text{-}\underline{\underline{CAT}}$ where A in $\underline{A}^N$ is a solution to (A_0, F):

$$\begin{tikzcd}
{[1]} \arrow[r, "o"] \arrow[dr, "A_0"] \arrow[ddr, "B_0"'] & {[N]} \arrow[r, "s"] \arrow[d, "A"] & {[N]} \arrow[d, "A"] \\
 & \underset{\sim}{A} \arrow[r, "F"] \arrow[d, "H"] & \underset{\sim}{A} \arrow[d, "H"] \\
 & \underset{\sim}{B} \arrow[r, "G"] & \underset{\sim}{B}
\end{tikzcd}$$ ■

1.3. COROLLARY. *If* $F: \underset{\sim}{A} \to \underset{\sim}{A}$, $G: \underset{\sim}{B} \to \underset{\sim}{B}$ *and* $H: \underset{\sim}{A} \to \underset{\sim}{B}$ *are* S*-indexed,* $HF = GF$, F *has solutions to all recursion problems, and* H *is onto on objects, then* G *has solutions to all recursion problems.* ■

This result will be applied in 2.12. We can also say something about functors which can be compared to bounded functors.

1.4. PROPOSITION. *Let* $\underset{\sim}{A}$ *be* $\underset{\sim}{S}$*-indexed with e-functor* E *and* F, G: $\underset{\sim}{A} \to \underset{\sim}{A}$.

1. *If* F *is mono bounded and* $EG \rightarrowtail EF$, *then* G *is mono bounded.*

2. *If* F *is epi-mono bounded and* $EF \twoheadrightarrow EG$, *then* G *is epi-mono bounded.*

3. *If* $\underset{\sim}{A} = \underset{\sim}{S}$, $E = id_{\underset{\sim}{S}}$, F *is mono bounded,* G *preserves epis and* $F \twoheadrightarrow G$, *then* G *is epi-mono bounded.*

PROOF. 1 and 2 are trivial (just use the same bounding objects as provided by the hypothesis). For 3, suppose B_X is the bound for X in $\underline{S}$, then observe that if $Y \twoheadleftarrow C \rightarrowtail B_X$, then

$$GY \twoheadleftarrow GC \twoheadleftarrow FC \rightarrowtail B_X$$

so B_X epi-mono bounds GY and this can be localized. ■

There are several other results concerning comparisons of bounded functors which are also easy consequences of the definition. For example, a mono bounded functor which preserves epis is epi-mono bounded.

We use that fact in the following application of 1.4, 3, to show that the «Kuratowski-finite subobjects» functor [5], denoted $X \mapsto K(X)$, is epi-mono bounded. For a start, we recall C. J. Mikkelsen's result that K is the free functor for the theory of $\vee$-semilattices. This theory is a quotient of the theory of monoids as Johnstone observed. [2, 9.20] so letting M denote the free monoid functor we have $M \twoheadrightarrow K$. Now M is mono bounded which may be seen in the proof of Theorem 3.2 of [7]. It remains to see that K preserves epimorphisms, but M preserves these so K does as well, being a quotient of an epi-preserving functor.

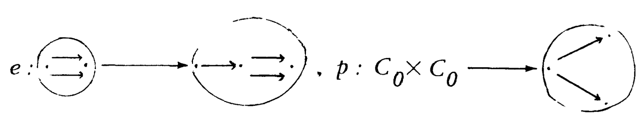
2. FINITE LIMITS AND AN APPLICATION OF INJECTIVE TOPOI.

The definition which follows includes several references to «hieroglyph» objects of diagrams with respect to an internal category

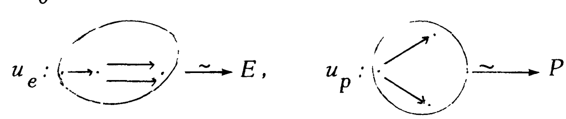
$$\mathrm{C}: \begin{tikzcd} C_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & C_1 \arrow[r, shift left=2] \arrow[r, shift right=2] & C_0 \arrow[l] \end{tikzcd}$$

in $\underset{\sim}{S}$. The objects of diagrams can all be defined as suitable inverse limits involving the morphisms defining C. This definition says that C has finite limits if it has canonical equalizers, binary products and terminal object. Its utility is Lemma 2.2.

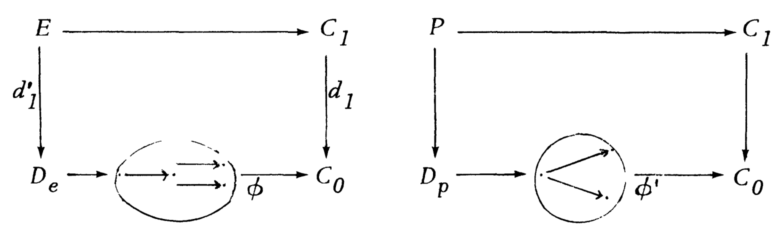
2.1. DEFINITION. C has *finite limits* iff there are morphisms



and $'1': 1 \to C_0$ and isomorphisms

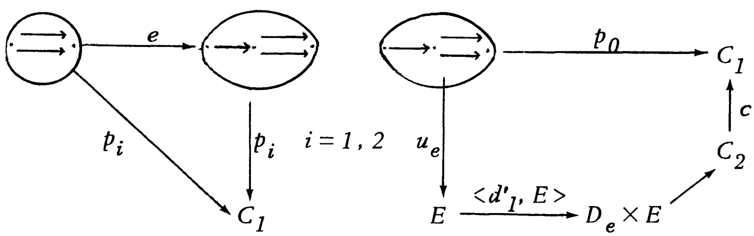


and $u_1: C_0 \to T$ where the following are pullbacks:



$$\begin{tikzcd} T \arrow[r] \arrow[d] & C_1 \arrow[d] \\ 1 \arrow[r, "'1'"] & C_0 \end{tikzcd}$$

$D_e = im(e)$, $D_p = im(p)$, ϕ projects to the middle object of the hieroglyph and ϕ' to the initial object. Moreover, several diagrams are required to commute, e. g. (for equalizers):



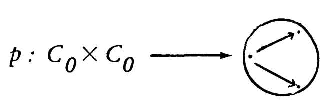
2.2. LEMMA. C *has finite limits iff* [C] *has (stable) finite limits.*

PROOF. ($\Rightarrow$) This is immediate. Indeed, for any I in $\underline{S}$ the data of 2.1 give, for example, to each pair of objects in $[C]^I$, two arrows of $[C]^I$, with common domain by composing with p. These are a product diagram. Equalizers and the terminal object are obtained from e and '1', so [C] has finite limits and these are obviously stable.

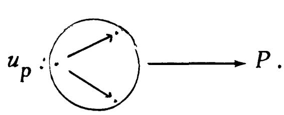
($\Leftarrow$) Let

$$C_0 \times C_0 \underset{p_2}{\overset{p_1}{\rightrightarrows}} C_0$$

be the generic pair of objects in $[C]^{C_0 \times C_0}$. These have a product diagram in $[C]^{C_0 \times C_0}$ which is the same thing as a map



The universal property of p may easily be checked to give precisely the required isomorphism



Similarly, the equalizer of the generic pair of arrows in [C]  and the terminal object in $[\mathrm{C}]^{1}$ provide the rest of the data to show that C has finite limits. ■

An indexed functor $F : \underset{\sim}{A} \to \underset{\sim}{B}$ will be said to be *left exact* if A has (stable) finite limits which are preserved by F. The full subcategory of $\underline{S}\text{-}\underline{\underline{CAT}}(\underset{\sim}{A}, \underset{\sim}{B})$ whose objects are such functors will be denoted by $Lex_{\underline{S}}(\underset{\sim}{A}, \underset{\sim}{B})$.

We assume again that $\underline{S}$ is a topos.

If C in $\underline{\underline{cat}}(\underline{S})$ has finite limits and $p : \underline{E} \to \underline{S}$ is a geometric morphism then $p^*\mathrm{C}$ in $\underline{\underline{cat}}(\underline{E})$ has finite limits, for the data of 2.1 are preserved (by any left exact functor). Recall (from e.g. [2, 4.31]) that the category $Flat(\mathrm{C}^{op}, \underline{S})$ is the category of flat presheaves on C, i.e. those whose corresponding discrete fibration is filtered (in the internal sense).

2.3. LEMMA. *If* C *in* $\underline{\underline{cat}}(\underline{S})$ *has finite colimits, then*

$$Flat(\mathrm{C}^{op}, \underline{S}) \approx Lex_{\underline{S}}([\mathrm{C}]^{op}, \underline{S}).$$

PROOF. If F is a flat internal presheaf on C with associated discrete fibration $\phi : \mathrm{F} \to \mathrm{C}$, then $-\otimes\mathrm{F} : \underset{\sim}{S}^{\mathrm{C}} \to \underset{\sim}{S}$ is a left exact $\underline{S}$-indexed functor as is the Yoneda functor $Y : [\mathrm{C}]^{op} \to \underset{\sim}{S}^{\mathrm{C}}$, but $F = (-\otimes\mathrm{F})Y$.

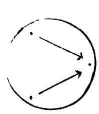
On the other hand, to $F : [\mathrm{C}]^{op} \to \underset{\sim}{S}$ we may associate a presheaf F whose family of values is

$$F^{C_0}(C_0) = f : F_0 \to C_0 \quad \text{in} \quad \underline{S}^{C_0}$$

and whose action is the transpose under the adjunction $\Sigma_{d_0} \dashv d_0^*$ of

$$F^{C_1}(C_1) : F^{C_1}(C_1) \to F^{C_1}(C_1) = F^{C_1}(d_1^* C_1) \to F^{C_1}(d_0^* C_0)$$
$$= d_1^* F^{C_0}(C_0) \to d_0^* F^{C_0}(C_0)$$

If F is left exact then the discrete fibration associated to F, denoted $\phi : \mathrm{F} \to \mathrm{C}$ has filtered domain. For example, the morphism

 $\longrightarrow F_0 \times F_0$

is split epi. Indeed, it is split by the morphism defined by the following natural transformations:

$$J \xrightarrow{\langle x, y\rangle} F_0 \times F_0 \quad \text{in } \underline{S}$$

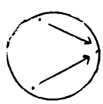
$$\begin{tikzcd} J \arrow[dr, "x'"] \arrow[drr, bend left, "x"] \arrow[ddr, "J"'] & & \\ & F^J A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{P.B.}"] & F_0 \arrow[d] \\ & J \arrow[r, "A"'] & C_0 \end{tikzcd} \qquad \begin{tikzcd} J \arrow[dr, "y'"] \arrow[drr, bend left, "y"] \arrow[ddr, "J"'] & & \\ & F^J B \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{P.B.}"] & F_0 \arrow[d] \\ & J \arrow[r] & C_0 \end{tikzcd}$$

where $A = fx,\ B = fy$

$$1 \xrightarrow{\langle x', y'\rangle} F^J A \times F^J B \quad \text{in } \underline{S}^J$$

$$\begin{tikzcd} & & F^J A \\ 1 \arrow[urr, "x'"] \arrow[r, "z"'] \arrow[drr, "y'"'] & F^J(A+B) \arrow[ur] \arrow[dr] & \\ & & F^J B \end{tikzcd}$$

since F is left exact

$J \longrightarrow$  in $\underline{S}$.

Similarly, preservation of equalizers and 1 provide splittings making the other required morphisms epic. ■

2.4. LEMMA. *If $p: \underline{E} \to \underline{S}$ is a geometric morphism and* C *in* $\underline{\underline{cat}}(\underline{S})$ *has finite limits, then* $Lex_{\underline{E}}([p^*\mathrm{C}], \underset{\sim}{E}) \approx Lex_{\underline{S}}([\mathrm{C}], \underset{\sim}{E})$.

PROOF. For any $\underline{E}$-indexed $F: [p^*\mathrm{C}] \to \underset{\sim}{E}$, define an $\underline{S}$-indexed functor

$$\bar{F}: [\mathrm{C}] \to \underset{\sim}{E} \quad \text{by} \quad \bar{F}^I(A) = F^{p^*I}(A)$$

for all I in $\underline{S}$ and A in $[\mathrm{C}]^I$. For an $\underline{S}$-indexed $G: [\mathrm{C}] \to \underset{\sim}{E}$ define

$$\bar{G}: [p^*\mathrm{C}] \to \underset{\sim}{E} \quad \text{by} \quad \bar{G}^J(A) = A^*\gamma$$

for all J in $\underline{E}$ and A in $[p^*\mathrm{C}]^J$ where $G^{C_0}(C_0) = \gamma : G_0 \to p^*C_0$ and the following is a pullback:

$$\begin{tikzcd} B \arrow[r] \arrow[d, "A^*\gamma"'] & G_0 \arrow[d, "\gamma"] \\ J \arrow[r, "A"] & p^*C_0 \end{tikzcd}$$

Extending these definitions to morphisms and verifying that they provide the required equivalences is routine. ■

Combining 2.3 and 2.4 gives immediately:

2.5. PROPOSITION. *If* $p: \underline{E} \to \underline{S}$ *is a geometric morphism and* C *in* $\underline{\underline{cat}}(\underline{S})$ *has finite limits, then*

$$Flat(p^*C^{op}, \underline{E}) \approx Lex_{\underline{S}}([C]^{op}, \underline{E}). \quad ■$$

The next several results serve to confirm P. Johnstone's suggestion that his characterization of bounded topoi over *set* which are injective with respect to inclusions remains valid over an arbitrary base [3]. Both 2.5 and 2.10 are essential. We recall, following Johnstone, the appropriate notion of injectivity. For a 1-arrow $f: \underline{A} \to \underline{B}$ in a 2-category $\underline{\underline{K}}$, denote

$$f^+: \underline{\underline{K}}(\underline{B}, \underline{E}) \to \underline{\underline{K}}(\underline{A}, \underline{E}) \quad \text{and} \quad f_+: \underline{\underline{K}}(\underline{E}, \underline{A}) \to \underline{\underline{K}}(\underline{E}, \underline{B})$$

the functors defined by composition. For a class $\underline{\underline{M}}$ of 1-arrows in $\underline{\underline{K}}$ an object $\underline{E}$ *is strongly injective* if for any $f: \underline{A} \to \underline{B}$ in $\underline{\underline{M}}$ there is a functor $\lambda_f: \underline{\underline{K}}(\underline{A}, \underline{E}) \to \underline{\underline{K}}(\underline{B}, \underline{E})$ and a natural isomorphism $a_f: f^+\lambda_f \to 1_{\underline{\underline{K}}(\underline{A}, \underline{E})}$. $\underline{E}$ in $\underline{\underline{TOP}}/\underline{S}$ is strongly injective if it is so, in this sense, with respect to inclusions.

2.6. PROPOSITION. *Let* C *be in* $\underline{\underline{cat}}(\underline{S})$ *and have finite limits, then for every geometric morphism* $f: \underline{F} \to \underline{E}$ *over* $\underline{S}$, *the functor* f^+ *has a right adjoint* λ_f *and* $\underline{S}^{C^{op}}$ *is strongly injective.*

PROOF. Notice that by Diaconescu's Theorem [1] and 2.5 we have

$$\underline{\underline{TOP}}/\underline{S}(F, \underline{S}^{C^{op}}) \approx Flat([p^*C], \underline{F}) \approx Lex_{\underline{S}}([C], \underline{F})$$

and then follow Johnstone's proof over *set* [3, 1.2]. ■

Our next goal is to show that a bounded $\underline{S}$-topos $p: \underline{E} \to \underline{S}$ may be included in presheaves on a category in $\underline{S}$ which may be taken to have finite limits. Now $\underline{E}$ has an object D of generators over $\underline{S}$, i.e. for any X in $\underline{E}$, the composite

$$p^*p_*(\tilde{X}^D) \times D \xrightarrow{\epsilon \times D} \tilde{X}^D \times D \xrightarrow{w} \tilde{X}$$

is epi, where ϵ is the counit of $p^* \dashv p_*$.

2.7. LEMMA. *If D is an object of generators for $\underline{E}$ over $\underline{S}$ and $j: D \rightarrowtail D'$ is monic, then so is D'.*

PROOF. Let X be in $\underline{E}$. $\tilde{X}$ is injective so there is a $\bar{j}: \tilde{X}^D \rightarrowtail \tilde{X}^{D'}$. Now consider

$$\begin{tikzcd} p^*p_*(\tilde{X}^D)\times D \arrow[r, "\epsilon\times D"] \arrow[d, "p^*p_*(\bar{j}\times j)"'] & \tilde{X}^D\times D \arrow[d, "\bar{j}\times j"'] \arrow[dr, "ev_D"] & \\ & & \tilde{X} \\ p^*p_*(\tilde{X}^{D'})\times D' \arrow[r, "\epsilon\times D'"] & \tilde{X}^{D'}\times D' \arrow[ur, "ev_{D'}"'] & \end{tikzcd}$$

The square commutes by naturality. Both

$$ev_{D'}(\bar{j}\times j)(\tilde{X}^j\times D) = ev_{D'}(\tilde{X}^{D'}\times j)$$

and $ev_D(\tilde{X}^j\times D)$ are transpose of $\tilde{X}^j$, so they are equal and cancelling $\tilde{X}^j\times D$ shows that the triangle commutes. But $ev_D(\epsilon\times D)$ is epi and hence so is $ev_{D'}(\epsilon\times D')$. ■

2.8. LEMMA. *If* D *in* $\underline{\underline{cat}}(\underline{E})$ *has finite limits then* p_*D *in* $\underline{\underline{cat}}(\underline{S})$ *has finite limits.*

PROOF. By 2.2 it is enough to show that $[p_*\mathrm{D}]$ has stable ($\underline{S}$-indexed) finite limits. We construct e. g. stable finite products in $[p_*\mathrm{D}]$. Let I be in $\underline{S}$ and $A, B: I \to p_*D_0$ be objects in $[p_*\mathrm{D}]^I$. Let $\bar{A}, \bar{B}: p^*I \to D_0$ correspond to A, B by adjointness, and $\bar{A}\times\bar{B}: p^*I \to D_0$ be their product in $[\mathrm{D}]^{p^*I}$. Now $\eta_I^* p_*(\bar{A}\times\bar{B})$ is an object of $[p^*\mathrm{D}]^I$ where $\eta_I: I \to p_*p^*I$ is the front adjunction, and it is easily shown to be a product of A and B. Stability follows by naturality of η. ■

2.9. LEMMA. *Let $K = L^N$ and let $S \rightarrowtail (\Omega^K)^*K$ be the generic subobject of K (in $\underline{S}/\Omega^K$). $\underset{\sim}{Full}(S)$, the full subcategory of $\underset{\sim}{S}$ determined by S, has finite limits.*

PROOF. We show that $\underset{\sim}{Full}(S)$ has binary products. Let p denote the isomorphism

$$K^2 \xrightarrow{\sim} L^{N\times 2} \xrightarrow{\sim} L^N = K.$$

We work at 1 for simplicity and let A, B be in $\underline{Full}(S)$, i.e.

$$A,\ B: 1 \to \Omega^K,\ \text{so}\ \ A^*S \rightarrowtail K\ \text{and}\ \ B^*S \rightarrowtail K.$$

Now $A^*(S)\times B^*(S)$ is an object of $\underline{S}$ with

$$A^*S\times B^*S \rightarrowtail K\times K \approx K^2 \xrightarrow{\sim p} K$$

giving $A\times B: 1 \to \Omega^K$ corresponding to the monic. Notice that $A^*S\times B^*S$ comes equipped with projections which give morphisms in $\underset{\sim}{Full}(S)$ making $A\times B$ the product of A and B.

$\underset{\sim}{Full}(S)$ also has equalizers and 1. Indeed, it is closed under subobjects in $\underline{S}$. ■

$\underset{\sim}{Full}(S)$ is small and $\underset{\sim}{Full}(S) \approx [\,Full(S)]$ where the latter is the internal full subcategory, so $Full(S)$ also has finite limits by 2.5.

2.10. THEOREM. *Let $p: \underline{E} \to \underline{S}$ be a bounded geometric morphism, then there is* C *in* $\underline{\underline{cat}}(\underline{S})$ *with finite limits such that* $\underline{E}$ *is a subtopos of* $\underline{S}^{C^{op}}$.

PROOF. It is well known that $\underline{E}$ is a subtopos of $\underline{S}^{C^{op}}$ if $C = p_*\mathrm{D}$ and $\mathrm{D}= Full(S)$ where S is the generic family of subobjects of an object of generators for $\underline{E}$ over $\underline{S}$ [2, 4.46]. Let D be an object of generators for E. By 2.7, $G = D^N$ is also an object of generators since $D \rightarrowtail D^N$. Let S be the generic family of subobjects of G. If $\mathrm{D} = Full(S)$ then D has finite limits by 2.9, hence so does $\mathrm{C} = p_*\mathrm{D}$ by 2.8. ■

2.11. COROLLARY. *$\underline{\underline{BTOP}}/\underline{S}$ has enough strong injectives.* ■

For the record, this means that Johnstone's characterization of injectives in $\underline{\underline{BTOP}}/Set$ [3, 1.4] extends to $\underline{\underline{BTOP}}/\underline{S}$, i.e. $\underline{E}$ is weakly injective (i.e. injective for inclusions in the category $\underline{BTOP}/\underline{S}$) iff $\underline{E}$ is a retract of $\underline{S}^{C^{op}}$ for some C with finite limits.

Returning finally to recursion problems, we obtain immediately:

2.12. PROPOSITION. *Let $f: \underline{E} \to \underline{E}$ be a morphism in $\underline{\underline{BTOP}}/\underline{S}$ and X an object of $\underline{E}$, then (X, f^*) has a solution.*

PROOF. By 2.11, $\underline{E}$ is a subtopos via $i: \underline{E} \underset{\longleftarrow}{\rightarrowtail} \underline{S}^{C^{op}} : a$, say, of an injective presheaf topos, so f has an extension to a geometric endomorphism $\hat{f}$ of $\underline{S}^{\mathbf{C}^{op}}$. Now $\hat{f}^*$ is epi-mono bounded [8, 2.3] and $a\hat{f}^* = f^*a$, so by 1.3 we are done. ■

2.13. COROLLARY. *If $f: \underline{E} \to \underline{E}$ is a morphism of $\underline{\underline{BTOP}}/\underline{S}$ then there is an indexed functor $\phi^*: \underset{\sim}{E} \to \underset{\sim}{E}^N$ such that ϕ^*X is a solution to (X, f^*) for X in $\underline{E}$.*

PROOF. To define ϕ^* on morphisms use 2.12 applied to $\underline{E}^2$ and uniqueness of solutions for functoriality. ■

We are now in a position to prove

2.14. THEOREM [4]. *$\underline{S}^N$ is the natural numbers object in $\underline{\underline{BTOP}}/\underline{S}$.*

PROOF. Let $x: \underline{S} \to \underline{E}$ and $f: \underline{E} \to \underline{E}$ be in $\underline{\underline{BTOP}}/\underline{S}$. By 2.13 there is an indexed functor $\phi^*: \underset{\sim}{E} \to \underset{\sim}{E}^N$ giving solutions to recursion problems, so by [8, 3.4] this is the inverse image of a geometric morphism $\underline{E}^N \to \underline{E}$ over $\underline{S}$. Moreover, the following commutes:

$$\begin{tikzcd}
S \arrow[r, "o"] \arrow[ddr, "x"'] & \underset{\sim}{S}^N \arrow[r, "s"] \arrow[d, "x^N"] & \underset{\sim}{S}^N \arrow[d, "x^N"] \\
& \underset{\sim}{E}^N \arrow[r, "s"] \arrow[d, "\phi"] & \underset{\sim}{E}^N \arrow[d, "\phi"] \\
& \underset{\sim}{E} \arrow[r, "f"] & \underset{\sim}{E}
\end{tikzcd}$$

Further, the composite $\phi\, x^N$ is unique in making both the triangle and the outside square commute. ■

REMARK. It should be pointed out that the methods developed in [8] and here show that $\underline{S}/N$ is the NNO in any subcategory of $\underline{\underline{TOP}}/\underline{S}$ for which all recursion problems posed by inverse images have solutions. The impact of 2.14 is to verify that $\underline{\underline{BTOP}}/\underline{S}$ is such a subcategory.

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