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NORMAL FORMS OF MATRICES IN TOPOI [1]

by Javad TAVAKOLI

0. INTRODUCTION.

In ordinary linear algebra every $m \times n$ matrix over a field is equivalent to one in normal (diagonal) form. The purpose of this paper is to examine this in an elementary topos with natural number object. We will give a positive answer to this problem if we are dealing with a geometric field (a commutative ring K in a topos $\underline{E}$, satisfying the axiom of non-triviality ($0 \neq 1$), is said to be a geometric field if

$$\underline{E} \models (a=0) \vee (a \in \ulcorner T \urcorner),$$

where T is the object of units of K [JN 2]) [2]. Our main theorem appears in Section 1, where we prove every linear transformation between finite dimensional vector spaces can be normalized. Also we show that if $K^{[p]} \approx K^{[q]}$, then $p = q$. In Section 2 we define rank for a linear transformation and introduce dimension for I-families of locally finite dimensional vector spaces. Finally it is shown that if

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

is an exact sequence of vector spaces, then if A_1 and A_2 are finite dimensional then A_3 is. If A_2 and A_3 are finite dimensional then A_1 is. If any two are locally finite dimensional then the other is and

$$dim(A_2) = dim(A_1) + dim(A_3).$$

Also we give an example to show that if A_1 and A_3 are finite dimensional A_2 is not necessarily.

[1] This research is part of the author's Ph. D. dissertation at Dalhousie University. The author wishes to express his deep gratitude to his research supervisor, Professor Robert PARE.

[2] The main definitions of fields are given in [JN 1] and [JN 2].

It is natural to ask whether the results obtained in this paper hold for the other types of fields. It is tempting to suggest that some variation of the results are also true for residue fields (for definition see [JN 2]).

All the concepts of this paper are considered in an elementary topos $\underline{E}$ with natural number object N and with geometric field K. The other notations can be found in [JN 1], [P&S] or [TV 1].

1. NORMAL FORM OF A LINEAR TRANSFORMATION.

(1.1) DEFINITION. Let $\phi : K^{[p]} \to K^{[q]}$ be a linear transformation between two finite dimensional vector spaces in $\underline{E}$. We say *ϕ is in normal form* if there exist natural numbers r, p' and q' such that

$$r+p' = p, \quad r+q' = q \quad (\text{i.e. } [r]\amalg[p'] = [p] \text{ and } [r]\amalg[q'] = [q])$$

and

$$\begin{array}{ccc} K^{[p]} & \xrightarrow{\quad\phi\quad} & K^{[q]} \\ {\scriptstyle (K^{i_1},K^{i_2})}\downarrow\approx & & \approx\downarrow{\scriptstyle (K^{j_1},K^{j_2})} \\ K^{[r]}\oplus K^{[p']} & \xrightarrow{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}} & K^{[r]}\oplus K^{[q']} \end{array}$$

commutes, where

$$[r] \xrightarrow{i_1} [p] \xleftarrow{i_2} [p'], \quad [r] \xrightarrow{j_1} [q] \xleftarrow{j_2} [q']$$

are the coproduct injections.

(1.2) REMARK. For any natural numbers p and q we have a family of normal forms. Consider the following pullback diagram

$$\begin{array}{ccc} N\times N\times N & \xrightarrow{(a\pi_{12},\, a\pi_{13})} & N\times N \\ {\scriptstyle (r,p',q')}\uparrow & \text{P.B.} & \uparrow{\scriptstyle (p,q)} \\ A & \longrightarrow & 1 \end{array}$$

where a is the addition operation on N. (Interpretation:

$$A = \{(r,p',q') \mid r+p' = p \text{ and } r+q' = q\}.)$$

Then in $\underline{E}/A$, $r+p' = A^*p$ and $r+q' = A^*q$. Now consider

$$\begin{array}{ccc} (A*K)^{[r]} \oplus (A*K)^{[p']} & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} & (A*K)^{[r]} \oplus (A*K)^{[q']} \\ \| \wr & & \| \wr \\ A*(K^{[p]}) & \xrightarrow{\quad \psi \quad} & A*(K^{[q]}) \end{array}$$

in $Vect_K(\underline{E})^A$. ψ is the required family of normal forms.

(1.3) LEMMA. *Let* p *be a natural number in* $\underline{E}$. *Then there is a morphism*

$$\Lambda : [1+p]*[1+p] \to [1+p]*[1+p] \quad in \quad \underline{E}/[1+p],$$

indexed by $[1+p]$ *such that*

$$\begin{array}{ccc} [1+p]*[1+p] & \xrightarrow{\quad \Lambda \quad} & [1+p]*[1+p] \\ \| \wr & & \nearrow \delta \\ [1+p]*1 + [1+p]*[p] & \xleftarrow{\ j_1\ } & [1+p]*1 \end{array}$$

commutes and $\Lambda^2 = 1_{[1+p]*[1+p]}$.

(Interpretation: Λ is a $[1+p]$-family of morphisms $\Lambda_i : [1+p] \to [1+p]$ such that for each $i \in [1+p]$,

$$\Lambda_i(0) = i, \ \Lambda_i(i) = 0 \ \text{ and } \Lambda_i(j) = j \ \text{ for } j \neq 0, i.)$$

PROOF. First we define an isomorphism

$$1+[p]+[p]+[p]+[c] \xrightarrow[a]{\approx} [1+p]\times[1+p]$$

$([p]\times[p] \approx [p]+[c])$ by:

$$1 \xrightarrow{(i_1, i_1)} [1+p]\times[1+p],$$

first $[p] \xrightarrow{(i_2, i_1)} [1+p]\times[1+p]$,

second $[p] \xrightarrow{(i_1, i_2)} [1+p]\times[1+p]$,

third $[p] \xrightarrow{\delta} [p]\times[p] \xrightarrow{i_2\times i_2} [1+p]\times[1+p]$,

and $[c] \xrightarrow{c} [p]\times[p] \xrightarrow{i_2\times i_2} [1+p]\times[1+p]$.

Now we define

$$\Lambda' : 1+[p]+[p]+[p]+[c] \to 1+[p]+[p]+[p]+[c]$$

by interchanging the first $[p]$ with the third $[p]$. It is obvious that Λ' is a morphism over $[1+p]$, i.e.,

$$\begin{array}{ccc} 1+[p]+[p]+[p]+[c] & \xrightarrow{\Lambda'} & 1+[p]+[p]+[p]+[c] \\ a \downarrow \wr & & \wr \downarrow a \\ [1+p]\times[1+p] & & [1+p]\times[1+p] \\ & \pi_1 \searrow \quad \swarrow \pi_1 & \\ & [1+p] & \end{array}$$

commutes. Therefore Λ' induces a morphism

$$\Lambda : [1+p]*[1+p] \to [1+p]*[1+p].$$

Since $\Lambda'^2 = 1$ then $\Lambda^2 = 1$. Also 1, the second $[p]$, and $[c]$ are constant under Λ' so the required diagram commutes. □

(1.4) DEFINITION. A linear transformation $\phi : K^{[p]} \to K^{[q]}$ is said to be *non-zero* if the pullback of $\bar{\phi}$ along $T \rightarrowtail K$ has global support, where $\bar{\phi}$ is given by

$$\begin{array}{cc} K^{[p]} \xrightarrow{\phi} K^{[q]} & \text{in } Vect_K(\underline{E}) \\ \hline [p] \to K^{[q]} & \text{in } \underline{E} \\ \hline [p]\times[q] \xrightarrow{\bar{\phi}} K & \text{in } \underline{E}\,. \end{array} \quad \begin{array}{c} \approx \\ \approx \end{array}$$

(1.5) LEMMA. *Let $H = Hom(K^{[1+p]}, K^{[1+q]})$ be the object of homomorphisms from $K^{[1+p]}$ to $K^{[1+q]}$, see [P & S], and*

$$\phi : H^*(K^{[1+p]}) \to H^*(K^{[1+q]})$$

be the generic one. Then there exists $U_1 + U_2 = H$ such that $i_1^*\phi$ is non-zero and $i_2^*\phi$ is zero, where

$$U_1 \xrightarrow{i_1} H \xleftarrow{i_2} U_2$$

are the injections.

PROOF. Consider the following pullback diagram

$$(*) \qquad \begin{array}{ccc} H^*([1+p]\times[1+q]) & \xrightarrow{\bar{\phi}} & H^*K \\ \uparrow & \text{P.B.} & \uparrow \\ l & \longrightarrow & H^*T \end{array}$$

where $\bar{\phi}$ is given by the following bijections

$$\frac{\dfrac{H^*(K^{[1+p]}) \xrightarrow{\ \phi\ } H^*(K^{[1+q]}) \quad \text{in } Vect_K(\underline{E})^H}{H^*[1+p] \to H^*(K^{[1+q]}) \quad \text{in } \underline{E}/H} \approx}{H^*([1+p]\times[1+q]) \xrightarrow{\ \bar{\phi}\ } H^*K \quad \text{in } \underline{E}/H} \approx \ .$$

In this diagram H^*T is a complemented subobject of H^*K so l is a complemented subobject of $H^*([1+p]\times[1+q])$, which is therefore a finite cardinal in $\underline{E}/H$ [JN 1]. Hence there exist

$$U_1 \xrightarrow{\ i_1\ } H \xleftarrow{\ i_2\ } U_2 \qquad \text{such that } U_1 + U_2 = H,$$

$i_1^* l$ non-zero in $\underline{E}/U_1$ and $i_2^* l$ is zero in $\underline{E}/U_2$ ([JN 1] Chapter 6), i.e., $i_1^* l$ has global support. Since $i_1^*\bar{\phi} = \overline{i_1^*\phi}$ so by applying i_1^* to (*) we get $i_1^*\phi$ non-zero (by Definition (1.4)). □

(1.6) LEMMA. *Let $\phi : K^{[1+p]} \to K^{[1+q]}$ be a non-zero linear transformation. Then there exist invertible homomorphisms*

$$P : K^{[1+p]} \to K \oplus K^{[p]} \quad \textit{and} \quad Q : K^{[1+q]} \to K \oplus K^{[q]}$$

and a homomorphism $f : K^{[p]} \to K^{[q]}$ such that

$$\begin{array}{ccc} K^{[1+p]} & \xrightarrow{\qquad \phi \qquad} & K^{[1+q]} \\ P \downarrow \approx & & \approx \downarrow Q \\ K \oplus K^{[p]} & \xrightarrow{\ \begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix}\ } & K \oplus K^{[q]} \end{array}$$

commutes.

PROOF. ϕ being non-zero means that the object l in the following diagram

$$\begin{array}{ccc} [1+p]\times[1+q] & \xrightarrow{\qquad \bar{\phi} \qquad} & K \\ \uparrow & \text{P.B.} & \uparrow \\ l & \xrightarrow{\qquad\qquad} & T \end{array}$$

has global support. Since l is a finite cardinal then it has a global element, i.e. we have $(i,j) : 1 \to [1+p]\times[1+q]$ such that $\bar{\phi}(i,j)$ is a unit. Apply i^* and j^* to the morphisms

$$\Lambda_p : [1+p]^*[1+p] \to [1+p]^*[1+p]$$

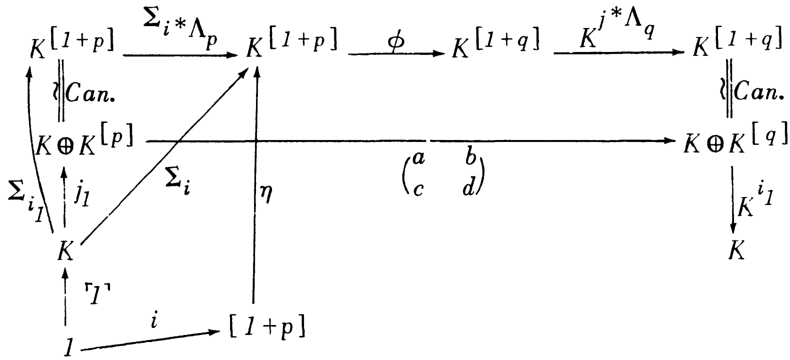
and

$$\Lambda_q : [1+q]*[1+q] \to [1+q]*[1+q]$$

(defined in Lemma (1.3)) to get

$$i^*\Lambda_p : [1+p] \to [1+p] \quad \text{and} \quad j^*\Lambda_q : [1+q] \to [1+q],$$

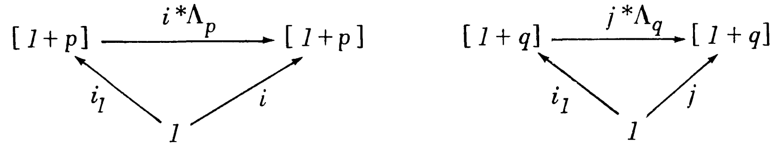
respectively. Now consider the following diagram



In this diagram $\Sigma_{(-)}$ is the functor defined in [TV 2] and j_1, i_1 are the injections. But

$$\Sigma_{i_1} = (K \xrightarrow{j_1} K \oplus K^{[p]} \underset{\approx}{\text{Can.}} K^{[1+p]})$$

and the diagrams



are commutative (Lemma (1.3)). Therefore

$$\Sigma_{i^*\Lambda_p i_1} = \Sigma_{i^*\Lambda_p} \Sigma_{i_1} = \Sigma_i \quad \text{and} \quad K^{i_1} K^{j^*\Lambda_q} = K^{i_1\, j^*\Lambda_q} = K^j .$$

Also, by definition of Σ_i, we have $\eta i = \Sigma_i \ulcorner 1 \urcorner$. Hence the transpose of

$$1 \xrightarrow{i} [1+p] \xrightarrow{\eta} K^{[1+p]} \xrightarrow{\phi} K^{[1+q]} \xrightarrow{K^j} K,$$

which is

$$1 \xrightarrow{j} [1+q] \xrightarrow{i \times [1+q]} [1+q] \times [1+p] \xrightarrow{\bar{\phi}} K,$$

factors through $T \rightarrowtail K$, since ϕ is non-zero; i.e. a is a unit. Hence we have the following diagram

$$
\begin{array}{ccc}
K\oplus K^{[p]} & \xrightarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} & K\oplus K^{[q]} \\
{\scriptstyle \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}} \uparrow & \nearrow {\scriptstyle \begin{pmatrix} 1 & b \\ -a^{-1}c & d \end{pmatrix}} & \\
K\oplus K^{[p]} & & \downarrow {\scriptstyle \begin{pmatrix} 1 & 0 \\ -a^{-1}c & 1 \end{pmatrix}} \\
{\scriptstyle \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}} \uparrow & & \\
K\oplus K^{[p]} & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & -ba^{-1}c+d \end{pmatrix}} & K\oplus K^{[q]}
\end{array}
$$

By letting

$$P^{-1} = \Sigma_{i^*\Lambda_p} Can.\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \\ -a^{-1}c & 1 \end{pmatrix} Can.K^{j^*\Lambda_q}$$

and $f = -ba^{-1}c + d$ we get the required result. □

(1.7) LEMMA. *Let p, $q: 1 \to N$ be such that there exist $U_1 \amalg U_2 = 1$ such that $U_1^* p = 0$ and $U_2^* q = 0$ (i.e. $\models p = 0 \vee q = 0$). Then*

$$Hom(K^{[p]}, K^{[q]}) = 1,$$

i.e. the only morphism $K^{[p]} \to K^{[q]}$ is the zero morphism.

PROOF.

$$U_1^* Hom(K^{[p]}, K^{[q]}) \approx Hom(U_1^* K^{U_1^*[p]}, U_1^* K^{U_1^*[q]}) \approx$$

$$Hom(0, U_1^* K^{U_1^*[q]}) \approx 1;$$

$$U_2^* Hom(K^{[p]}, K^{[q]}) \approx Hom(U_2^* K^{U_2^*[p]}, U_2^* K^{U_2^*[q]}) \approx$$

$$Hom(U_2^* K^{U_2^*[p]}, 0) \approx 1.$$

Since $\underline{E}/U_1 \times \underline{E}/U_2 \approx \underline{E}$, then $Hom(K^{[p]}, K^{[q]}) = 1$. □

(1.8) THEOREM. *Let $\bar{p}$, $\bar{q}$ be any two natural numbers in $\underline{E}$. Then every linear transformation $U: K^{[\bar{p}]} \to K^{[\bar{q}]}$ is equivalent to one in normal form.*

PROOF. We will prove this by induction. Let p and q be natural numbers in $\underline{E}$ satisfying the condition of Lemma (1.7). In what follows the constant natural numbers 1^*p and 1^*q are also denoted by p and q, respectively. E. g. in the definition of i_1 below, $B^*K^{[b+p]}$ means $B^*K^{[b+B^*p]}$. Consider

$$\begin{array}{ccc} N^*N \times N^*N \approx N^*(N\times N) & \xrightarrow{\quad N^*a \quad} & N^*N \\ \uparrow & \text{PB} & \uparrow n \\ b & \longrightarrow & 1 \end{array}$$

in $\underline{E}/N$. Then $b: \Sigma_N b = B \to N$ is the object consisting of all (r, l) such that $r+l = n$, i.e. if $m: 1 \to N$ then

$$\frac{(m) \longrightarrow (b)}{1 \xrightarrow{(r,l)} N\times N, \quad r+l = m} \approx \quad .$$

Let r_0 and l_0 be the generic natural numbers such that $r_0 + l_0 = b$. Let

$$i_1 = Iso^B(B^*K^{[b+p]}, B^*K^{[r_0]} \oplus B^*K^{[l_0+p]})$$

and

$$i_2 = Iso^B(B^*K^{[b+q]}, B^*K^{[r_0]} \oplus B^*K^{[l_0+q]}) \ .$$

Consider $d = \Sigma_b i_1 \underset{B}{\times} i_2$ in $\underline{E}/N$ which has the following universal property. If $m: 1 \to N$ then

$$\frac{\dfrac{(m) \longrightarrow (d)}{(m) \xrightarrow{a} (b) \text{ and } (a) \longrightarrow i_1 \underset{B}{\times} i_2} \approx}{\begin{array}{l} 1 \xrightarrow{(r,l)} N\times N,\ r+l = m \text{ and } \theta_1 : 1^*K^{[m+p]} \xrightarrow{\approx} 1^*K^{[r]} \oplus 1^*K^{[l+p]} \\ \qquad\qquad\qquad\qquad \text{and } \theta_2 : 1^*K^{[m+q]} \xrightarrow{\approx} 1^*K^{[r]} \oplus 1^*K^{[l+q]} \end{array}} \approx \ .$$

There is a morphism $\pi: (d) \to (h)$ given by the following natural transformations. If $m: 1 \to N$:

$$\frac{\dfrac{\dfrac{(m) \longrightarrow (d)}{1^*K^{[m+p]} \xrightarrow[\approx]{\theta_1} 1^*K^{[r]} \oplus 1^*K^{[l+p]} \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} 1^*K^{[r]} \oplus 1^*K^{[l+q]} \xrightarrow[\approx]{\theta_2^{-1}} 1^*K^{[m+q]}} \approx}{m^*(N^*K^{[n+p]}) \xrightarrow{\theta_2^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \theta_1} m^*(N^*K^{[n+q]})} \approx}{(m) \longrightarrow (h)} \ .$$

It suffices to show that π is a split epimorphism. Consider

$$\begin{array}{ccc} t = split^N(\pi) & \longrightarrow & (d)^{(h)} \\ \downarrow & \text{PB} & \downarrow \pi^{(h)} \\ 1 & \longrightarrow & (h)^{(h)} \end{array}$$

in $\underline{E}/N$. We want to show $t = split^N(\pi)$ has a global element.

$$0^*h = 0^*Hom(N^*K^{[n+p]}, N^*K^{[n+q]}) \approx Hom(K^{[p]}, K^{[q]}) = 1$$

(Lemma (1.7)). Also we get a global element for 0^*d by the following

$$\frac{\dfrac{1 \xrightarrow{(0,0)} N\times N,\ \theta_1 = 1_{K^{[p]}} : K^{[p]} = K^{[p]},\ \theta_2 = 1_{K^{[q]}} : K^{[q]} = K^{[q]}}{(0) \longrightarrow (d)} \approx}{1 \longrightarrow 0^*(d)} \approx .$$

But since

$$\begin{array}{ccc} 0^*split^N(\pi) \approx split(0^*\pi) & \longrightarrow & 0^*(d) \approx 0^*(d)^{0^*(h)} \\ \downarrow & & \downarrow {\scriptstyle 0^*\pi^{0^*(h)}} \\ 1 & = & 1 \approx 0^*(h)^{0^*(h)} \end{array}$$

is a pullback then we have a global element $1 \to 0^*split^N(\pi)$. We need only to show that there is a morphism

$$t = split^N(\pi) \to s^*t = s^*split^N(\pi).$$

To do this it is enough to show there exists a morphism $\gamma : t \times s^*h \to s^*d$ such that $(s^*\pi)\gamma$ is the projection, because if we have such a map then

$$(t \xrightarrow{\bar{\gamma}} s^*d^{s^*h} \xrightarrow{s^*\pi^{s^*h}} s^*h^{s^*h}) = (t \to 1 \to s^*h^{s^*h}),$$

i.e. we get $t \to s^*t$. For simplicity of notation we denote the functor

$$(t)\times(\ ) : \underline{E}/N \to \underline{E}/N$$

by $(\bar{\ })$. Let Φ be the generic homomorphism for

$$s^*h = Hom(N^*K^{[1+n+p]}, N^*K^{[1+n+q]}).$$

Let $\Sigma_N s^*h = A$, then by Lemma (1.5) there exists $\mu_1 + \mu_2 = s^*h$ such that Φ is non-zero on μ_1 and it is zero on μ_2. If

$$\begin{array}{ccc} U_1 & \overset{a_1}{\rightarrowtail} & A \\ & {\scriptstyle \mu_1} \searrow \quad \swarrow {\scriptstyle s^*h} & \\ & N & \end{array}$$

is the injection then, by Lemma (1.6), there exist homomorphisms P, Q, and f such that P and Q are invertible and

$$(*)\qquad \begin{array}{ccc} (U_1^*K)^{[1+\mu_1+p]} & \xrightarrow{a_1^*\Phi} & U_1^*K^{[1+\mu_1+q]} \\ P\downarrow\approx & & \approx\downarrow Q \\ U_1^*K\oplus U_1^*K^{[\mu_1+p]} & \xrightarrow{\begin{pmatrix}1&0\\0&f\end{pmatrix}} & U_1^*K\oplus U_1^*K^{[\mu_1+q]} \end{array}$$

commutes. The homomorphism f gives us a morphism $\rho:(\mu_1)\to(h)$ such that $\rho^*\phi = f$, where ϕ is the generic homomorphism for h. On the other hand by definition of t we have

$$\frac{(\overline{h}) \longrightarrow (d)}{\overline{H}\xrightarrow{(r,l)} N\times N,\ r+l=\bar{h},\ \theta_1:\overline{H}^*K^{[\bar{h}+p]}\xrightarrow{\approx}\overline{H}^*K^{[r]}\oplus\overline{H}^*K^{[l+p]}\ \approx}$$

and $\theta_2:\overline{H}^*K^{[\bar{h}+q]}\xrightarrow{\approx}\overline{H}^*K^{[r]}\oplus\overline{H}^*K^{[l+p]}$

such that

$$\theta_2^{-1}\begin{pmatrix}1&0\\0&0\end{pmatrix}\theta_1:\overline{H}^*K^{[\bar{h}+p]}\to\overline{H}^*K^{[\bar{h}+q]}$$

is $\pi_2^*\phi$ where $\pi_2:(\overline{h})\to(h)$ is the projection (because by definition of $\pi:(d)\to(h)$ and the fact that $(\overline{h})\to(d)\xrightarrow{\pi}(h)$ is the projection π_2). Now apply $\bar{\rho}^*$ to $\theta_2^{-1}\begin{pmatrix}1&0\\0&0\end{pmatrix}\theta_1=\pi_2^*\phi$ to get

$$\bar{\rho}^*(\theta_2^{-1}\begin{pmatrix}1&0\\0&0\end{pmatrix}\theta_1)=\bar{\rho}^*\pi_2^*\phi .$$

Also by pulling back the diagram (*) along the projection $\pi_2':(\overline{\mu_1})\to(\mu_1)$, we get

$$\begin{array}{ccc} \overline{U}_1^*K^{[1+\bar{\mu}_1+p]} & \xrightarrow{\pi_2'^*a_1^*\Phi} & \overline{U}_1^*K^{[1+\bar{\mu}_1+q]} \\ P'=\bar{\mu}_1^*P\ \downarrow\approx & & \approx\downarrow \bar{\mu}_1^*Q=Q' \\ \overline{U}_1^*K\oplus\overline{U}_1^*K^{[\bar{\mu}_1+p]} & \xrightarrow{\begin{pmatrix}1&0\\0&\pi_2'^*f\end{pmatrix}} & \overline{U}_1^*K\oplus\overline{U}_1^*K^{[\bar{\mu}_1+q]} \end{array} .$$

But

$$\begin{array}{ccc} (\overline{\mu_1}) & \xrightarrow{\bar{\rho}} & (\overline{h}) \\ \pi_2'\downarrow & & \downarrow\pi_2 \\ (\mu_1) & \xrightarrow{\rho} & (h) \end{array}$$

commutes, so

$$\pi_2'^*f = \pi_2'^*\rho^*\phi = \bar{\rho}^*\pi_2^*\phi = \bar{\rho}^*(\theta_2^{-1}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\theta_1) .$$

On the other hand

$$(r+l)\bar{\rho} = r\bar{\rho} + l\bar{\rho} = (\bar{h})\bar{\rho} = \bar{\mu}_1 .$$

Therefore $1+r\bar{\rho}+l\bar{\rho} = 1+\bar{\mu}_1$ and so we have the following natural isomorphism

$$\bar{U}_1 \xrightarrow{(1+r\bar{\rho},\, l\bar{\rho})} N\times N \text{ such that } 1+r\bar{\rho}+l\bar{\rho} = 1+\bar{\mu}_1 ,$$

$$\tilde{\theta}_1 = (1\oplus\bar{\rho}^*\theta_1)P' : \bar{U}_1^* K^{[1+\bar{\mu}_1+p]} \xrightarrow{\approx} \bar{U}_1^*K\oplus\bar{U}_1^*K^{[r\bar{\rho}]}\oplus\bar{U}_1^*K^{[l\bar{\rho}+p]} \approx$$
$$\approx \bar{U}_1^*K^{[1+r\bar{\rho}]}\oplus\bar{U}_1^*K^{[l\bar{\rho}+p]}$$

and

$$\hat{\theta}_2 = (1\oplus\bar{\rho}^*\theta_2)\, Q' : \bar{U}_1^*K^{[1+\bar{\mu}_1+q]} \xrightarrow{\approx} \bar{U}_1^*K\oplus\bar{U}_1^*K^{[r\bar{\rho}]}\oplus\bar{U}_1^*K^{[l\bar{\rho}+q]}$$
$$\approx \bar{U}_1^*K^{[1+r\bar{\rho}]}\oplus\bar{U}_1^*K^{[l\bar{\rho}+q]}$$

$$\frac{\quad}{(1+\bar{\mu}_1) \longrightarrow (d)}\ \approx$$
$$\frac{\quad}{\Sigma_s(\overline{\mu_1}) \longrightarrow (d)}\ \approx$$
$$(\overline{\mu_1}) \xrightarrow{\nu_1} s^*(d)$$

such that

$$\begin{array}{ccc} (\overline{\mu_1}) & \xrightarrow{\nu_1} & s^*(d) \\ \downarrow{\scriptstyle \pi_2'} & & \downarrow{\scriptstyle s^*\pi} \\ (\mu_1) & \xrightarrow{a_1} & s^*(h) \end{array}$$

commutes, because Φ is the generic homomorphism and

$$\pi_2'^*a_1^*\Phi = (\tilde{\theta}_2)^{-1}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}(\tilde{\theta}_1)^{-1} .$$

Also we have a morphism $(\overline{\mu_2}) \to s^*(d)$ defined as follows:

$$(0, 1+\bar{\mu}_2) : \bar{U}_2 \to N\times N \quad \text{and}$$

$$\xi_1 : \bar{U}_2^*K^{[1+\bar{\mu}_2+p]} \xrightarrow[\text{Can iso}]{1\approx} \bar{U}_2^*K^{[0]}\oplus\bar{U}_2^*K^{[1+\bar{\mu}_2+p]} ,$$

$$\xi_2 : \bar{U}_2^*K^{[1+\bar{\mu}_2+q]} \xrightarrow[\text{Can iso}]{1\approx} \bar{U}_2^*K^{[0]}\oplus\bar{U}_2^*K^{[1+\bar{\mu}_2+q]}$$
$$\overline{\qquad\qquad}\ \approx$$

$$\frac{\dfrac{(1+\bar{\mu}_2) \longrightarrow (d)}{\Sigma_s(\overline{\mu_2}) \longrightarrow (d)}}{(\overline{\mu_2}) \xrightarrow{\nu_2} s^*(d)} \quad \begin{matrix}\approx \\ \approx\end{matrix} \quad .$$

Since the 1 in $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ is the identity on $\bar{U}^*K^{[0]} \to \bar{U}^*K^{[0]}$, which is equal to zero, and Φ is zero on μ_2 we have

$$\xi_2^{-1}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\xi_1 = \xi_2^{-1}\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\xi_1 = \pi_2''^{*}a_2^{*}\Phi = 0$$

where $\bar{\pi}_2'': (\overline{\mu_2}) \to (\mu_2)$ is the projection. Hence by definition of π

$$\begin{array}{ccc} (\overline{\mu_2}) & \xrightarrow{\nu_2} & s^*(d) \\ {\scriptstyle \pi_2''}\downarrow & & \downarrow{\scriptstyle s^*\pi} \\ (\mu_2) & \xrightarrow{a_2} & s^*(h) \end{array}$$

commutes. So we have a morphism

$$\overline{s^*(h)} = (\overline{\mu_1}) + (\overline{\mu_2}) \xrightarrow{\gamma = \binom{\nu_1}{\nu_2}} s^*(d)$$

such that $(s^*\pi)\gamma : \overline{s^*(h)} \to s^*(h)$ is the projection.

Now let $\bar{p}$, $\bar{q}$ be any two natural numbers in $\underline{E}$. There exists $T_1 + T_2 = 1$ such that $\bar{p} \leq \bar{q}$ on T_1 and $\bar{p} > \bar{q}$ on T_2, i.e. there are natural numbers r_1 and r_2 such that

$$T_1^*\bar{q} = r_1 + T_1^*\bar{p} \quad \text{and} \quad T_2^*\bar{p} = r_2 + T_2^*\bar{q}.$$

Consider two natural numbers

$$1 = T_1 + T_2 \quad \begin{matrix} \xrightarrow{\bar{r}_1 = \binom{r_1}{0}} \\ \xrightarrow[\bar{r}_2 = \binom{0}{r_2}]{} \end{matrix} \quad N$$

and a natural number

$$\bar{l}: 1 = T_1 + T_2 \xrightarrow{\binom{T_1^*\bar{p}}{T_2^*\bar{q}}} N \ .$$

Then $\bar{l} + \bar{r}_1 = \bar{q}$ and $\bar{l} + \bar{r}_2 = \bar{p}$ (we can interpret $\bar{l}$ as the minimum value of $\bar{p}$ and $\bar{q}$). But $\bar{r}_1$, $\bar{r}_2$ satisfy the condition of Lemma (1.7) because $T_1^*\bar{r}_2 = 0$ and $T_2^*\bar{r}_1 = 0$, so if we apply $\bar{l}^*$ to the above argument we get the result, i.e. every linear transformation $U: K^{[\bar{p}]} \to K^{[\bar{q}]}$ is equivalent to one in normal form. This completes the proof. □

(1.9) COROLLARY. *Any monomorphism* $\phi: K^{[p]} \rightarrowtail K^{[p]}$ *is an isomorphism.*

PROOF. By Theorem (1.8) there are natural numbers r, p_1, p_2 such that

$$\begin{array}{ccc} K^{[p]} & \overset{\phi}{\rightarrowtail} & K^{[p]} \\ \theta_1 \| \wr & & \wr \| \theta_2 \\ K^{[r]} \otimes K^{[p_1]} & \xrightarrow{\psi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} & K^{[r]} \oplus K^{[p_2]} \end{array}$$

commutes and $r + p_1 = p = r + p_2$. Since ϕ is mono, then ψ is. Hence the kernel of ψ, i.e. $K^{[p_1]}$, is the 0 vector space, i.e. $p_1 = 0$; but $p_1 = p_2$, so $p_2 = 0$. This means ψ is an identity on $K^{[r]}$ which implies that ϕ is an isomorphism. □

(1.10) COROLLARY. *If* W *is any vector space which satisfies*

$$Iso(0, W) \approx 0, \quad \text{then} \quad Mon(K^{[p]} \oplus W, K^{[p]}) \approx 0$$

for any natural number p.

PROOF. It is obvious that we have the following pullback diagram

$$\begin{array}{ccc} 0 & \rightarrowtail & W \\ \downarrow & \text{PB} & \downarrow \\ K^{[p]} & \overset{i}{\rightarrowtail} & K^{[p]} \otimes W. \end{array}$$

Given any I-element of $Mon(K^{[p]} \otimes W, K^{[p]})$, we have

$$\frac{I \longrightarrow Mon(K^{[p]} \oplus W, K^{[p]}) \text{ in } \underline{E}}{I^*(K^{[p]}) \oplus I^*W \overset{\phi}{\rightarrowtail} I^*K^{[p]} \text{ in } Vect_K(\underline{E})^I} \approx .$$

Then we have the monomorphism

$$I^*(K^{[p]}) \overset{I^*i}{\rightarrowtail} I^*(K^{[p]}) \oplus I^*W \overset{\phi}{\rightarrowtail} I^*(K^{[p]})$$

in $Vect_K(\underline{E})^I$ which is an isomorphism, by Corollary (1.9). Therefore ϕ is an isomorphism, i.e. I^*i is an isomorphism. Now apply I^* to the above diagram to get $0 \approx I^*W$ in $Vect_K(\underline{E})^I$, which is equivalent to

$$I \to Iso(0, W) \approx 0 \; ; \quad \text{i.e. } I \approx 0 .$$

Therefore $Mon(K^{[p]} \oplus W, K^{[p]}) \approx 0$ (let $I = Mon(K^{[p]} \oplus W, K^{[p]})$ and ϕ to be a generic monomorphism). □

(1.11) PROPOSITION. *If* $\phi : K^{[p]} \rightarrowtail K^{[q]}$ *is a monomorphism then* $p \leq q$.

PROOF. Let $U_1 + U_2 = 1$ in $\underline{E}$ such that $p \leq q$ on U_1 and $p > q$ on U_2. Then in $\underline{E}/U_2$ there exists a natural number r such that $U_2^* p = r + U_1^* q$, and so

$$Mon(U_2^* K^{[U_2^* p]}, U_2^* K^{[U_2^* q]}) \approx Mon((U_2^* K)^{[U_2^* q]} \oplus (U_2^* K)^{[r]}, (U_2^* K)^{[U_2^* q]})$$

has a global element $U_2^* \phi$, which is impossible by Corollary (1.10), unless $r = 0$. Hence $U_2 = 0$, i.e. $p \leq q$. □

(1.12) PROPOSITION. *If* $K^{[p]} \approx K^{[q]}$ *then* $p = q$.

PROOF. Let $\phi : K^{[p]} \to K^{[q]}$ be the isomorphism, then by Theorem (1.8) there are natural numbers r, p', q' such that $p = r + p'$, $q = r + q'$ and

$$\begin{array}{ccc} K^{[p]} & \xrightarrow{\quad \phi \quad} & K^{[q]} \\ \theta_1 \Big\| \wr & & \wr \Big\| \theta_2 \\ K^{[r]} \oplus K^{[p']} & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \psi} & K^{[r]} \oplus K^{[q']} \end{array}$$

commutes. ϕ is an isomorphism implies ψ is, so the kernel of ψ is the zero vector space, i.e. $K^{[p']} \approx 0$, so $p' = 0$. On the other hand since the image of ψ is $K^{[r]}$, then $K^{[r]} \approx K^{[r]} \oplus K^{[q']}$ and, by Corollary (1.10) $q' = 0$ so $r = p = q$. □

(1.13) COROLLARY. *Every epimorphism* $\phi : K^{[p]} \twoheadrightarrow K^{[p]}$ *is an isomorphism.*

PROOF. It is easy to see that finite cardinals are internally projective, see [JN1], and therefore locally projective. Thus ϕ splits locally, i.e. there exists $I \twoheadrightarrow 1$ such that $I^*\phi$ splits. This means there is a mono m in $Vect_{I^*K}(\underline{E})^I$ such that $I^*\phi . m = 1_{I^*(K^{[p]})}$; by Corollary (1.9), m is an isomorphism. Therefore $I^*\phi$ is an iso. Since I^* reflects isomorphisms, then ϕ is an isomorphism. □

(1.14) COROLLARY. *If $Iso(0, W) \approx 0$ for W a vector space in $\underline{E}$, then $epi(K^{[p]}, W \oplus K^{[p]}) \approx 0$ for any finite cardinal $[p]$ in E.*

PROOF. The proof is similar to Corollary (1.10). □

NOTE. Corollary (1.14) shows that if $K^{[p]} \to K^{[q]}$ is an epimorphism, then $q \leq p$. □

(1.15) COROLLARY. *Let V be a locally finite dimensional (l.f.d.) vector space and $\phi : V \to V$ be a linear transformation.*

(i) If ϕ is mono, then it is an isomorphism.

(ii) If ϕ is epi, then it is an isomorphism.

PROOF. By definition of l.f.d. there exists

$$I \twoheadrightarrow 1 \quad \text{such that} \quad I^*V = (I^*K)^{[p]} \quad \text{in} \quad Vect_{I^*K}(\underline{E})^I .$$

(i) If ϕ is mono then $I^*\phi : (I^*K)^{[p]} \rightarrowtail (I^*K)^{[p]}$ is also mono; then by Corollary (1.9) $I^*\phi$ is an isomorphism. Hence ϕ is an isomorphism.

(ii) If ϕ is epi, then $I^*\phi : (I^*K)^{[p]} \to (I^*K)^{[p]}$ is epi. By Corollary (1.13), $I^*\phi$ is an isomorphism. Therefore ϕ is an isomorphism. □

2. RANK OF A LINEAR TRANSFORMATION.

Let $\phi : K^{[p]} \to K^{[q]}$ be a linear transformation. Then there exist natural numbers r, p' and q' such that $p = r + p'$, $q = r + q'$ and

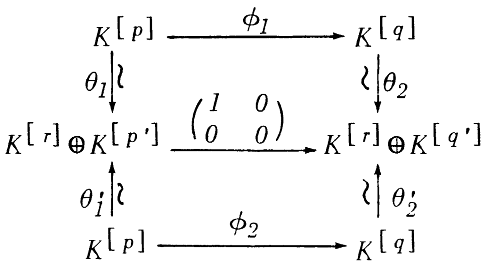
$$\begin{array}{ccc} K^{[p]} & \xrightarrow{\quad \phi \quad} & K^{[q]} \\ \Vert \wr & & \wr \Vert \\ K^{[r]} \oplus K^{[p']} & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} & K^{[r]} \oplus K^{[q']} \end{array}$$

(Theorem (1.8)). This shows that the image of ϕ is $K^{[r]}$. If the image of ϕ is also $K^{[r']}$ for some natural number r', then $K^{[r]} \approx K^{[r']}$ and so $r = r'$ (by Proposition (1.12)), i.e. r is the unique natural number with the above property.

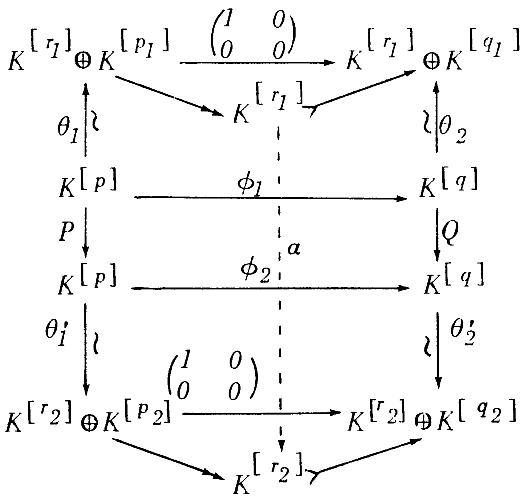
(2.1) DEFINITION. Let $\phi : K^{[p]} \to K^{[q]}$ be a linear transformation from $K^{[p]}$ to $K^{[q]}$ in $\underline{E}$. Then the natural number $r: 1 \to N$, which is given above, is called the *rank of* ϕ and is denoted by $r(\phi)$.

(2.2) THEOREM. *Let* $K^{[p]} \underset{\phi_2}{\overset{\phi_1}{\rightrightarrows}} K^{[q]}$ *be two linear transformations in* $\underline{E}$. *Then* $r(\phi_1) = r(\phi_2)$ *iff there exist two invertible linear transformations* $P : K^{[p]} \to K^{[p]}$ *and* $Q : K^{[q]} \to K^{[q]}$ *such that* $Q\phi_1 = \phi_2 P$.

PROOF. Suppose $r(\phi_1) = r(\phi_2) = r$. Then, by Definition (2.1) the following is commutative



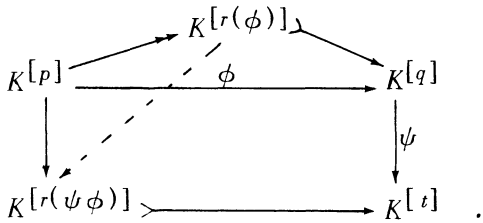
Now by taking $P = \theta_1'^{-1} . \theta_1$ and $Q = \theta_2'^{-1} . \theta_2$ we are done. Conversely, suppose there are invertible linear transformations P and Q such that $Q\phi_1 = \phi_2 P$. Apply Definition (2.1) to ϕ_1 and ϕ_2 to get the following diagram



where $r_1 = r(\phi_1)$ and $r_2 = r(\phi_2)$. By the diagonal lemma there exists a unique homomorphism $a: K^{[r_1]} \to K^{[r_2]}$ such that the resulting diagrams commute. Therefore a is an isomorphism and then $r_1 = r_2$, by Proposition (1.12). □

(2.3) COROLLARY. *Let $\phi: K^{[p]} \to K^{[q]}$ and $\psi: K^{[q]} \to K^{[t]}$ be two linear transformations. Then $r(\psi \cdot \phi) \leq r(\phi)$ and $r(\psi \cdot \phi) \leq r(\psi)$.*

PROOF. Apply Definition (2.1) to ϕ and $\psi\phi$ to get a commutative diagram

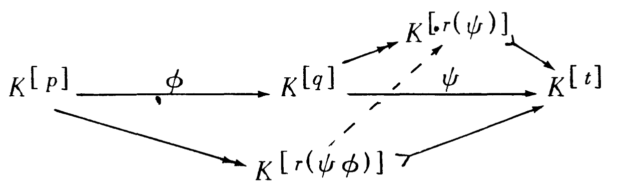


By the Diagonal Lemma there exists a unique homomorphism

$$K^{[r(\phi)]} \to K^{[r(\psi\phi)]}$$

such that the resulting diagrams commute, and is an epimorphism. So by the Note after Corollary (1.14) $r(\psi\phi) \leq r(\phi)$.

Now apply Definition (2.1) to ψ and $\psi\phi$ to get a commutative diagram



Then by the diagonal lemma there exists a unique homomorphism

$$K^{[r(\psi\phi)]} \to K^{[r(\psi)]}$$

such that the resulting diagrams commute, and is a monomorphism. So, by Proposition (1.11), $r(\psi\phi) \leq r(\psi)$. □

(2.4) THEOREM. *Any finite dimensional subspace of a finite dimensional vector space has a finite dimensional complement.*

PROOF. Let $K^{[p]}$ be a finite dimensional subspace of $K^{[q]}$, i.e. there is a monomorphism $\phi: K^{[p]} \rightarrowtail K^{[q]}$. Apply Theorem (1.8) to get

$$\begin{array}{ccc} K^{[p]} & \xrightarrow{\quad\phi\quad} & K^{[q]} \\ \| \wr & & \wr \| \\ K^{[r]} \oplus K^{[p_1]} & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} & K^{[r]} \oplus K^{[q_1]} \end{array}$$

where $r = r(\phi)$. Since ϕ is monomorphism, then $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is monomorphism. So $K^{[p_1]} \approx 0$, i.e. $p_1 = 0$. Therefore

$$K^{[p]} \approx K^{[r]} \text{ and } K^{[p]} \oplus K^{[q_1]} \approx K^{[q]}. \quad \Box$$

(2.5) DEFINITION. Let I be an object of $\underline{E}$. A vector space V in $Vect_K(\underline{E})^I$ is said to be an *I-family of locally finite dimensional vector spaces* if there exist $\alpha : J \twoheadrightarrow I$ and a natural number $p : J \to N$ in $\underline{E}/J$ such that $\alpha^* V \approx (J^* K)^{[p]}$ in $Vect_K(\underline{E})^J$.

(2.6) THEOREM. *Let V be an I-family of locally finite dimensional vector spaces. Then there exists a unique morphism $p' : I \to N$, such that $p'\alpha = p$, where α and p are given above.*

PROOF. Let

$$J' \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} J \xrightarrow{\quad\alpha\quad} I$$

be the kernel pair of α. Then we have

$$\begin{array}{l} \pi_1^* \alpha^* V \approx \pi_1^*((J^*K)^{[p]}) \approx (J'^*K)^{[p\pi_1]} \\ \quad \| \wr \\ \pi_2^* \alpha^* V \approx \pi_2^*((J^*K)^{[p]}) \approx (J'^*K)^{[p\pi_2]} \end{array} .$$

Therefore $(J'^*K)^{[p\pi_1]} \approx (J'^*K)^{[p\pi_2]}$ in $Vect_K(\underline{E})^{J'}$ and, by Proposition (1.12), $p\pi_1 = p\pi_2$. Since α is a coequalizer of (π_1, π_2), hence there exists a unique morphism $p' : I \to N$ such that

$$\begin{array}{ccccc} J' & \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} & J & \xrightarrow{\alpha} & I \\ & & & \searrow^{p} & \downarrow{p'} \\ & & & & N \end{array}$$

commutes, i.e. $p'\alpha = p$. $\Box$

(2.7) DEFINITION. The natural number $p': I \to N$ given in Theorem (2.6) is called the *dimension of* V and is denoted by $dim(V)$. In particular, if $I = 1$, i.e. if V is locally finite dimensional, then V has a dimension, namely $p': I \to N$.

(2.8) THEOREM. *Let V and V' be locally finite dimensional vector spaces such that V is a subspace of V', i.e. there is a monomorphism $\phi: V \rightarrowtail V'$. If $dim(V) = dim(V')$, then $V \approx V'$.*

PROOF. Let

$$I^*V = (I^*K)^{[p]} \text{ in } Vect_{I^*K}(\underline{E})^I, \text{ where } a: I \twoheadrightarrow 1,$$
$$J^*V' = (J^*K)^{[q]} \text{ in } Vect_{J^*K}(\underline{E})^J, \text{ where } \beta: J \twoheadrightarrow 1.$$

By definition of dimension we have

$$\begin{array}{ccc} I & \xrightarrow{a} & 1 \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & N & \end{array} \quad \text{and} \quad \begin{array}{ccc} J & \xrightarrow{\beta} & 1 \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle q'} & \\ & N & \end{array}$$

where $p' = dim(V)$ and $q' = dim(V')$. Since $q' = p'$, by assumption, then we have

$$\begin{array}{ccccc} I\times J & \xrightarrow{\pi_1} & I & \xrightarrow{a} & 1 \\ {\scriptstyle \pi_2}\searrow & & {\scriptstyle p}\searrow & & \swarrow{\scriptstyle p'} \\ & J & \xrightarrow{q} & N & \\ & {\scriptstyle \beta}\searrow & & \nearrow{\scriptstyle q'} & \\ & & 1 & & \end{array}$$

which implies $p\pi_1 = q\pi_2$. Apply $(I\times J)^*$ to ϕ to get

$$\begin{array}{ccc} (I\times J)^*V & \overset{(I\times J)^*\phi}{\rightarrowtail} & (I\times J)^*V' \\ \| \wr & & \wr \| \\ \pi_1^*a^*V & \overset{(I\times J)^*\phi}{\rightarrowtail} & \pi_2^*\beta^*V' \\ \| \wr & & \wr \| \\ \pi_1^*((I^*K)^{[p]}) & \overset{(I\times J)^*\phi}{\rightarrowtail} & \pi_2^*((J^*K)^{[q]}) \\ \| \wr & & \wr \| \\ ((I\times J)^*K)^{[p\pi_1]} & \overset{(I\times J)^*\phi}{\rightarrowtail} & ((I\times J)^*K)^{[q\pi_2]} \end{array} .$$

But, by Corollary (1.9), $(I\times J)^*\phi$ is an isomorphism. Then ϕ is an isomorphism. □

(2.9) COROLLARY. *Let $\phi : V \to V'$ be a linear transformation with V and V' locally finite dimensional vector spaces, then*

(i) $dim(V) \leq dim(V')$ if ϕ is mono,

(ii) $dim(V) \geq dim(V')$ if ϕ is epi,

(iii) $dim(V) = dim(V')$ if ϕ is iso.

PROOF. (i) Since V and V' are l.f.d. then there are objects $\alpha : I \twoheadrightarrow 1$, $\beta : J \twoheadrightarrow 1$ such that

$$I^*V = (I^*K)^{[p]} \quad \text{and} \quad J^*V' = (J^*K)^{[q]},$$

where $p : I \to N$, $q : J \to N$ are natural numbers in $\underline{E}/I$ and $\underline{E}/J$, respectively. Suppose ϕ is a monomorphism, then

$$\begin{array}{ccc} (I\times J)^*V & \xrightarrow{(I\times J)^*\phi} & (I\times J)^*V' \\ \| \wr & & \| \wr \\ ((I\times J)^*K)^{[p\pi_1]} & \xrightarrow{(I\times J)^*\phi} & ((I\times J)^*K)^{[q\pi_2]} \end{array}$$

is a monomorphism, where π_i's are the projections. Then, by Proposition (1.11), $p\pi_1 \leq q\pi_2$ as natural numbers in $\underline{E}/I\times J$. If $W \rightarrowtail N\times N$ represents «$\leq$» on N, then

$$\begin{array}{ccc} I\times J & \xrightarrow{(\alpha\pi_1,\, \beta\pi_2)} & 1 \\ \downarrow & \swarrow & \downarrow (p', q') \\ W & \rightarrowtail & N\times N \end{array}$$

commutes, where $dim(V) = p'$, $dim(V') = q'$ are given by

$$\begin{array}{ccc} I & \xrightarrow{\alpha} & 1 \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & N & \end{array} \quad \text{and} \quad \begin{array}{ccc} J & \xrightarrow{\beta} & 1 \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle q'} & \\ & N & \end{array}$$

respectively. Then (p', q') factors through $W \rightarrowtail N\times N$, i.e. $p' \leq q'$.

(ii) The proof is similar to (i).

(iii) With the same notation as in (i), if ϕ is an isomorphism then $(I\times J)^*\phi$ is an isomorphism hence, Proposition (1.12), $p\pi_1 = q\pi_2$ i.e. $(p\pi_1, q\pi_2)$ factors through the diagonal subobject of $N\times N$. Then there is a morphism $1 \to \Delta$ which makes

$$\begin{array}{ccc} I\times J & \xrightarrow{(\alpha\pi_1,\beta\pi_2)} & I \\ \downarrow & \swarrow & \downarrow{\scriptstyle (p',q')} \\ \Delta & \rightarrowtail & N\times N \end{array}$$

commute, i.e. $p' = q'$. □

(2.10) COROLLARY. *Every locally finite dimensional subspace of a l.f.d. vector space is locally complemented.*

PROOF. Let S be a subspace of V. Since S and V are locally finite dimensional, then there exist $J \twoheadrightarrow 1$, $I \twoheadrightarrow 1$, $p: J \to N$ and $q: I \to N$ such that $J^*S \approx (J^*K)^{[p]}$ and $I^*V \approx (I^*K)^{[q]}$. Hence

$$\begin{array}{ccc} (J\times I)^*S & \subset & (J\times I)^*V \\ \| \wr & & \wr \| \\ ((J\times I)^*K)^{[p\pi_1]} & \subset & ((J\times I)^*K)^{[q\pi_2]}, \end{array}$$

where π_i's are the projections. Apply Theorem (2.4) to this subspace (i.e. $(J\times I)^*S \subset (J\times I)^*V$) to get

$$((J\times I)^*K)^{[p\pi_1]} \oplus ((J\times I)^*K)^{[t]} \approx ((J\times I)^*K)^{[q\pi_2]},$$

where $t: J\times I \to N$ is a natural number in $\underline{E}/J\times I$. □

(2.11) THEOREM. *Any complemented subspace of a locally finite dimensional vector space is locally finite dimensional.*

PROOF. Let V be a locally finite dimensional vector space and $V_1 \subset V$ such that $V = V_1 \oplus V_2$, for some vector space V_2. Then there exist $I \twoheadrightarrow 1$ and $p: I \to N$ such that $I^*V \approx (I^*K)^{[p]}$ in $Vect_K(\underline{E})^I$. So we have

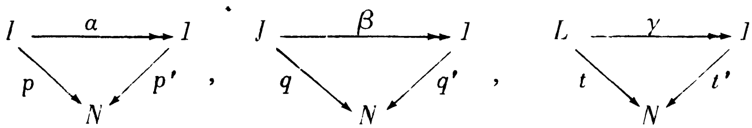
$$I^*V \approx (I^*K)^{[p]} \xrightarrow{\ \pi\ } I^*V_1 \xrightarrow{\ i\ } (I^*K)^{[p]}$$

where π is projection and i is injection. But $i\pi$ is a linear transformation on $(I^*K)^{[p]}$, so the image should be finite dimensional (i.e. $I^*V_1 \approx (I^*K)^{[r]}$, where r is the rank of $i\pi$). Therefore V_1 is locally finite dimensional. □

(2.12) COROLLARY. *Let V, V_1 and V_2 be locally finite dimensional* *vector spaces such that* $V = V_1 \oplus V_2$. *Then*

$$dim(V) = dim(V_1) + dim(V_2).$$

PROOF. Let p', q', t' be the dimensions of V, V_1 and V_2, respectively. Then there exist



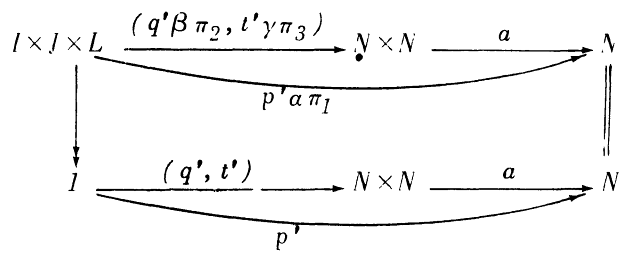
such that

$$I^*V \approx (I^*K)^{[p]}, \quad J^*V_1 \approx (J^*K)^{[q]}, \quad L^*V_2 \approx (L^*K)^{[t]}.$$

Apply $(I\times J\times L)^*$ to $V = V_1 \oplus V_2$ to get

$$\begin{array}{ccc} (I\times J\times L)^*V & = & (I\times J\times L)^*V_1 \oplus (I\times J\times L)^*V_2 \\ \wr\| & & \wr\| \qquad\qquad \wr\| \\ \pi_1^*\alpha^*V & = & \pi_2^*\beta^*V_1 \oplus \pi_3^*\gamma^*V_2 \\ \wr\| & & \wr\| \qquad\qquad \wr\| \\ ((I\times J\times L)^*K)^{[p\pi_1]} & = & ((I\times J\times L)^*K)^{[q\pi_2]} \oplus ((I\times J\times L)^*K)^{[t\pi_3]} \\ & & \wr\| \\ & & ((I\times J\times L)^*K)^{[q\pi_2 + t\pi_3]}, \end{array}$$

where π_i's are the projections. Therefore $p\pi_1 = q\pi_2 + t\pi_3$ (Proposition (1.12)) and by the following commutative diagram



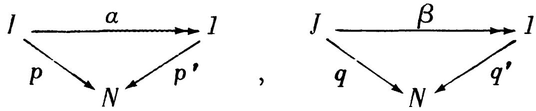
we have $p' = q' + t'$. □

(2.13) PROPOSITION. *Let* V *and* W *be locally finite dimensional vector spaces. Then* $dim(V \otimes W) = (dim(V))(dim(W))$; *for the definition of tensor product, see* [TV1] *Chapter II.*

PROOF. If

$$dim(V) = p' \quad \text{and} \quad dim(W) = q',$$

then there exist



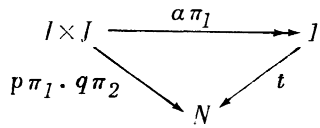
such that $I^*V \approx (I^*K)^{[p]}$ and $J^*W \approx (J^*K)^{[q]}$. Hence

$$(I\times J)^*V \approx ((I\times J)^*K)^{[p\pi_1]} \quad\text{and}\quad (I\times J)^*W \approx ((I\times J)^*K)^{[q\pi_2]}$$

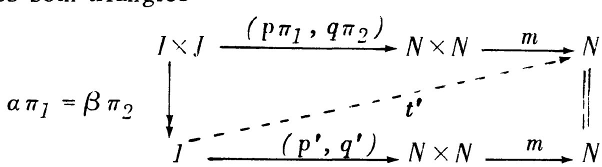
where π_i's are the projections. Since tensor product is preserved by the inverse image of a geometric morphism, then we have

$$\begin{aligned}(I\times J)^*(V\otimes W) &= (I\times J)^*V \otimes (I\times J)^*W \approx \\ &\approx ((I\times J)^*K)^{[p\pi_1]} \otimes ((I\times J)^*K)^{[q\pi_2]}\end{aligned}$$

which is isomorphic to $((I\times J)^*K)^{[p\pi_1\cdot q\pi_2]}$ (see [TV1] Chapter II). Thus there exists a unique $t: 1\to N$ such that



commutes (Definition (2.7)). Also, there exists a unique $t': 1\to N$ which makes both triangles



commute, where m is the multiplication on N (diagonal Lemma). In particular, $t'\alpha\pi_1 = p\pi_1\cdot q\pi_2$. But by uniqueness of t, $t' = t = p'.q'$, i.e., $dim(V\otimes W) = (dim(V))(dim(W))$. □

The next theorem summarizes some of the theorems and corollaries.

(2.14) THEOREM. *Let* $0\to A_1\to A_2\to A_3\to 0$ *be an exact sequence of* K*-vector spaces in* $\underline{E}$.

1. *If* A_1 *and* A_2 *are finite dimensional then* A_3 *is.*
2. *If* A_2 *and* A_3 *are finite dimensional then* A_1 *is.*

3. *If any two are locally finite dimensional, then the other is, and* $dim(A_2) = dim(A_1) + dim(A_3)$.

4. *If* A_1 *and* A_3 *are finite dimensional,* A_2 *is not necessarily.*

The following is an example for (4). Let $K \overset{1}{\underset{1}{\rightrightarrows}} K$ be a geometric field in $Set^{\cdot \rightrightarrows \cdot}$, where K is a field in Set, and let A_2 be

$$K^2 \overset{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}{\underset{\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}}{\rightrightarrows}} K^2 .$$

It is obvious that A_2 is not finite dimensional (but it is locally finite dimensional because if $U = (1 \overset{0}{\underset{1}{\rightrightarrows}} 2)$ is in $Set^{\cdot \rightrightarrows \cdot}$ then $Set^{\cdot \rightrightarrows \cdot}/U \xrightarrow{\simeq} Set^{\cdot \rightrightarrows \cdot}$ and so the image of $U * A_2$ under this equivalence is

$$\begin{array}{ccc} & \nearrow^{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}} & K^2 \\ K^2 & & \\ & \searrow_{\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}} & K^2 \end{array}$$

which is finite dimensional in $Set^{\cdot \rightrightarrows \cdot}$. But we have the following exact sequence in $Set^{\cdot \rightrightarrows \cdot}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{i_1} & K^2 & \xrightarrow{\pi_2} & K & \longrightarrow & 0 \\ \Big\Downarrow & & 1\Big\Downarrow 1 & & \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\Big\Downarrow\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} & & 1\Big\Downarrow 1 & & \Big\Downarrow \\ 0 & \longrightarrow & K & \xrightarrow{i_1} & K^2 & \xrightarrow{\pi_2} & K & \longrightarrow & 0 . \end{array}$$

□

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