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IN WHICH CATEGORIES ARE FIRST-ORDER AXIOMATIZABLE HULLS CHARACTERIZABLE BY ULTRAPRODUCTS?

by Bui Huy HIEN and I. SAIN

In Andreka-Nemeti [1] the class $STr(C)$ of all small trees over C is defined for an arbitrary category C. Throughout the present paper C denotes an arbitrary category. In Definition 4 of [1] on page 367 the injectivity relation

$$\models\ \subseteq (Ob\, C) \times (STr(C))$$

is defined. Intuitively, the members of $STr(C)$ represent the formulas, and $\models$ represents the validity relation between objects of C considered as models and small trees of C considered as formulas. If $\phi \in STr(C)$ and $a \in Ob\, C$ then the statement $a \models \phi$ is associated to the model theoretic statement «the formula ϕ is valid in the model a». It is proved there that the Łos Lemma is true in every category C if we use the above quoted concepts. To this the notion of an ultraproduct $P_{i \in I} a_i / U$ of objects $\langle a_i \mid i \in I \rangle \in {}^I Ob\, C$ of C is defined in [1], in [2] and in [7] Definition 12. Then the problem was asked in [1] («Open Problem 1» on page 375) «for which categories is the characterization theorem of axiomatizable hulls of classes of models $Mod\, Th\, K = Uf\, Up\, K$ true?», where the operators Uf and Up on classes of models is defined on page 319 of the book [3], but here we recall them in Definition 6 of the present paper. Of course, here in the definition of Uf and Up on classes $K \subset Ob\, C$ of objects of C we have to replace the standard notion of ultraproducts of models by the above quoted category theoretic ultraproduct $P_{i \in I} a_i / U$ of objects of C, see Definitions 4 and 6 in the present paper.

For the definitions of the class $STr(C)$ and the injectivity relation $\models$ the reader is referred to [1]. We note that the relation $\models$ is defined between objects of C and elements of $STr(C)$.

DEFINITION 1. Let C be an arbitrary category and let $K \subset Ob\,C$ and $T \subset STr(C)$ be arbitrary classes. Let $a \in Ob\,C$ and $\phi \in STr(C)$. Then we define :

(i) $K \models T$ iff $(\forall b \in K)(\forall \psi \in T)\ b \models \psi$.

(ii) $K \models \phi$ iff $K \models \{\phi\}$, and $a \models T$ iff $\{a\} \models T$.

(iii) $Mod\,T \overset{d}{=} \{b \in Ob\,C \mid b \models T\}$.

(iv) $Th\,K \overset{d}{=} \{\psi \in STr(C) \mid K \models \psi\}$.

(v) $a \equiv_{ee} b$ iff $Th\{a\} = Th\{b\}$.

(vi) $Ee\,K \overset{d}{=} \{b \in Ob\,C \mid (\exists a \in K)\ b \equiv_{ee} a\}$.

In the present paper we characterize those categories in which $Mod\,Th\,K = Ee\,Up\,K$ holds for all $K \subset Ob\,C$.

Note that the above introduced notations $Mod\,T$ and $Th\,K$ are sloppy since the precise notation would be $Mod_C\,T$ and $Th_C\,K$ since e. g. $Mod_C\,T$ is a function of both C *and* T. We hope that context will help.

Strongly small objects of C were defined in [1], [7] Definition 13 and [2]. We shall use this notion. We note that in the textbook [4] in item 22E there on page 155 strongly small objects were defined under the name *strongly finitary objects.*

Let $(I, \leq)$ be an arbitrary preordered set, i. e. a small category in which there are no parallel arrows. Diagrams indexed by $(I, \leq)$ will be denoted by

$$\langle a_i \xrightarrow{b_j^i} a_j \mid i, j \in I,\ i \leq j \rangle \text{ or shortly } \langle b_j^i \mid i \leq j \in I \rangle .$$

I. e., let $F : (I, \leq) \to C$ be a functor. Now,

$$F \overset{d}{=} \langle F(i) \xrightarrow{F(i,j)} F(j) \mid i, j \in I,\ i \leq j \rangle .$$

The colimit of this diagram F is denoted by $\langle b^i : F(i) \to b \rangle_{i \in I}$, where $\langle F(i) \xrightarrow{b^i} \rangle_{i \in I}$ is the cocone part and b the object part of the colimit.

DEFINITION 2 (Nemeti-Sain [7], page 556). An object *a is strongly small* (for short s. small) if the functor $Hom(a, \cdot)$ is continuous (i. e. preserves direct limits).

NOTATION. s. small objects will be denoted by $\otimes$-s. $\otimes \xrightarrow{f}$ means that

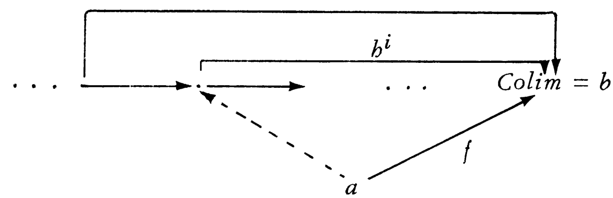
$dom(f)$ is s. small and we use $\xrightarrow{f} \otimes$ similarly.

REMARK. From the above definition it follows that the object a is s. small iff for any directed diagram $<h_j^i \mid i \leq j \in I>$ with colimiting cocone

$$c \stackrel{d}{=} << h^i >_{i \in I}, b>,$$

conditions (i) and (ii) below are satisfied:

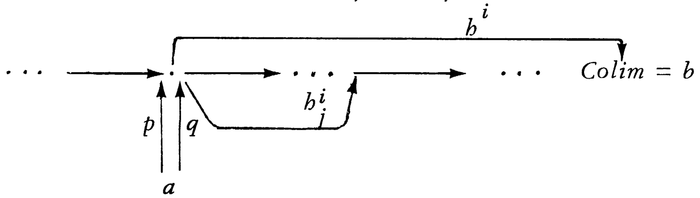
(i) Every morphism $f: a \to b$ cofactors through the cocone c.



(ii) To any pair

$$a \underset{q}{\overset{p}{\rightrightarrows}} \quad \text{such that } h^i . p = h^i . q \text{ for some } i \in I,$$

there exists a $j \in I$ such that $h_j^i . p = h_j^i . q$.

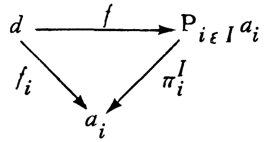


We note that limits and colimits are always small in this paper. E. g., $Hom(\otimes, -)$ does not necessarily preserve large direct limits.

An object is called *small* if it satisfies (i) of the above remark.

DEFINITION 3. Let C be an arbitrary category. We say that C has only *set-many nonisomorphic strongly small objects* iff there is a *set* $B \subset Ob\,C$ such that every strongly small object of C is isomorphic to some element of B.

NOTATIONS connected to products: The product $P_{i \in I} a_i$ of a family of objects $<a_i>_{i \in I}$ will also be (ambiguously) denoted by P_I. We use the notation π_i^I for the i-th member of the cone of projections belonging to the product P_I. I. e., the «product cone» is $<P_I, <\pi_i^I>_{i \in I}>$. By the definition of a product, a cone $\langle f_i : d \to a_i \rangle_{i \in I}$ induces a unique morphism $f : d \to \mathrm{P}_I$, such that the diagrams



commute for each $i \in I$ (provided that the product exists). We shall denote this induced morphism $f : d \to \mathrm{P}_I$ by $\dot{\Pi}_{i \in I} f_i$. Sometimes, though, it is better to write $\dot{\Pi}\, c = \dot{\Pi} \langle d, \langle f_i \rangle_{i \in I} \rangle$. E. g. $\dot{\Pi} \langle d, \emptyset \rangle$ is the unique element of $Hom(d, e)$ where e is the terminal object $\mathrm{P}_{i \in \emptyset} a_i$.

DEFINITION 4 ([1, 2, 7, 8]). Let $\langle a_i \rangle_{i \in I}$ be a family of objects. Let U be a set of subsets of I (i.e., $U \subset SbI$ is arbitrary). Now, consider all the products P_X ($\overset{d}{=} \mathrm{P}_{i \in X} a_i$) for the sets $X \in U$. If $X, Y \in U$ and $Y \supset X$ then the morphism induced by the cone of projections of P_Y into the product P_X is denoted by π_X^Y. I. e. $\pi_X^Y \overset{d}{=} \dot{\Pi}_{i \in X} \pi_i^Y$. By this we have defined a diagram of «products and projections»

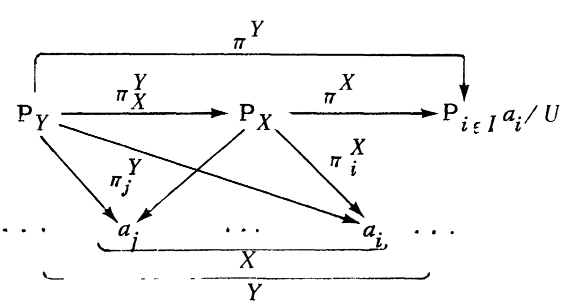
$$\langle \pi_X^Y : \mathrm{P}_Y \to \mathrm{P}_X \mid X, Y \in U, Y \supset X \rangle.$$

Note that this diagram is indexed by the poset $(U, \supset)$. (This poset consists of U ordered by the inverse $\supset$ of the inclusion relation $\subset$.) The *colimit* of the above diagram is denoted by

$$\langle \pi^Y : \mathrm{P}_Y \to (\mathrm{P}_{i \in I} a_i / U) \rangle_{Y \in U}.$$

If U is a *filter*, then $\mathrm{P}_{i \in I} a_i / U$ is called a *reduced product of* $\langle a_i \rangle_{i \in I}$. If U is an *ultrafilter*, then $\mathrm{P}_{i \in I} a_i / U$ is called an *ultraproduct*.

The next figure illustrates the definition.



DEFINITION 5. We say that *ultraproducts exist in* C iff for every *set* I and for all $a \in {}^I(Ob\,C)$ and for every ultrafilter U on I the ultraproduct $P_{i \in I} a_i / U$ exists in C.

DEFINITION 6. Let $K \subset Ob\,C$ be arbitrary. Then

(i) $Up\,K \overset{d}{=} \{ P_{i\in I} a_i/U \mid I$ is a set, $\{a_i\}_{i \in I} \subset K$, U is an ultrafilter on I and the ultraproduct $P_{i\in I} a_i/U$ exists in $C \}$.

(ii) $Uf K \overset{d}{=} \{ b \in Ob\,C \mid Up\{b\} \cap K \neq 0 \}$.

THEOREM 1. *Let C be an arbitrary category. Assume that conditions (i)-(iii) below hold. Let $K \subset Ob\,C$ be an arbitrary class. Then*

$$Mod\,Th\,K = Ee\;Up\,K. \quad (\textit{That is: } \text{«}Mod\,Th\text{»} = \text{«}Ee\,Up\text{» on } C.)$$

(i) C has only set-many nonisomorphic strongly small objects.

(ii) Ultraproducts exist in C (the small ones only, Definition 3).

(iii) C has an initial object.

PROOF is that of Theorem 1 in [5]. □

Theorems 2, 3 below state that both conditions (i) and (ii) are needed in Theorem 1 above.

THEOREM 2 *(necessity of (i) in Theorem 1). There exists a category C in which all ultraproducts exist and C has an initial object, but*

$$Mod\,Th\,K \neq Ee\;Up\,K \quad \textit{for some } K \subset Ob\,C.$$

That is, C satisfies (ii) and (iii) of Theorem 1 but not its conclusion.

PROOF. Let $\infty \overset{d}{=} \{1\}$. Let Ord be the class of all ordinals. Then we have $\infty \notin Ord$. Let $Ord+1 \overset{d}{=} Ord \cup \{\infty\}$. Let $\leq\, \subset {}^2(Ord+1)$ be defined by

$$\leq \overset{d}{=} \{<\beta,\infty> \mid \beta \in Ord+1\} \cup \{<\beta,\alpha> \mid \alpha \in Ord \text{ and } (\beta \in \alpha \text{ or } \beta = \alpha)\}.$$

Then $P \overset{d}{=} <Ord+1, \leq>$ is an ordered class. Hence P may be considered as a category with $Ob\,P = Ord+1$.

FACT 1. The s. small objects of P are exactly the successor ordinals and 0. Hence there is a proper class of *nonisomorphic* s. small objects. The *initial* object of P is 0.

LEMMA 2. *Let $\phi \in STr(P)$. Then* $[\,Ord \models \phi \;\Rightarrow\; \infty \models \phi\,]$.

PROOF of Lemma 2. Assume $Ord \models \phi$. By $\phi \in STr(P)$, all objects occurring in ϕ are s. small, hence ∞ does not occur in ϕ. Since only set-many objects can occur in ϕ we conclude that

$$(\exists \kappa \in Ord)(\text{for every object } a \text{ occurring in } \phi \text{ we have } a < \kappa).$$

Then ϕ is related to κ exactly the same way as it is related to ∞. Hence $\kappa \models \phi$ implies $\infty \models \phi$. But $Ord \models \phi$ implies $\kappa \models \phi$.

COROLLARY 3. $\infty \in Mod\, Th(Ord)$.

LEMMA 4. *Let* $\alpha, \beta \in Ord+1$. *Then* $Th(\alpha) = Th(\beta)$ *iff* $\alpha = \beta$.

PROOF of Lemma 4. Assume $\alpha \neq \beta$. Then $\alpha < \beta$ or $\beta < \alpha$, assume $\alpha < \beta$. Clearly $<\alpha+1, \emptyset> \in STr(P)$ since $\alpha + 1$ is s. small. Then

$$\alpha \not\models <\alpha+1, \emptyset> \quad \text{while} \quad \beta \models <\alpha+1, \emptyset>$$

since $Hom(\alpha+1, \beta) \neq 0$ by $\alpha + 1 \leq \beta$.

Clearly, all *reduced products exist in* P since suprema and infima of sub*sets* of $(Ord+1)$ do exist in $(Ord+1, \leq)$. Obviously, $Up\, Ord = Ord$, in P since by ultraproducts we understand ultraproducts of *sets* of objects only. Hence by Lemma 4 we have $Ee\, Up\, Ord = Ord$ in P. Thus

$$Mod\, Th\, Ord = Ord + 1 \neq Ord = Ee\; Up\, Ord$$

is proved to hold in P. QED of Theorem 2.

THEOREM 3. *There is a category* C *and a class* K *of objects of* C *such that (i) and (iii) of Theorem 1 hold as well as (I) and (II) below:*

(I) $Mod\, Th\, K \underset{\neq}{\supset} Ee\, Up\, K$.

(II) Let Up^w *denote the formation of weak ultraproducts which were introduced in [11] under the name «universal ultraproducts». Then*

$$Mod\, Th\, K \underset{\neq}{\supset} Ee\, Up^w K.$$

PROOF. Let C be the subcategory of *Sets* (category of sets and maps) such that $Ob\, C = Ob\, Sets$ and

$$Mor\, C = \{ f \in Mor(Sets) \mid (|dom\, f| \geq \omega \text{ and } |cod\, f| \geq \omega) \text{ or } dom\, f = 0 \text{ or } f = 1_A \text{ for some set } A \}.$$

Let $K = \{ A \in Ob\, C \mid |A| < \omega \}$. We claim that

$$Up^{w} K = K \quad \text{and} \quad Ee\, K = K, \quad \text{hence} \quad Ee\, Up^{w} K = K.$$

But an object A is s. small in C iff $|A| < \omega$. Since the formula $\langle A, O \rangle$ is not valid in K and since there are no other non-trivial formulas, we have $Mod\, Th\, K = Ob\, C$. Obviously (i) and (ii) of Theorem 1 hold in C.

If C is an arbitrary category and $K \subset Ob\, C$, then

$$Mod\, Th\, K \supset Ee\, Up\, K \supset Uf\, Up\, K \quad \text{(by [1])}. \quad \square$$

PROPOSITION 4. *The conditions of Theorem 1 are not the best possible, namely: There exists a category C such that all three conditions (i), (ii) and (iii) of Theorem 1 fail but the conclusion of Theorem 1 is true.*

PROOF. Let C be a large discrete category. That is $Ob\, C$ is a proper class (not a set) and the only morphisms are identities. Then every object of C is s. small. Thus there is a proper class of nonisomorphic s. small objects. Further ultraproducts do not exist in C, since there are no non-identity morphisms. Let $K \subset Ob\, C$. We claim that $Mod\, Th\, K = K$. Let $a \in Ob\, C$. Assume $a \notin K$. Then $\langle a, 0 \rangle \in S\, Tr(C)$, namely $\langle a, 0 \rangle$ is the one-element tree with root a and no branches. Clearly

$$a \not\models \langle a, 0 \rangle \quad \text{and} \quad (\forall b \in Ob\, C)(b \neq a \Rightarrow b \models \langle a, 0 \rangle).$$

Thus $K \models \langle a, 0 \rangle$ proving that $a \notin Mod\, Th\, K$. $\square$

PROBLEMS. (i) Improve Theorem 1. Find a sharper characterization of those categories in which $Mod\, Th = Ee\, Up$.

(ii) Under what conditions is $Mod\, Th = Uf\, Up$ true?

(iii) Is there a category C satisfying (i) and (ii) of Theorem 1 in which $Mod\, Th\, K \neq Ee\, Up\, K$ for some $K \subset Ob\, C$? This is solved by I. Sain affirmatively, see [5] Theorem 2.

For the category Lf_{α} of locally finite cylindric algebras, see [6] or in the textbook on representable cylindric algebras [3] page 321. The following is a corollary of results in [6] and Theorem 1 above. For a motivation we note that Lf_{α} is exactly the class of algebras obtained from classical first-order theories, as it was proved in Proposition 1 of [6].

COROLLARY 5. *Let α be any ordinal and $K \subset Lf_\alpha$ be arbitrary. Then in the category Lf_α we have $Mod\,Th\,K = Ee\,Up\,K$.*

PROBLEM. Is $Mod\,Th\,K = Uf\,Up\,K$ true in Lf_α ?

For a comprehensive study of our subject see [9]. The fact that $STr(C)$ corresponds exactly to the class of first-order formulas is proved in [10].

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