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MODULO C HOMOTOPY

by J. R. DENNETT

This work arose from an attempt to understand how the ideas of localization and completion relate to p equivalences as defined by Serre i.e. maps $f: X \to Y$ such that $f^*: H^*(Y, Z_p) \to H^*(X, Z_p)$ is an isomorphism. Localization is usually set up by defining for a space X its localization EX, i.e. in a sense enlarging the category by including more objects. It seemed to me that it might be fruitful to keep the same objects but to change the morphisms by defining a mod$p$ homotopy relation.

In suitable circumstances the localization can also be regarded as a functor $\eta : \mathfrak{A} \to \mathfrak{A}/\Sigma$ where $\mathfrak{A}/\Sigma$ is the category of fractions with respect to a suitable morphisms class [3]. This suggests taking Σ to be the class of p equivalences and defining maps f and g to be *mod$p$ homotopic* if $\eta(f) = \eta(g)$. This is just the situation studied by Bauer and Dugundji [1] although not in the cases (e.g. p equivalences) in which I was interested. They define morphisms f and g in $\mathfrak{A}$ to be Σ-homotopic if $\eta(f) = \eta(g)$, and show, for example, that if $\mathfrak{A}$ is the category of topological spaces and continuous maps and Σ is the class of homotopy equivalences then Σ-homotopy coincides with the usual notion of homotopy.

In this note we investigate the homotopy relation, in the homotopy category of pointed topological spaces, determined by the class of morphisms which induce $\mathcal{C}$ isomorphisms in homology, where $\mathcal{C}$ is a Serre class of abelian groups. Since this class admits a calculus of left fractions, the homotopy relation has another description in terms of equalisers. In the category of 1-connected spaces this class also admits a calculus of right fractions and so the homotopy relation has a description in terms of coequalisers too. This mod $\mathcal{C}$ homotopy relation enables us to define mod $\mathcal{C}$ homotopy groups. If we take $\mathcal{C}$ to be the Serre class of all finite abelian groups with p torsion (p prime) and work in the category of 1-connected finite CW-complexes, then the mod $\mathcal{C}$ homotopy groups are the p components of the usual homotopy groups.

Let $\mathcal{T}$ denote the category of pointed topological spaces and continuous base point preserving maps and let $\bar{\mathcal{T}}$ denote the homotopy category of $\mathcal{T}$. Let $\mathcal{C}$ denote a Serre class of abelian groups and let Σ denote the set of maps in $\mathcal{T}$ which induce $\mathcal{C}$ isomorphisms in integral homology. Let $\bar{\Sigma}$ denote the image of Σ in $\bar{\mathcal{T}}$.

THEOREM 1. *$\bar{\Sigma}$ admits a calculus of left fratctions in $\bar{\mathcal{T}}$.*

PROOF. The constructions in Lemma 3.6 of [2] work in this situation.

Let $\mathcal{T}_1$ denote the category of 1-connected pointed topological spaces and let Σ_1 denote the set of maps in $\mathcal{T}_1$ which induce $\mathcal{C}$ isomorphisms of all homotopy groups. Let $\bar{\Sigma}_1$ denote the image of Σ_1 in $\bar{\mathcal{T}}_1$, the homotopy category of $\mathcal{T}_1$.

THEOREM 2. *$\bar{\Sigma}_1$ admits a calculus of right fractions in $\bar{\mathcal{T}}_1$.*

PROOF. (i) It is obvious that $\bar{\Sigma}_1$ contains identity maps and is closed under composition.

(ii) Suppose that we have

$$\begin{tikzcd} & Z \arrow[d, "f"] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

where $f \in \bar{\Sigma}_1$. Replace f and g by fibrations and pullback to

$$\begin{tikzcd} W \arrow[r, "r"] \arrow[d, "s"'] & Z \arrow[d, "f"] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

where $s\colon W \to X$ is the induced fibration. Since $f \in \bar{\Sigma}_1$,

$$\pi_i(F) \in \mathcal{C} \quad \text{for } i \geq 1$$

where F is the fibre of f. This implies that $s_{\#}\colon \pi_i(W) \to \pi_i(X)$ is a $\mathcal{C}$ isomorphism for $i \geq 2$, $\pi_1(W) \in \mathcal{C}$ and $\pi_0(W) = 0$. Now apply Corollary 8 page 444 of [5] to get $s'\colon W' \to W$ where W' is 1-connected and

$$s'_{\#}\colon \pi_i(W') \approx \pi_i(W) \quad \text{for } i \geq 2.$$

(iii) Suppose that we have

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{\;\;r\;\;} Z$$

where $rf \approx rg$ and $r \in \bar{\Sigma}_1$. We may assume that r is a fibration. Let S be

$$\{(y_1, \omega, y_2) \in Y \times Z^I \times Y \mid \omega(0) = r(y_1),\ \omega(1) = r(y_2)\}$$

and define

$$\alpha : Y^I \to S \text{ by } \alpha(\lambda) = (\lambda(0), r\lambda, \lambda(1)).$$

By Corollary 10 page 416 of [5] α is a (Serre) fibration and the fibre is ΩF_r where F_r is the fibre of r. The homotopy sequence of the fibration r gives

$$\pi_i(F_r) \in \mathcal{C} \text{ for } i \geq 1 \text{ and } \pi_0(F_r) = 0.$$

Therefore $\pi_i(\Omega F_r) \in \mathcal{C}$ for $i \geq 0$. Define

$$\beta : X \to S \text{ by } \beta(x) = (f(x), F(x), g(x))$$

where $F : X \to Z^I$ is given by the homotopy between rf and rg. Pullback

$$\begin{tikzcd} & Y^I \arrow[d, "\alpha"] \\ X \arrow[r, "\beta"] & S \end{tikzcd}$$

to

$$\begin{tikzcd} W \arrow[r, "G"] \arrow[d, "h"'] & Y^I \arrow[d, "\alpha"] \\ X \arrow[r, "\beta"] & S \end{tikzcd} .$$

G yields a homotopy between fh and gh. Moreover the homotopy sequence of the fibration h shows that $h_{\#}: \pi_i(W) \to \pi_i(X)$ is a $\mathcal{C}$ isomorphism for $i \geq 2$ and $\pi_1(W) \in \mathcal{C}$. Replace W by its path component containing the base point and, as before, approximate by $s' : W' \to W$ where W' is 1-connected and $s'_{\#}: \pi_i(W') \approx \pi_i(W)$ for $i \geq 2$.

Theorem 1 also holds in $\bar{\mathcal{T}}_1$ and in the following categories:

$\bar{\mathfrak{W}}$ = homotopy category of CW complexes,

$\bar{\mathfrak{W}}_1$ = homotopy category of 1-connected CW-complexes,

$\bar{\mathfrak{F}}$ = homotopy category of spaces of finite type,

$\bar{\mathfrak{F}}_1$ = homotopy category of 1-connected spaces of finite type,

$\overline{\mathfrak{F}\mathfrak{W}}$ = homotopy category of CW-complexes of finite type,

$\overline{\mathfrak{F}\mathfrak{W}}_1$ = homotopy category of 1-connected CW-complexes of finite type.

Theorem 2 holds in $\bar{\mathfrak{W}}_1$ since having obtained W' we can find a CW complex K and a weak homotopy equivalence from K to W'.

PROPOSITION 3. *Theorem 2 holds in* $\bar{\mathfrak{F}}_1$ *and in* $\overline{\mathfrak{F}\mathfrak{W}}_1$.

PROOF. The construction of W in (ii) and (iii) does not depend on $\mathcal{C}$. If we take $\mathcal{C}$ to be the Serre class of finitely generated abelian groups and work in the categories $\bar{\mathfrak{F}}_1$ or $\overline{\mathfrak{F}\mathfrak{W}}_1$ then any map induces $\mathcal{C}$ isomorphisms of homotopy. But $\pi_i(W')$ is $\mathcal{C}$ isomorphic to $\pi_i(X)$ where X is of finite type. Therefore W' is of finite type.

If $\mathcal{C}$ is an acyclic ideal of abelian groups and we work in one of the categories of 1-connected spaces, then $\Sigma = \Sigma_1$ and $\bar{\Sigma}$ admits a calculus of left and right fractions. Let us call a map in Σ a $\mathcal{C}$ *equivalence.*

DEFINITION [1]. Suppose that f and $g: X \to Y$ in $\mathcal{T}_1$. Then f *is mod* $\mathcal{C}$ *homotopic to* g if $\eta(f) = \eta(g)$ where $\eta: \mathcal{T}_1 \to \mathcal{T}_1/\Sigma$ is the localization functor. We write $f \underset{\mathcal{C}}{\approx} g$.

PROPOSITION 4. *(i)* $f \underset{\mathcal{C}}{\approx} g$ *iff there is a* $\mathcal{C}$ *equivalence* $h: Y \to Z$ *in* $\mathcal{T}_1$ *such that* $hf \approx hg$.

(ii) $f \underset{\mathcal{C}}{\approx} g$ *iff there is a* $\mathcal{C}$ *equivalence* $k: W \to X$ *in* $\mathcal{T}_1$ *such that* $fk \approx gk$. *(Here «≈» denotes the usual homotopy relation.)*

PROOF. Let $\bar{\eta}: \bar{\mathcal{T}}_1 \to \bar{\mathcal{T}}_1/\bar{\Sigma}_1$ be the localization functor and let $\pi: \mathcal{T}_1 \to \bar{\mathcal{T}}_1$ be the natural surjection. If $f \approx g$ in $\mathcal{T}_1$ then $\eta(f) = \eta(g)$, so that η induces $\tilde{\eta}: \bar{\mathcal{T}}_1 \to \mathcal{T}_1/\Sigma_1$. By the universal property for η there exists a functor $\theta: \mathcal{T}_1/\Sigma_1 \to \bar{\mathcal{T}}_1/\bar{\Sigma}_1$ such that $\theta\eta = \bar{\eta}\pi$. By the universal property for $\bar{\eta}$ there exists a functor

$$\psi: \bar{\mathcal{T}}_1/\bar{\Sigma}_1 \to \mathcal{T}_1/\Sigma_1 \quad \text{such that } \psi\bar{\eta} = \tilde{\eta}.$$

Then θ and ψ give an equivalence $\mathcal{T}_1/\Sigma_1 \approx \bar{\mathcal{T}}_1/\bar{\Sigma}_1$.

$$\begin{tikzcd} \mathcal{T}_1 \arrow[r, "\pi"] \arrow[d, "\eta"'] & \bar{\mathcal{T}}_1 \arrow[d, "\bar{\eta}"] \arrow[dl, "\bar{\bar{\eta}}"'] \\ \mathcal{T}_1/\Sigma_1 \arrow[r, shift left, "\theta"] & \bar{\mathcal{T}}_1/\bar{\Sigma}_1 \arrow[l, shift left, "\psi"] \end{tikzcd}$$

Therefore,

$$\begin{aligned} f \underset{\mathcal{C}}{\approx} g &\iff \eta(f) = \eta(g) \iff \theta\eta(f) = \theta\eta(g) \\ &\iff \bar{\eta}(\bar{f}) = \bar{\eta}(\bar{g}) \text{ where } \bar{f} = \pi(f) \\ &\iff \text{there exists } \bar{h} \text{ in } \bar{\Sigma}_1 \text{ such that } \bar{h}\bar{f} = \bar{h}\bar{g} \end{aligned}$$

(since $\bar{\Sigma}_1$ admits a calculus of left fractions in $\bar{\mathcal{T}}_1$)

$$\begin{aligned} &\iff \text{there exists } h \text{ in } \Sigma_1 \text{ such that } hf \approx hg \\ &\iff \text{there exists } k \text{ in } \Sigma_1 \text{ such that } fk \approx gk \end{aligned}$$

(by Part (iii) of Theorem 2).

Clearly mod $\mathcal{C}$ homotopy is an equivalence relation and behaves correctly under composition.

Let $[X, Y]_{\mathcal{C}}$ denote the set of mod $\mathcal{C}$ homotopy classes of maps from X to Y and let $[f]_{\mathcal{C}}$ denote the mod $\mathcal{C}$ homotopy class of a map f. If $f: X \to Y$ in $\mathcal{T}_1$ then f induces mappings

$$f_{\#}: [Z, X]_{\mathcal{C}} \to [Z, Y]_{\mathcal{C}}, \quad f^{\#}: [Y, Z]_{\mathcal{C}} \to [X, Z]_{\mathcal{C}}$$

for any Z in $\mathcal{T}_1$.

Let SX denote the (reduced) suspension of X and ΩY denote the space of loops on Y.

THEOREM 5. *$[SX, Y]_{\mathcal{C}}$ is a group. It is abelian if $X = SZ$ or $Y = \Omega Z$. If $\Omega Y \in \mathcal{T}_1$ then $[X, \Omega Y]_{\mathcal{C}}$ is a group and there is an isomorphism*

$$[SX, Y]_{\mathcal{C}} \approx [X, \Omega Y]_{\mathcal{C}}.$$

PROOF. If $f, g: SX \to Y$ let $f * g$ denote $(f \vee g)v: SX \to Y$ where $v: SX \to SX \vee SX$ is the comultiplication. Define $[f]_{\mathcal{C}} * [g]_{\mathcal{C}}$ to be $[f * g]_{\mathcal{C}}$. It suffices to show that if $f_1 \underset{\mathcal{C}}{\approx} f_2$ and $g_1 \underset{\mathcal{C}}{\approx} g_2$ then $f_1 * g_1 \underset{\mathcal{C}}{\approx} f_2 * g_2$.

Suppose there are $\mathcal{C}$ equivalences $h: Y \to Z$ and $k: Y \to W$ such that $h f_1 \approx h f_2$ and $k g_1 \approx k g_2$. By Theorem 1 there exist $\mathcal{C}$ equivalences r and s such that $rh \approx sk$.

$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[d, "k"'] & Z \arrow[d, "r"] \\ W \arrow[r, "s"] & V \end{tikzcd}$$

Then

$$rh(f_1 \vee g_1)\, v = ((rh\, f_1) \vee (rh\, g_1))v \approx ((rh\, f_1) \vee (sk\, g_1))v$$

$$\approx (rh\, f_2) \vee (sk\, g_2))v \approx ((rh\, f_2) \vee (rh\, g_2))v = rh(f_2 \vee g_2)v.$$

Since rh is a $\mathcal{C}$ equivalence $f_1 * g_1 \underset{\mathcal{C}}{\approx} f_2 * g_2$.

If $\Omega Y \in \mathcal{T}_1$ then a similar argument using Theorem 2 shows that the usual operation on $[X, \Omega Y]$, the group of homotopy classes of maps from X to ΩY, yields a group structure on $[X, \Omega Y]_{\mathcal{C}}$.

Let $\theta: [SX, Y] \to [X, \Omega Y]$ and $\psi: [X, \Omega Y] \to [SX, Y]$ be the usual isomorphisms. To complete the proof of the theorem it is sufficient to show that θ and ψ preserve mod $\mathcal{C}$ homotopy. Suppose that $f, g: SX \to Y$ and $f \underset{\mathcal{C}}{\approx} g$, i.e. there is a $\mathcal{C}$ equivalence

$$h: Y \to Z \quad \text{such that} \quad F: hf \approx hg.$$

The homotopy $F: SX \times I \to Z$ yields a homotopy $\bar{F}: X \times I \to \Omega Z$, and $\bar{F}: \Omega h \theta f \approx \Omega h \theta g$. Since Ωh is a $\mathcal{C}$ equivalence, $\theta f \underset{\mathcal{C}}{\approx} \theta g$. Similarly, by taking a $\mathcal{C}$ equivalence on the left, ψ also preserves mod $\mathcal{C}$ homotopy.

DEFINITION. For $n \geq 1$ the n^{th} *mod $\mathcal{C}$ homotopy group* $\pi_n^{\mathcal{C}}(X)$ is $[S_n, X]_{\mathcal{C}}$ It is abelian if $n > 1$.

Then $f: X \to Y$ in $\mathcal{T}_1$ induces $f_{\#}: \pi_n^{\mathcal{C}}(X) \to \pi_n^{\mathcal{C}}(Y)$. Also there is a canonical epimorphism $\alpha: \pi_n(X) \to \pi_n^{\mathcal{C}}(X)$.

DEFINITION. $f: X \to Y$ in $\mathcal{T}_1$ is a *mod $\mathcal{C}$ homotopy equivalence* if there exists $g: Y \to X$ in $\mathcal{T}_1$ such that

$$fg \underset{\mathcal{C}}{\approx} 1_Y \quad \text{and} \quad gf \underset{\mathcal{C}}{\approx} 1_X.$$

We now consider the situation in $\mathcal{H}$, category of finite 1-connected CW-complexes. Let P be a (possibly empty) subset of the primes and let $\mathcal{C}_P$ be the class of finite abelian groups without P torsion. Then $\mathcal{C}_P$ equivalences are precisely P equivalences [4]. Let $\pi^{\mathrm{P}}(X)$ denote $\pi^{\mathcal{C}_{\mathrm{P}}}(X)$ and let $\underset{\mathrm{P}}{\approx}$ denote $\underset{\mathcal{C}_{\mathrm{P}}}{\approx}$. In the homotopy category $\overline{\mathcal{H}}$ the homotopy classes of P equivalences admit a calculus of left fractions and (as in Proposition 4) we have $f \underset{\mathrm{P}}{\approx} g$ iff there is a P equivalence h in $\mathcal{H}$ such that $hf \approx hg$. For X in $\mathcal{H}$ let X_{P} denote the localization [4] and $i_X : X \to X_{\mathrm{P}}$ the canonical inclusion.

PROPOSITION 6. *Suppose that $f, g: X \to Y$ in $\mathcal{H}$. Then $f \underset{\mathrm{P}}{\approx} g$ iff $i_Y f \approx i_Y g$.*

PROOF. Suppose that $f \approx_{\mathrm{P}} g$, i.e. there exists a P equivalence $h: Y \to Z$ in $\mathcal{H}$ such that $hf \approx hg$. Then we have the commutative diagram

$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[d, "i_Y"'] & Z \arrow[d, "i_Z"] \\ Y_{\mathrm{P}} \arrow[r, "h_{\mathrm{P}}"] & Z_{\mathrm{P}} \end{tikzcd}$$

where h_{P} is a homotopy equivalence [4, Theorem 2.4]. Therefore

$$i_Y f \approx h_{\mathrm{P}}^{-1} i_Z h f \approx h_{\mathrm{P}}^{-1} i_Z h g \approx i_Y g.$$

Conversely, suppose that $H: X \times I \to Y_{\mathrm{P}}$ is a homotopy between $i_Y f$ and $i_Y g$. Since $X \times I$ is a finite CW-complex, $H: X \times I \to Y_\lambda$ where Y_λ is a finite CW-complex occuring in the construction of Y_{P}. Thus $H: if \approx ig$ where $i: Y \to Y_\lambda$ is the inclusion and a P equivalence.

THEOREM 7. $\pi_n^{\mathrm{P}}(X) \approx (\pi_n(X))_{\mathrm{P}}$, *the* P *component of* $\pi_n(X)$.

PROOF. It follows from Proposition 6 that there is a well defined monomorphism $\beta : \pi_n^{\mathrm{P}}(X) \to \pi_n(X_{\mathrm{P}})$ such that the diagram

$$\begin{tikzcd} \pi_n(X) \arrow[r, "a"] \arrow[dr, "(i_X)_\#"'] & \pi_n^{\mathrm{P}}(X) \arrow[d, "\beta"] \\ & \pi_n(X_{\mathrm{P}}) \end{tikzcd}$$

commutes. Thus

$$\pi_n^{\mathrm{P}}(X) \approx \frac{\pi_n(X)}{\ker a} \approx \frac{\pi_n(X)}{\ker (i_X)_\#}.$$

Now $\pi_n(X_{\mathrm{P}}) \approx \pi_n(X) \otimes \mathrm{Q}_{\mathrm{P}}$, where Q_{P} is the ring of rationals which, in their lowest form, have denominator prime to p for all p in P and $(i_X)_\#$ is

$$1 \otimes i: \pi_n(X) \otimes Z \to \pi_n(X) \otimes Q_P,$$

where $i: Z \to Q_P$ is the inclusion [4, Theorem 2.5]. Hence

$$\pi_n(X) \approx (\pi_n(X))_P.$$

THEOREM 8. *If $f: X \to Y$ in $\mathcal{H}$ is a mod $\mathcal{C}_p$ homotopy equivalence for all primes p, then f is a homotopy equivalence.*

PROOF. For each prime p there exists $g_p: Y \to X$ such that $fg_p \underset{p}{\approxeq} 1_Y$ and $g_p f \underset{p}{\approxeq} 1_X$. By Proposition 6, $i_Y f g_p \approx i_Y$ and $i_X g_p f \approx i_X$. Therefore

$$g_p^* f^* i_Y^* = i_Y^*: H^*(Y_p; Z_p) \to H^*(Y; Z_p)$$

$$\text{and} \quad f^* g_p^* i_X^* = i_X^*: H^*(X_p; Z_p) \to H^*(X; Z_p).$$

Since i_Y^* and i_X^* are isomorphisms so is $f^*: H^*(Y; Z_p) \to H^*(X; Z_p)$. Thus f is a p equivalence. Since this holds for all primes p, f is a homotopy equivalence.

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