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INTRODUCTION.

If X and Y are objects of a category $\underline{A}$, then $\underline{A}(X, Y)$ denotes the set of morphisms from X to Y in $\underline{A}$.

Let $\underline{A}$ be a category with finite limits. An object Y of $\underline{A}$ is *cartesian* if the functor $- \times Y : \underline{A} \to \underline{A}$ has a right adjoint, i. e. there is a functor $(\ )^Y : \underline{A} \to \underline{A}$ together with bijections $\theta_{X,Y} : \underline{A}(X \times Y, Z) \to \underline{A}(X, Z^Y)$ natural in X and Y. $\underline{A}$ is said to be *cartesian closed* if every object of A is cartesian.

Cartesian objects in many non-cartesian closed categories have been studied, for example, topological spaces by Day and Kelly [2], uniform spaces by Niefield [12], locales by Hyland [6], and toposes by Johnstone and Joyal [9].

If T is a fixed object of a category $\underline{A}$, then $\underline{A}/T$ is the category whose objects are morphisms $Y \to T$ of $\underline{A}$, and morphisms are commutative triangles in $\underline{A}$ over T. If $\underline{A}$ has finite limits, then so does $\underline{A}/T$; thus, one can also study cartesian objects over T. This has been done for topological spaces by Booth and Brown [1] and by Niefield [12], and for uniform spaces, affine schemes, and certain locales and toposes by Niefield [12, 13, 14].

In this paper we consider a property of a base object T that insures that any morphism $Y \to T$ is cartesian in $\underline{A}/T$, provided Y is cartesian in $\underline{A}$. We begin with a general result (Theorem 2.1) which is later interpreted in the categories of topological spaces (Section 3), locales (Section 4) and toposes (Section 5). In view of the fact that we tend to suppress the projection $Y \to T$, and speak only of objects over T, this seems to be a desirable property to require of a reasonable base object.

2. CARTESIAN DIAGONALS.

Let $\underline{A}$ be a category with finite limits, and let T be a fixed object of $\underline{A}$. If Y is an object over T, we can factor its projection p as

$$Y \xrightarrow{\ (1_Y, p)\ } Y \times T \xrightarrow{\ \pi_2\ } T$$

where $(1_Y, p)$ denotes the graph of p, and π_2 is the second projection. Now, suppose Y is cartesian in $\underline{A}$. Then $Y \times T$ is cartesian over T (since pulling back along any morphism preserves cartesian objects [12], 1.4), and cartesianness is a transitive relation [12], 1.3; hence, Y is cartesian over T (via p) provided that Y is cartesian over $Y \times T$ (via the graph of p). But, again by [12], 1.4, this is always the case if T is cartesian over $T \times T$ (via the diagonal) since the diagram

is a pullback. Thus, we obtain the following theorem.

THEOREM 2.1. *If Y is cartesian in $\underline{A}$, and T has a cartesian diagonal, then Y is cartesian over T via any morphism $Y \to T$ of $\underline{A}$.*

3. TOPOLOGICAL SPACES.

Let Top denote the category of topological spaces and continuous maps, and let T be a fixed topological space.

DEFINITION 3.1. A subset A of T is *locally closed* if $A = U \cap F$, where U is open and F is closed in T.

It is an easy exercise to show that A is a locally closed subset of T iff $A = U \cap \bar{A}$, for some open subset U of T, where $\bar{A}$ denotes the closure of A in T.

LEMMA 3.2. *The following are equivalent:*

(a) T has a cartesian diagonal.

(b) $\Delta_T = \{(t, t) \mid t \in T\}$ is a locally closed subset of $T \times T$, i. e.,

T has a locally closed diagonal.

　　(c) T is a locally Hausdorff space, i. e., every point of T admits a Hausdorff neighborhood.

　　(d) T is a union of Hausdorff open subspaces.

PROOF. The equivalence of (a) and (b) is precisely Corollary 2.7 of [12], and (c) <=> (d) is an easy exercise. We shall show that (b) and (c) are equivalent.

　　Suppose that Δ_T is a locally closed subset of $T \times T$. Then $\Delta_T = \bar{\Delta}_T \cap U$, for some open subset U of $T \times T$. Now, if $t \in T$, there is an open subset V of T such that $(t, t) \in V \times V \subset U$. We claim that V is Hausdorff. If $t_1, t_2 \in V$ and $t_1 \neq t_2$, then $(t_1, t_2) \notin \bar{\Delta}_T$ (since $V \times V \cap \bar{\Delta}_T \subset \Delta_T$) and hence, t_1 and t_2 can be separated by disjoint open subsets as desired.

　　Conversely, suppose every element of T admits a Hausdorff neighborhood. Then there is a family $\{U_\alpha\}$ of Hausdorff open subsets, such that $T = \cup_\alpha U_\alpha$. Let $U = \cup_\alpha U_\alpha \times U_\alpha$. We claim that $\Delta_T = \bar{\Delta}_T \cap U$. Clearly, $\Delta_T \subset \bar{\Delta}_T \cap U$. We shall show that if $(s, t) \notin \Delta_T$, then we have $(s, t) \notin \bar{\Delta}_T \cap U$. Suppose that $(s, t) \notin \Delta_T$, and $(s, t) \in U$. Then

$$(s, t) \in U_\alpha \times U_\alpha \quad \text{for some } \alpha \quad \text{and } s \neq t ;$$

hence, since U_α is Hausdorff, there are disjoint open subsets V and W of U_α such that $(s, t) \in V \times W$. Now, if $(s, t) \in \bar{\Delta}_T$, then $V \times W$ meets Δ_T contradicting the disjointness of V and W. Therefore, if $(s, t) \notin \Delta_T$, then $(s, t) \notin \bar{\Delta}_T \cap U$; and the proof is complete.

　　Clearly, any Hausdorff space is locally Hausdorff. An example of a non-Hausdorff such space is obtained by identifying two copies of the unit interval $[0, 1]$ along $[0, 1)$, i. e., the pushout of the inclusion $[0, 1) \to [0, 1]$ along itself.

　　Next, we interpret Theorem 2.1 in *Top* . If Y is a cartesian space, it is not difficult to show that Z^Y can be identified with $Top(Y, Z)$, and the bijection

$$\theta_{X, Y} : Top(X \times Y, Z) \to Top(X, Z^Y)$$

is given by

$[\theta_{X,Y}(f)](x) : Y \to Z$ is the map $y \mapsto f(x,y)$, where $f: X \times Y \to Z$.

Thus, the study of cartesian spaces reduces to the well-known problem of defining a suitable topology for $Top(Y, Z)$ [3]. But, $- \times Y$ has a right adjoint iff it preserves quotient maps (by Freyd's Special Adjoint Functor Theorem [4]). Such spaces Y were characterized by Day and Kelly [2] as those for which the lattice $O(Y)$ of open subsets of Y is a continuous lattice (in the sense of Scott [15]). In particular, if Y is locally compact (i. e., every point has arbitrarily small compact neighborhoods), then Y is cartesian in Top. Moreover, if Y is Hausdorff, or more generally sober, then the converse also holds [5].

THEOREM 3.3. *If Y is locally compact (in the above sense) and T is locally Hausdorff, then Y is cartesian over T via any projection.*

PROOF. This easily follows from Theorem 2.1 by Lemma 3.2 and the above remarks.

In the remainder of this section we investigate some properties of locally Hausdorff spaces.

PROPOSITION 3.4. *Every locally Hausdorff space is T_1.*

PROOF. Suppose that T is locally Hausdorff, and U is an open subset of $T \times T$ such that $\Delta_T = \bar{\Delta}_T \cap U$. Suppose that $s, t \in T$ and $s \neq t$. We shall show that s admits an open neighborhood that misses t. If (s, t) is not in $\bar{\Delta}_T$, then s and t can be separated; and we are done. Thus, we can assume that $(s, t) \in \bar{\Delta}_T$. Now, since $\Delta_T \subset U$, there is an open neighborhood V of s such that $V \times V \subset U$; and $t \notin V$, for if $t \in V$, then

$$(s, t) \in V \times V \cap \bar{\Delta}_T \subset \Delta_T.$$

Therefore, T is T_1.

Recall that a closed subset of a space T is *irreducible* if it cannot be expressed as a union of two closed proper subsets. Note that the closure of a point of T is irreducible. We say that T is a *sober space* if every nonempty irreducible closed subset is the closure of a unique point of T. Note that sober «lies between» T_0 and Hausdorff, but is incomparable

with T_1. A well-known example of a non-sober T_1 space is the natural numbers N with the «complements of finite subsets» topology.

PROPOSITION 3.5. *Every locally Hausdorff space is sober.*

PROOF. Suppose that T is locally Hausdorff. We shall show that every irreducible closed subset F contains at most one point. Let U be an open subset of $F \times F$ such that $\Delta_F = \bar{\Delta}_F \cap U$. If F contains more than one point, then $U \smallsetminus \bar{\Delta}_F$ is nonempty, for if $U \subset \bar{\Delta}_F$, then F is discrete, and hence not irreducible. Let $(s, t) \in U \smallsetminus \bar{\Delta}_F$. Then there are disjoint open neighborhoods V and W of s and t, respectively, in F. But, $F \smallsetminus V$ and $F \smallsetminus W$ are proper subsets of F and closed subsets of T, and

$$F = (F \smallsetminus V) \cup (F \smallsetminus W);$$

contradicting the irreducibility of F. Therefore, T is a sober space.

PROPOSITION 3.6. *Let T be a locally compact locally Hausdorff space. Then a subspace Y of T is locally compact iff it is locally closed.*

PROOF. Suppose Y is a locally compact subspace of a locally Hausdorff space T. Then the inclusion $Y \to T$ is cartesian by Theorem 3.3; hence, Y is a locally closed subspace of T by [12], 2.7.

Conversely, suppose Y is a locally closed subspace of T. Then Y is cartesian over T (via the inclusion) by [12], 2.7, and T is cartesian in *Top* by transitivity. But, Y is a sober space by Proposition 3.5, and every cartesian sober space is locally compact [5]. This completes the proof.

Note that we did not use the local compactness of T in the first part of the above proof, i.e., we showed that any locally compact subspace of a locally Hausdorff space is locally closed.

4. LOCALES.

Let *Loc* denote the category of locales and morphisms in the «geometric» direction.

One can consider locally closed (i. e. pullbacks of open and closed) sublocales of a locale, and so it makes sense to talk about a locale whose

diagonal is locally closed. But, if T is a space, then the product $O(T) \times O(T)$ of the locale of opens of T with itself in Loc is not necessarily isomorphic to $O(T \times T)$; hence, T may be locally Hausdorff, while $O(T)$ does not have a locally closed diagonal in Loc. Note that if T is a locally compact sober space, then $O(T) \times O(T) \approx O(T \times T)$ [7], and it follows that T is locally Hausdorff iff $O(T)$ has a locally closed diagonal. For a further discussion of this matter, we refer the reader to [8].

DEFINITION 4.1. A locale L is *locally strongly Hausdorff* if L has a locally closed diagonal.

LEMMA 4.2. *A locale L is locally strongly Hausdorff iff L has a cartesian diagonal.*

PROOF. This follows directly from Definition 4.1 since the inclusion of a sublocale is cartesian iff it is locally closed [13].

Now, Hyland [6] showed that a locale A is cartesian in Loc iff it is locally compact, i.e. a continuous lattice. Putting this together with Theorem 2.1 and the above lemma we get:

THEOREM 4.3. *If A is locally compact, and L is locally strongly Hausdorff in Loc, then A is cartesian over L via any morphism $A \to L$ in Loc.*

5. TOPOSES.

Let $B Top / Sets$ denote the 2-category of Grothendieck toposes i.e. bounded toposes over $Sets$, geometric morphisms, and natural transformations. Now, $B Top / Sets$ has finite limits [10]; hence, one can consider cartesian toposes in the appropriate 2-categorical sense.

If $\underline{S}$ is any topos, then the notion of an open or a closed subtopos, hence a locally closed subtopos makes sense [10].

DEFINITION 5.1. A Grothendieck topos $\underline{S}$ is *locally strongly Hausdorff* if the diagonal $\underline{S} \to \underline{S} \times_{Sets} \underline{S}$ is a locally closed inclusion.

If A is a locally strongly Hausdorff locale, then the topos $Sh A$ of set-valued sheaves on A is a locally strongly Hausdorff topos, since

$Sb\,A \to Sb\,(\,A \times A\,)$ is a locally closed inclusion, and

$$Sb\,(\,A \times A\,) \approx Sb\,A \times_{Sets} Sb\,A.$$

In particular, if T is a locally compact sober space, then $Sb\,T$ is locally strongly Hausdorff iff T is a locally Hausdorff space. Of course, if T is not locally compact, then T may be locally Hausdorff without $Sb\,T$ being locally strongly Hausdorff.

LEMMA 5.2. *Let $\underline{S}$ be a Grothendieck topos. Then $\underline{S}$ is locally strongly Hausdorff iff the diagonal $\underline{S} \to \underline{S} \times_{Sets} \underline{S}$ is a cartesian inclusion in $\underline{B\,Top}/\underline{S} \times_{Sets} \underline{S}$.*

PROOF. The inclusion of a subtopos is cartesian iff it is locally closed [13]; hence, the desired result follows.

In [9], Johnstone and Joyal showed that a Grothendieck topos $\underline{E}$ is cartesian in $\underline{B\,Top}/Sets$ iff it is a continuous category. Noting that the remarks leading up to Theorem 2.1 are valid for $\underline{B\,TOP}/Sets$ we obtain the following theorem.

THEOREM 5.4. *Let $\underline{E}$ and $\underline{S}$ be Grothendieck toposes. If $\underline{E}$ is a contin-uous category and $\underline{S}$ is a locally strongly Hausdorff topos, then any geo-metric morphism $\underline{E} \to \underline{S}$ is cartesian in $\underline{B\,Top}/S$.*

We conclude with another interesting property of locally compact locally Hausdorff spaces. Although every locally Hausdorff space Y is cart-esian as a space, it is not necessarily the case that $Sb\,Y$ is cartesian as a Grothendieck topos. The spaces Y such that $Sb\,Y$ is cartesian are precisely the metastably locally compact spaces [9]. In particular, $Sb\,Y$ is cartesian if Y is stably locally compact, i. e.,

$U_1 \ll V_1$ and $U_2 \ll V_2$ in Y implies $U_1 \cap U_2 \ll V_1 \cap V_2$ in Y,

where $\ll$ denotes the usual « way below » relation.

LEMMA 5.3. *If Y is a locally compact Hausdorff space, then Y is stably locally compact.*

PROOF. Recall that in a locally compact space $U \ll V$ iff there is a com-pact subset C such that $U \subset C \subset V$ [5]. Now, suppose

$$U_1 \ll V_1 \quad \text{and} \quad U_2 \ll V_2 \quad \text{in} \quad Y.$$

Then there are compact subsets C_1 and C_2 such that

$$U_1 \subset C_1 \subset V_1 \quad \text{and} \quad U_2 \subset C_2 \subset V_2.$$

But, $C_1 \cap C_2$ is compact since Y is Hausdorff, and

$$U_1 \cap U_2 \subset C_1 \cap C_2 \subset V_1 \cap V_2.$$

Therefore, $U_1 \cap U_2 \ll V_1 \cap V_2$; and it follows that Y is stably locally compact.

THEOREM 5.4. *If* Y *is a locally compact locally Hausdorff space, then* $Sh\,Y$ *is cartesian in* $\underline{\underline{B\,Top}}\,/\,Sets$.

PROOF. If Y is a locally compact locally Hausdorff space, then Y has an open cover consisting of locally compact Hausdorff spaces. Now, meta-stable local compactness is a local property (Y satisfies it iff Y has an open cover consisting of such spaces [9]); hence, the result follows from the above lemma.

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