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NOTION OF TOPOLOGY FOR BICATEGORIES

by R. BETTI and A. CARBONI *)

INTRODUCTION.

We deal with the approach to categories based on a bicategory introduced by Walters [3, 4] and Betti & Carboni [1]. In view of further developments, where B-category theory becomes relevant to cohomology (Street [2]) and other geometrical applications (as Walters' «glueing data») it seems useful to have an abstract notion of topology for general bicategory.

Here such a notion is given in term of a «closure operator» for a locally partially ordered bicategory, even if it can be easily generalized to any locally left exact bicategory. The theorem is that in the wellknown case when B = $Rel(C)$ (the base bicategory for presheaves) closure operators on B are exactly Grothendieck topologies on C.

Moreover, on the basis of the given definition and of a previous paper [1] an intrinsic notion of sheaf for a closure operator is given. These notions arose in helpful conversations with R.H. Street, while he was visiting Milan.

1. We recall that $Rel(C)$ is the bicategory defined as follows (Walters [4], for C any locally small category):

objects of $Rel(C)$ are those of C, 1-cells $u \mapsto v$ are cribles of spans $u \leftarrow w \rightarrow v$, 2-cells are inclusions.

$Rel(C)$ is a symmetric bicategory, and R^o denotes the opposite 1-cell of R.

DEFINITION. Let B be a bicategory, locally an inf-semilattice. A *closure operator* on B is a locally left exact lax idempotent monad in B which is the identity on objects.

Any Grothendieck topology J on C determines a closure operator

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on *Rel(C)* by defining

$$\bar{R} = \{ \; u \leftarrow p \rightarrow v \; \mid \text{ there exists a cover } (p_i \to p)_i \text{ such that } \; \begin{matrix} & p_i & \\ & \downarrow & \\ & p & \\ u \swarrow & & \searrow v \end{matrix} \; \in R \; \} \, ,$$

as in Walters [4] where all stated properties are proved, but the locally left exactness $\overline{R \wedge S} = \bar{R} \wedge \bar{S}$: just observe that if U and V are coverings of u , then also $U \wedge V$ is a covering.

Let us observe that the axioms of a closure operator cannot be strengthened by requiring strict functoriality. Indeed, let C be a regular category and J the regular epi-topology; if f is a regular epi, then the crible generated by $<f, f>$ contains its closure iff f has a section.

Our aim is to show that in fact any closure operator on *Rel(C)* can be obtained in the previous way from a unique Grothendieck topology J on C.

THEOREM. *Closure operators on Rel(C) correspond bijectively to Grothendieck topologies on C.*

PROOF. If $Rel(C) \rightrightarrows Rel(C)$ is a closure operator, define the covering cribles as those cribles

$$U = \{ h_i : u_i \to u \}$$

such that the 1-cell $R_U = <h_i, h_i>$ satisfies $1_u \subset \bar{R}_U$. With this definition, we get a topology on C :

i) trivially the maximal crible covers.

ii) Let U be a covering crible of u and $f: v \to u$ a morphism of C ; observe that

$$R_{f^* U} = 1 \wedge f . R_U . f^o$$

(for terminology and properties of *Rel(C)* see Walters [1]), then

$$1 \subset f f^o \subset f . \bar{R}_U . f^{\,o} \subset \overline{\bar{f} . \bar{R}_U . \bar{f}^o} = \overline{f . R_U . f^o}$$

(the last equality easily follows from the identity $\overline{\bar{R}.\bar{S}} = \overline{R.S}$, which is a direct consequence of the axioms). Hence

$$1 \subset 1 \wedge \overline{f.R_U.f^o} \subset \bar{1} \wedge \overline{f.R_U.f^o} = \overline{1 \wedge f.R_U.f^o}$$

(by left exactness). So $1 \subset \bar{R}_{f^*U}$ and f^*U covers.

iii) Let U be a covering crible of u and V a crible such that, for each $f \in U$, f^*V covers, we have

$$\overline{f^o.f} \subset \overline{f^o.\overline{f.R_V.f^o}.f} \quad \text{(because } f^*V \text{ covers)}$$

$$= \overline{f^o.f.R_V.\,f^o.f} \subset \bar{R}_V \quad \text{(because } f^o.f \subset 1 \text{)}.$$

So

$$R_U = \bigvee_{f \in U} f^o f \subset \bar{R}_V ,$$

hence $\bar{R}_U \subset \bar{R}_V$ and thus $1 \subset \bar{R}_V$.

It is straightforward to verify that if J is a topology on C and $\bar{\ }$ is the associated closure operator then J is a J-cover iff $1 \subset \bar{R}_U$. Conversely, given a closure operator on $Rel(C)$, we need to show

$$\bar{T} = \left\{ u \xleftarrow{\;h\;} w \xrightarrow{\;k\;} v \;\middle|\; \begin{array}{l} \text{there exists a crible } U \text{ on } w \text{ such that} \\ 1 \subset \bar{R}_U \text{ and } h^o.R_U.k \subset T \end{array} \right\}$$

for each 1-cell T of $Rel(C)$. In one direction we have

$$<h, k> = h^o.k \subset \overline{h^o.k} \subset \overline{h^o.\bar{R}_U.k} = \overline{h^o.R_U.k} \subset \bar{T}.$$

In the other one, define R_U as $1 \wedge h.T.k^o$. Then $1 \subset \bar{R}_U$ for

$$1 \subset h.h^o.k.k^o \subset h.\bar{T}.k^o \subset \overline{h.\bar{T}.k^o} = \overline{h.T.k^o}.$$

Moreover

$$h^o.(1 \wedge h.T.k^o).k \subset h^o.h.T.k^o.k \subset T.$$

In the same way we can translate notions relative to Grothendieck topologies in this more «algebraic» context. For instance it is easy to prove the following

PROPOSITION. *The following conditions are equivalent:*

i) Representables are J-sheaves.

ii) If R is a «partial map» (i. e. $R^o R \subset 1$), then

$$\bar{R}.\bar{S} = \overline{R.S} \quad \text{and} \quad \bar{1} = 1.$$

PROOF. A compatible family $u_\alpha \to v$ on the covering $u_\alpha \to u$ gives rise to a partial map $R: u \nrightarrow v$. The hypothesis implies that R is a map.

Given any closure operator $(\bar{\ }): \mathrm{B} \to \mathrm{B}$, a new bicategory $\bar{\mathrm{B}}$ is defined by taking the same objects as B and, as 1-cells, the closed ones, i.e. locally the algebras for the idempotent monad induced by the closure operator. In $\bar{\mathrm{B}}$ the composition is defined by $\overline{R.S}$; identities for such composition are closures of the old ones.

A pair of morphisms of bicategories is obtained: $\mathrm{B} \underset{(\bar{\ })}{\overset{i}{\rightleftarrows}} \bar{\mathrm{B}}$, such that $(\bar{\ })$ is locally left exact left adjoint to i. Observe that i is really a lax morphism (see the above counterexample), while $(\bar{\ })$ is a strict one (homomorphism). The induced change of base B-*Cat* $\to$ $\bar{\mathrm{B}}$-*Cat* is also denoted by $(\bar{\ })$.

The result of [1] motivates the following definitions:

DEFINITION 1. Let $\mathrm{B} \to \bar{\mathrm{B}}$ be a closure operator and X a B-category. A bimodule $R: Y \nrightarrow X$ *covers* X if $Y \subset \overline{R.R^o}$ and $R^o.R \subset \bar{X}$.

The above definition amounts to require that $\bar{R}: \bar{Y} \nrightarrow \bar{X}$ has a right adjoint in $\bar{\mathrm{B}}$-*Cat*.

DEFINITION 2. Let X be a B-category. X is a *sheaf* if each covering $R: Y \nrightarrow X$ is representable by a functor.

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