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A NOTE ON THE GENERALIZED REFLEXION OF GUITART AND LAIR

by G. M. KELLY *

By a *weak reflexion* of a locally-small category $\mathcal{A}$ onto a full subcategory $\mathcal{B}$ we mean the assigning to each $A \in \mathcal{A}$ of a *small* projective cone π_A, with vertex A and with base in $\mathcal{B}$, such that $\mathcal{A}(\pi_A, B)$ is a colimit-cone in **Set** for each $A \in \mathcal{A}$ and each $B \in \mathcal{B}$. When each π_A has its base indexed by a discrete category, π is a *multi-reflexion* in the sense of Diers [1]; it is an actual reflexion if moreover each of these discrete categories is 1.

For example, let $\mathcal{A}$ be the category of commutative rings. When $\mathcal{B}$ consists of local rings, a weak reflexion is given by taking for π_A the cone of localizations $A \to A_{\mathbf{p}}$ of A; its base is indexed by the ordered set of prime ideals $\mathbf{p}$ of A. When $\mathcal{B}$ consists of the fields, a multi-reflexion is given by the discrete cone $A \to A/\mathbf{m}$ where $\mathbf{m}$ runs through the maximal ideals of A. When $\mathcal{B}$ consists of the rings A with $2A = 0$, an actual reflexion is given by $A \to A/2A$.

Guitart and Lair study in [4] the existence of weak reflexions when $\mathcal{B}$ is given as follows. *We have a set* $\Theta = \{\theta_\beta\}$ *of projective cones*

$$\theta_\beta : \Delta N_\beta \to T_\beta : \mathcal{L}_\beta \to \mathcal{A}$$

in $\mathcal{A}$, *where* ΔN_β *denotes the functor constant at* N_β; *and* $\mathcal{B}$ *consists of those* $A \in \mathcal{A}$ *for which each* $\mathcal{A}(\theta_\beta, A)$ *is a colimit-cone in* **Set**. They further restrict themselves to the special case in which *each generator of each cone* θ_β *is an epimorphism in* $\mathcal{A}$.

Each of the examples above is of this kind. For local rings there are two cones θ_1 and θ_2 in $\mathcal{A}$; θ_1 is the pushout diagram of the two (epimorphic) maps

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$$\mathbf{Z}[x,y]/(xy-1) \leftarrow \mathbf{Z}[x] \rightarrow \mathbf{Z}[x,y]/((1-x)y-1),$$

while θ_2 is the cone of vertex 0 over the empty diagram. For fields there are again two cones: θ_2 as above and θ_3 the discrete cone

$$\mathbf{Z} \leftarrow \mathbf{Z}[x] \rightarrow \mathbf{Z}(x).$$

For rings with $2A = 0$, there is a single cone θ_4 whose base is indexed by 1, namely $\mathbf{Z} \rightarrow \mathbf{Z}/2\mathbf{Z}$.

We suppose henceforth that $\mathfrak{B}$ is given as above. We recall that, for a regular cardinal α, an object $A \in \mathfrak{A}$ is called *α-presentable* if $\mathfrak{A}(A,-): \mathfrak{A} \rightarrow \mathbf{Set}$ preserves α-filtered colimits. Guitart and Lair sketch in [4] a rather complicated proof by transfinite induction of the following: *There is a weak reflexion π of $\mathfrak{A}$ onto $\mathfrak{B}$ if $\mathfrak{A}$ is cocomplete, if each $\mathfrak{L}_\beta$ is small, and if there is a regular cardinal α such that each N_β and each $T_\beta L$ is α-presentable. Moreover π can be taken to be a multi-reflexion if each $\mathfrak{L}_\beta$ is discrete.*

The α-presentability hypothesis is a strong one; hardly any objects are α-presentable in the category of topological spaces or in the dual of an algebraic category. By analogy with the case where each $\mathfrak{L}_\beta$ is 1 - the «orthogonal subcategory problem» of [2] - this hypothesis should not be needed when the generators of the cones θ_β are epimorphic: at least if $\mathfrak{A}$ is cowellpowered, which is not a grave restriction. By the same analogy, there should be a simple and direct proof in this case. We now verify that this is so, and that moreover the base of each cone π_A may then be taken to be an ordered set.

We refer to [5] for the notion of *strong monomorphism*, and for the fact that epimorphisms and strong monomorphisms constitute a factorization system (see [2]) on $\mathfrak{A}$ if $\mathfrak{A}$ admits finite limits and all intersections of strong monomorphisms, or if $\mathfrak{A}$ admits finite colimits and all cointersections of epimorphisms; certainly, therefore, if $\mathfrak{A}$ is complete and wellpowered, or cocomplete and cowellpowered.

THEOREM 1. *Let the full subcategory $\mathfrak{B}$ of the locally-small category $\mathfrak{A}$*

be determined as above by a set Θ *(not necessarily small) of cones* θ_β *(not necessarily small), where each generator of each* θ_β *is epimorphic in* $\mathcal{A}$. *Let epimorphisms and strong monomorphisms constitute a factorization system on* $\mathcal{A}$, *and let* $\mathcal{A}$ *be cowellpowered.*

For each $A \in \mathcal{A}$ *denote by* S_A *the small category whose objects are (a set of representatives of) the epimorphisms* $p: A \to C$ *in* $\mathcal{A}$ *with domain* A *and codomain in* $\mathcal{B}$, *and whose maps* $p \to p'$ *are the maps* $q: C \to C'$ *with* $qp = p'$ *; clearly* S_A *is an ordered set. Let* $d_A: S_A \to \mathcal{B} \subset \mathcal{A}$ *be the projection functor sending* $p: A \to C$ *to* C, *and let*

$$\pi_A: \Delta A \to d_A: S_A \to \mathcal{A}$$

be the cone whose p*-th component is* p *itself.*

Then an object B *of* $\mathcal{A}$ *lies in* $\mathcal{B}$ *if and only if each* $\mathcal{A}(\pi_A, B)$ *is a colimit-cone in* **Set**.

PROOF. The essential observation is that $\mathcal{B}$ is closed in $\mathcal{A}$ under strong subobjects. To see this it suffices to consider a single cone $\theta: \Delta N \to T$ of Θ, with epimorphic generators $\theta_i: N \to T_i$. Let $j: D \to B$ be a strong monomorphism in $\mathcal{A}$, with $B \in \mathcal{B}$. By the diagonal-fill-in property for epimorphisms and strong monomorphisms, the diagram

$$\begin{array}{ccc} \mathcal{A}(T_i, D) & \xrightarrow{\mathcal{A}(\theta_i, D)} & \mathcal{A}(N, D) \\ {\scriptstyle \mathcal{A}(T_i, j)}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathcal{A}(N, j)} \\ \mathcal{A}(T_i, B) & \xrightarrow[\mathcal{A}(\theta_i, B)]{} & \mathcal{A}(N, B) \end{array}$$

is a pullback in **Set**. Since colimits are universal in **Set**, and since $\mathcal{A}(\theta_i, B)$ is a colimit-cone in **Set**, so is $\mathcal{A}(\theta_i, D)$; so that $D \in \mathcal{B}$.

It is now easy to see that $\mathcal{A}(\pi_A, B)$ is a colimit-cone for $B \in \mathcal{B}$. For let $f: A \to B$, and let f factorize as an epimorphism $p: A \to C$ followed by a strong monomorphism $j: C \to B$. Since $C \in \mathcal{B}$ by the above, p is a generator of π_A through which f factorizes. If f also factorizes as $g\,p'$ through another generator $p': A \to C'$ of π_A, the diagonal-fill-in property applied to $gp' = jp$ gives a $q: C' \to C$ with $qp' = p$ and $jq = g$. Hence $\mathcal{A}(\pi_A, B)$ is a colimit-cone.

Conversely, if $\mathcal{A}(\pi_A, B)$ is a colimit-cone for each A, then $\mathcal{A}(\pi_B, B)$ is a colimit-cone; so that $1: B \to B$ factorizes as $1 = jp$ for some epimorphism $p: B \to C$ with $C \in \mathcal{B}$. But then the epimorphism p, being a coretraction, is invertible; and $B \in \mathcal{B}$. □

THEOREM 2. *Add to the hypotheses of Theorem 1 the completeness of* $\mathcal{A}$, *and suppose each cone* θ_β *to have a discrete base* $\mathcal{L}_\beta$. *Then the restriction of* π_A *to a suitable full subcategory of* S_A *gives a multi-reflexion of* $\mathcal{A}$ *onto* $\mathcal{B}$.

PROOF. Since connected limits commute with discrete colimits in **Set**, we have $\mathcal{B}$ closed in $\mathcal{A}$ under connected limits. For each connected component δ of S_A, therefore, the limit of $d_A | \delta : \delta \to S_A \to \mathcal{A}$ is an object E_δ of $\mathcal{B}$; and the $p: A \to C$ of S_A induce a map $r_\delta : A \to E_\delta$. Let this factorize as the epimorphism $s_\delta : A \to K_\delta$ followed by the strong monomorphism $k_\delta : K_\delta \to E_\delta$. Then $K_\delta \in \mathcal{B}$, and s_δ is an object of S_A; clearly the greatest object of the ordered set S_A which belongs to δ. It is now evident that any $f: A \to B$ with $B \in \mathcal{B}$ factorizes uniquely through some s_δ, and through one only. □

We include for completeness the classical:

THEOREM 3. *If each* $\mathcal{L}_\beta = 1$ *in Theorem 2,* $\mathcal{B}$ *is closed under limits in* $\mathcal{A}$, *and we get an actual reflexion* ρ_A *of* $\mathcal{A}$ *onto* $\mathcal{B}$, *where* ρ_A *is the epimorphic part of the factorization of* $A \to \lim d_A$ *into an epimorphism followed by a strong monomorphism.* □

We end by observing that the cowellpoweredness hypothesis of Theorem 1 does hold in the example to which Guitart and Lair give most prominence - that of the algebras for a *mixed sketch* S. By this is meant a small category $\mathcal{S}$ in which are given a small set $\Phi = \{\phi_\alpha\}$ of small projective cones and a small set $\Psi = \{\psi_\beta\}$ of small inductive cones; unlike Guitart and Lair, we do not ask the ϕ_α to be limit-cones nor the ψ_β to be colimit-cones. The category S-*Alg* of S-algebras is the full subcategory of $[\mathcal{S}, \mathbf{Set}]$ given by those $A: \mathcal{S} \to \mathbf{Set}$ for which each $A\phi_\alpha$ is a limit-cone and each $A\psi_\beta$ is a colimit-cone. The sketch S is *projective* when the set Ψ is empty; write S_0 for the projective sketch obtained from S by discarding Ψ. It is classical that categories of the form S_0-*Alg* are the *locally presentable* ones of Gabriel-Ulmer [3]; and that such a category is reflective in $[\mathcal{S}, \mathbf{Set}]$, and is therefore complete and cocomplete.

Let $Z: \mathcal{S}^{op} \to S_0\text{-}Alg$ be the composite of the Yoneda embedding $Y: \mathcal{S}^{op} \to [\mathcal{S}, \mathbf{Set}]$ and the reflexion $R: [\mathcal{S}, \mathbf{Set}] \to S_0\text{-}Alg$. Clearly $\mathcal{B} =$ S-*Alg* is the full subcategory of $\mathcal{A} = S_0\text{-}Alg$ consisting of those objects A such that $\mathcal{A}(-, A)$ sends the projective cone $\theta_\beta = Z\psi_\beta$ of $\mathcal{A}$ to a colimit-cone in **Set** for each β. Note that each generator of θ_β is epimorphic if each generator of ψ_β is monomorphic.

Finally, observe that $\mathcal{A}$ is cowellpowered by Satz 7.14 of [3], an account of which in English can be found in Section 8.6 of [6].

BIBLIOGRAPHY.

1. Y. DIERS, Familles universelles de morphismes, *Ann. Soc. Sci. Bruxelles* 93 (1979), 175-195.
2. P. J. FREYD & G. M. KELLY, Categories of continuous functors I, *J. Pure Appl. Algebra* 2 (1972), 169-191; Erratum 4 (1974), 121.
3. P. GABRIEL & F. ULMER, Lokal präsentierbare Kategorien, *Lecture Notes in Math.* 221, Springer-Verlag, 1971.
4. R. GUITART & C. LAIR, Existence de diagrammes localement libres, *Diagrammes* 6 (1981).
5. G. M. KELLY, Monomorphisms, epimorphisms, and pull-backs, *J. Austral. Math. Soc.* 9 (1969), 124-142.
6. G. M. KELLY, Structures defined by finite limits in the enriched context I, *Cahiers Top. et Géom. Diff.* XXIII-1 (1982), 3-42.

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