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## **Abstract pro arrows I**

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**ABSTRACT PROARROWS I**

by R. J. WOOD

**0. INTRODUCTION.**

In any 2-category  $\mathcal{K}$  one can do a certain amount of «formal category theory». For example one can define extensions, liftings, adjoints, monads, etc... and prove various simple propositions about them. There are now numerous examples in the literature. We draw the reader's attention to Propositions 1 and 2 in [S & W], for they will be used in the present paper. Most such results, with minor modifications, also hold in the more general context of an arbitrary bicategory.

Clearly  $\mathcal{K}$  has to possess additional structure and/or enjoy certain properties in order to capture within it many results which hold in *CAT*. Identification and study of several structures and properties has been carried out in various papers. «Yoneda structures» (Street and Walters [S & W]) «the calculus of modules» (Street [S<sub>1</sub>]) and «elementary cosmoi» (Street [S<sub>2</sub>]) provide examples of what we have in mind. Indeed, this paper leans heavily on the considerations of the above authors.

Let *set* denote the category of sets in some universe and let *SET* denote the category of sets in another universe which contains the first as an element. If  $CAT = cat(SET)$ , then *set* is an object of *CAT*, but *SET* is not. Let  $\mathcal{K} = CAT$  and let *A* and *B* be objects of  $\mathcal{K}$ ; then  $\mathcal{K}(A, B) = B^A$ , the usual functor category. (It is also an object of  $\mathcal{K}$ .) The category of profunctors from *A* to *B* is given by

$$\mathfrak{M}(A, B) = SET^{B^{op} \times A}.$$

Of fundamental importance is the full subcategory  $m(A, B) = set^{B^{op} \times A}$ . We have

$$\mathcal{K}^{coop}((set^A)^{op}, B) = m(A, B) = \mathcal{K}(A, set^{B^{op}}).$$

A cospan  $A \xrightarrow{f} M \xleftarrow{g} B$  in  $CAT$  is said to be *admissible* if the composite

$$B^{op} \times A \xrightarrow{g^{op} \times f} M^{op} \times M \xrightarrow{M(-, -)} SET$$

is in  $m(A, B)$ . An arrow  $B \xrightarrow{g} M$  is *admissible* if the cospan  $M \rightarrow M \xleftarrow{g} B$  is.

Abstractly given admissible arrows and, for admissible  $A$ , abstract Yoneda arrows  $A \rightarrow \mathcal{P}A$  were the starting point for the axiomatic investigations in [S&W]. In subsequent work we will require admissible cospans,  $\mathcal{P}$  and a left 2-adjoint of  $\mathcal{P}$  for the 2-categories  $\mathcal{K}$  that we consider. This suggests a slight retreat from [S&W] to a middle-of-the-road abstract  $m$  as above.  $m$  is also easier to work with than  $\mathcal{P}$  in the same way that monoidal categories are easier to work with than closed categories. Recasting [S&W] in this guise shows that size considerations are quite a secondary feature for many formal concepts and we thus arrive at  $\mathfrak{M}$ , the subject of this paper.

More explicitly, our starting data is a  $\mathcal{K}$ , together with a homomorphism of bicategories  $( )_* : \mathcal{K} \rightarrow \mathfrak{M}$ , subject to axioms suggested by  $CAT \rightarrow PROF$  ( $PROF$  being the bicategory of profunctors mentioned above). Simple variants of this paradigmatic example are provided by

$$\mathcal{K} = V-CAT \quad (\mathfrak{M} = \text{enriched profunctors})$$

$$\mathcal{K} = S\text{-indexed-CAT} \quad (\mathfrak{M} = \text{indexed profunctors}).$$

We should point out though that like  $\mathcal{P}$  in [S&W] our data is not grounded in a universal property, so that even for the  $\mathcal{K}$  above considerable flexibility is possible. With slight modifications the examples in [S&W] apply here. Considering  $set$  as a locally discrete bicategory,  $set \rightarrow rel$ , where  $rel$  is the bicategory of sets and relations, yields an example as does also  $SET \rightarrow CAT \rightarrow PROF$ .

A remark about our first axiom is in order. We have demanded that  $\mathfrak{M}$  be biclosed and yet our results about *proarrows* presented here do not depend on it. Elsewhere we will make use of this axiom and others relating  $\mathfrak{M}$  to the spans and to the cospans of  $\mathcal{K}$ . For the present Axiom 1 serves only as a notational convenience (as does Axiom 4) so the following is also an example:  $\mathcal{K}$  = the opposite of the 2-category of toposes and geometric morphisms,  $\mathfrak{M}$  = the 2-category of toposes and left exact functors.

Regarding a left exact functor between toposes as a progeometric morphism provides further insight into the «glueing construction» and the «left exact cotriple construction». Indeed, the corresponding constructions for profunctors are well known. These matters will be dealt with axiomatically in the sequel mentioned above.

All definitions in this paper are relative to  $( )_*$  as introduced in Section 1.  $\mathcal{K}$  denotes a fixed bicategory, although in practice it is usually a 2-category. Even when the latter is the case,  $\mathcal{M}$  is usually just a bicategory. We have suppressed all mention of the coherent isomorphisms.

Thanks go to Bob Paré for helpful discussions.

**1. A B S T R A C T P R O A R R O W S.**

A homomorphism of bicategories (see [B])  $( )_* : \mathcal{K} \rightarrow \mathcal{M}$  is said to equip  $\mathcal{K}$  with abstract proarrows if the following axioms are satisfied:

AXIOM 1.  $\mathcal{M}$  is biclosed.

AXIOM 2. For every arrow  $f$  in  $\mathcal{K}$ ,  $f_*$  has a right adjoint  $f^*$  in  $\mathcal{M}$ .

AXIOM 3.  $( )_*$  is locally fully faithful.

Any homomorphism of bicategories  $( )_* : \mathcal{K} \rightarrow \mathcal{M}$  admits a factorization  $\mathcal{K} \rightarrow \mathcal{I} \rightarrow \mathcal{M}$  where the objects of  $\mathcal{I}$  are those of  $\mathcal{K}$ ,

$$\mathcal{I}(A, B) = \mathcal{M}(A_*, B_*)$$

and the rest of the data is obvious.

PROPOSITION 1. *With notation as above, if  $\mathcal{K} \rightarrow \mathcal{M}$  equips  $\mathcal{K}$  with abstract proarrows, so does  $\mathcal{K} \rightarrow \mathcal{I}$ . ■*

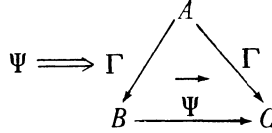
We henceforth assume

AXIOM 4. The objects of  $\mathcal{M}$  are those of  $\mathcal{K}$  and  $( )_*$  is the identity on objects.

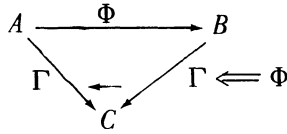
Horizontal composition in  $\mathcal{M}$  will be denoted by  $\otimes$  and all composites in  $\mathcal{K}$  and in  $\mathcal{M}$  will be written in diagrammatic order. For arrows  $\Phi : A \rightarrow B$ ,  $\Psi : B \rightarrow C$  and  $\Gamma : A \rightarrow C$  in  $\mathcal{M}$ , the following will serve to illustrate our notation for the biclosed structure :

$$\frac{\frac{\Phi \longrightarrow \Psi \Longrightarrow \Gamma}{\Phi \otimes \Psi \longrightarrow \Gamma}}{\Psi \longrightarrow \Gamma \longleftarrow \Phi}$$

TAUTOLOGY 2.

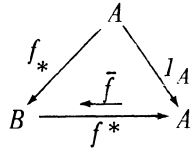


is a right lifting in  $\mathfrak{M}$  and



is a right extension in  $\mathfrak{M}$ . ■

For  $f: A \rightarrow B$  in  $\mathcal{K}$ , we denote the unit of the adjunction  $f_* \dashv f^*$  in  $\mathfrak{M}$  by  $\bar{f}: I_A \rightarrow f_* \otimes f^*$ . Recall from [S&W] that



is the unit for an adjunction iff the diagram is an absolute left lifting iff the diagram is an absolute left extension. Thus, for a transformation

$$\begin{array}{ccc} & f & \\ A & \xrightarrow{\quad} & B \\ & \downarrow \tau & \\ & g & \end{array}$$

in  $\mathcal{K}$  we can define  $\tau^*$  to be the unique transformation in  $\mathfrak{M}$  satisfying the equality

This yields a homomorphism  $( )^*: \mathcal{K}^{coop} \rightarrow \mathfrak{M}$ .

Axiom 3 asserts that, for all objects  $A$  and  $B$  and all arrows  $f, g: A \rightrightarrows B$  in  $\mathcal{K}$ , there is a bijection as indicated by the first horizontal line below. The second bijection is then trivial.

$$\frac{\frac{f_* \longrightarrow g_*}{f \longrightarrow g}}{g^* \longrightarrow f^*}$$

For  $f: A \rightarrow M$  and  $g: M \leftarrow B$  in  $\mathcal{K}$  it will sometimes be illuminating to write

$$f_* = M[1, f], \quad g^* = M[g, 1] \quad \text{and} \quad f_* \otimes g^* = M[g, f].$$

PROPOSITION 3 (*Yoneda*). For  $b: X \rightarrow B$  in  $\mathcal{K}$ :

(i) For  $\Phi: A \rightarrow B$  in  $\mathfrak{M}$ ,  $b_* \Longrightarrow \Phi \approx \Phi \otimes b^*$ .

(ii) For  $\Psi: A \leftarrow B$  in  $\mathfrak{M}$ ,  $\Psi \Longleftarrow b^* \approx b_* \otimes \Psi$ .

PROOF. (i)

$$\frac{\frac{\Gamma \longrightarrow \Phi \otimes b^*}{\Gamma \otimes b_* \longrightarrow \Phi}}{\Gamma \longrightarrow b_* \Longrightarrow \Phi}$$

(ii) Similarly. ■

COROLLARY 4. For  $b: X \rightarrow B$  and  $j: A \rightarrow B$  in  $\mathcal{K}$ ,

$$b_* \Longrightarrow j_* \approx B[b, j] \approx b^* \Longleftarrow j^*. \quad \blacksquare$$

The above isomorphisms of arrows in  $\mathfrak{M}$  are a strong intemalization of the bijections which precede Proposition 3.

In any bicategory, an arrow  $\Delta: X \rightarrow Y$  is said to be *left* (resp. *right*) *continuous* if  $\Delta$  respects all right liftings (resp. extensions). So, in symbols,  $\Delta$  is left (resp. right) continuous iff  $\Delta \otimes (\Psi \Longrightarrow \Gamma) \approx \Psi \Longrightarrow (\Delta \otimes \Gamma)$  (resp.  $(\Gamma \Longleftarrow \Phi) \otimes \Delta \approx (\Gamma \otimes \Delta) \Longleftarrow \Phi$ ) for all appropriate  $\Gamma$  and  $\Psi$  (resp.  $\Gamma$  and  $\Phi$ ). In one form or another the following «very formal adjoint arrow Theorem» is classical:

LEMMA 5. In any biclosed bicategory an arrow is a left (resp. right) adjoint iff it is left (resp. right) continuous.

PROOF. The «only if» part is true in any bicategory. Conversely assume that  $\Delta: X \rightarrow Y$  is left continuous. Define  $\Gamma = \Delta \Longrightarrow 1_Y: Y \rightarrow X$ .  $\Delta$  respects the lifting so, by (a dual of) Proposition 2 in [S&W],  $\Delta \dashv \Gamma$ . ■

## 2. INDEXED LIMITS.

For  $\Phi: M \rightarrow A$  in  $\mathfrak{M}$  and  $s: A \rightarrow C$  in  $\mathcal{K}$ , a  $\Phi$ -indexed colimit for  $s$

is an arrow  $\Phi \cdot s: M \rightarrow C$  in  $\mathcal{K}$  and a transformation

$$\begin{array}{ccc}
 & C & \\
 (\Phi \cdot s)^* \swarrow & & \searrow s^* \\
 M & \xrightarrow[\Phi]{} & A
 \end{array}$$

such that the indicated diagram is right lifting. So if  $\Phi \cdot s$  exists it is characterized, uniquely up to isomorphism, by  $(\Phi \cdot s)^* \approx \Phi \implies s^*$ .

For  $\Psi: A \rightarrow M$  in  $\mathcal{M}$  and  $s: A \rightarrow C$  in  $\mathcal{K}$ , a  $\Psi$ -indexed limit for  $s$  is an arrow  $\{\Psi, s\}: M \rightarrow C$  in  $\mathcal{K}$  and a transformation

$$\begin{array}{ccc}
 A & \xrightarrow{\Psi} & M \\
 s_* \searrow & \xleftarrow{\pi} & \searrow \{\Psi, s\}_* \\
 & C &
 \end{array}$$

such that the indicated diagram is a right extension. If  $\{\Psi, s\}$  exists we have  $\{\Psi, s\}_* \approx s_* \Leftarrow \Psi$ .

For  $\mathcal{K}$  as in [S&W] it may happen that  $\Phi: M \rightarrow A$  in  $\mathcal{M}$  can be replaced by  $j: M \rightarrow \mathcal{P}A$  in  $\mathcal{K}$ . When this is the case our definition of  $\Phi \cdot s$  easily translates into the definition of  $col(j, s)$  given there (if  $\Phi \cdot s$  is «admissible»). The way in which  $col(j, s)$  captures the colimit-like notions of enriched category theory has been discussed in [S&W]. A more detailed account of an important special case is given in [B&K]. Here we will just indicate how the above notion of limit applies to «ordinary limits» in  $CAT$ ,  $V-CAT$ ,  $S$ -indexed- $CAT$  (see [P&S]), etc. A similar discussion can be found in [B&K].

For many  $\mathcal{K}$  we have  $CAT \times \mathcal{K} \rightarrow \mathcal{K}$ ,  $(D, X) \mapsto DX$  satisfying

$$\frac{DX \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \\ \xrightarrow{\quad} \end{array} Y \text{ in } \mathcal{K}}{D \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \\ \xrightarrow{\quad} \end{array} \mathcal{K}(X, Y) \text{ in } CAT} ,$$

i. e.,  $\mathcal{K}$  is « $CAT$ -tensored», or we have some fragment of such. In this case the «ordinary» limit  $lim f: X \rightarrow Y$  of a  $D$ -diagram  $f: D \rightarrow \mathcal{K}(X, Y)$  of objects of  $Y$  defined over  $X$  is given by  $\{(! X)_* \cdot \bar{f}\}$ , where  $! X: DX \rightarrow X$  in  $\mathcal{K}$  is given by tensoring  $!: D \rightarrow 1$  with  $X$  and  $f: DX \rightarrow Y$  is the transpose of  $f$ .

$\{\Psi, s\}$  may be said to exist *weakly* if  $\{\Psi, s\}_*$  enjoys the universal

property merely with respect to arrows in  $\mathfrak{M}$  of the form  $t_*: M \rightarrow C$ . Let  $\mathcal{K} = \mathbf{V}\text{-CAT}$ , for suitable  $\mathbf{V}$ , in the discussion above and let  $(\ )_{\mathcal{K}}: \mathcal{K} \rightarrow \mathfrak{M}$  be as expected. Then weak existence of  $\{(!X)_{*}, \tilde{f}\}$ , for  $X$  the unit  $\mathbf{V}$ -category, coincides with existence of  $\lim f$  in  $Y_0$ , the underlying category of  $Y$ , while true existence coincides with the usual notion of existence in  $Y$ . We will not pursue the question «When does weak existence imply existence?» in this paper.

We return now to our general considerations.

PROPOSITION 6. For  $\Gamma: N \rightarrow M$  in  $\mathfrak{M}$ ,  $\Phi: M \rightarrow A$  in  $\mathfrak{M}$  and  $s: A \rightarrow C$  in  $\mathcal{K}$ , if  $\Phi \cdot s$  exists, then  $\Gamma \cdot (\Phi \cdot s) \approx (\Gamma \otimes \Phi) \cdot s$  when either side exists. Similarly, for suitable  $\Psi$  and  $\Delta$  in  $\mathfrak{M}$ ,  $\{\Delta, \{\Psi, s\}\} \approx \{\Psi \otimes \Delta, s\}$  when either side exists.

PROOF. In general:

$$\Gamma \Rightarrow (\Phi \Rightarrow \Lambda) \approx (\Gamma \otimes \Phi) \Rightarrow \Lambda \quad \text{and} \quad (\Lambda \Leftarrow \Psi) \Leftarrow \Delta \approx \Lambda \Leftarrow (\Psi \otimes \Delta).$$

Using the first isomorphism the first result follows from

$$(\Gamma \cdot (\Phi \cdot s))^* \approx \Gamma \Rightarrow (\Phi \Rightarrow s^*) \quad \text{and} \quad ((\Gamma \otimes \Phi) \cdot s)^* \approx (\Gamma \otimes \Phi) \Rightarrow s^*.$$

Similarly, the second result follows using the second isomorphism. ■

PROPOSITION 7. For  $a: X \rightarrow A$  in  $\mathcal{K}$  and  $s: A \rightarrow C$  in  $\mathcal{K}$ :

$$a_* \cdot s \approx a s \approx \{a^*, s\}.$$

PROOF.

$$(a s)^* \approx s^* \otimes a^* \approx a_* \Rightarrow s^* \approx (a_* \cdot s)^*,$$

where the second isomorphism is the Yoneda isomorphism of Proposition 3. ■

An arrow  $f: C \rightarrow D$  in  $\mathcal{K}$  is said to *preserve*  $\Phi \cdot s$  (resp.  $\{\Psi, s\}$ ) when the defining lifting (resp. extension) diagram is respected by  $f^*$  (resp.  $f_*$ ).

PROPOSITION 8. If  $f$  has a right (resp. left) adjoint in  $\mathcal{K}$ , it preserves any indexed colimits (resp. limits) for which this makes sense.

PROOF. If  $f \dashv u$ , then  $f^* \dashv u^*$  and Lemma 5 applies. Similarly if  $t \dashv f$ ,



$$t_* \dashv f_*$$

PROPOSITION 9 (*Formal criterion for representability*). For  $\Phi: A \rightarrow B$  in  $\mathbb{M}$ , the following are equivalent:

- (i)  $\Phi \approx f_*$  for  $f: A \rightarrow B$  in  $\mathbb{K}$ .
- (ii)  $\Phi \cdot l_B$  exists, is isomorphic to  $f$  and  $\Phi$  is left continuous.
- (iii)  $\Phi \cdot l_B$  exists, is isomorphic to  $f$  and  $\Phi$  respects the lifting which defines  $\Phi \cdot l_B$ .

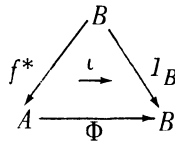
PROOF. (i)  $\rightarrow$  (ii).

$$\Phi \cdot l_B \approx f_* \cdot l_B \approx f$$

by Proposition 7 and since  $\Phi \approx f_* \dashv f^*$ ,  $\Phi$  is left continuous by Lemma 5.

(ii)  $\rightarrow$  (iii). Trivial.

(iii)  $\rightarrow$  (i). We have that



is a right lifting respected by  $\Phi$ . Applying Proposition 2 of [S&W],  $\Phi \dashv f^*$ . But  $f_* \dashv f^*$ , hence  $\Phi \approx f_*$ . ■

For arrows

$$A \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{u} \end{array} B$$

in  $\mathbb{K}$ , an immediate consequence of Axioms 2 and 3 is that  $f \dashv u$  in  $\mathbb{K}$  iff  $f_* \approx u_*$  in  $\mathbb{M}$ . (Recall that the latter in our «hom» notation is  $A[f, l] \approx B[l, u]$ . It follows that  $A[bf, a] \approx B[b, au]$  for all  $a: X \rightarrow A$  and  $b: Y \rightarrow B$ .) We leave the reader the simple task of formulating the duals of Proposition 9 and its corollary below.

COROLLARY 10 (*Formal adjoint arrow Theorem*). For  $f: B \rightarrow A$  in  $\mathbb{K}$ , the following are equivalent:

- (i)  $f$  has a right adjoint  $u$  in  $\mathbb{K}$ .
- (ii)  $f^* \cdot l_B$  exists, is isomorphic to  $u$  and  $f$  preserves all indexed colimits.

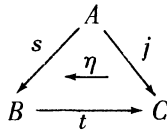
(iii)  $f^* \cdot l_B$  exists, is isomorphic to  $u$  and  $f$  preserves it. ■

We should perhaps remark here that « $f$  preserves all indexed colimits» is in general a weaker statement than « $f^*$  is left continuous», since there may be few indexed colimits with codomain  $B$ . The proof of Proposition 8 shows of course that left adjoints have the stronger property.

Following [L] and [S<sub>1</sub>] an object  $B$  in  $\mathcal{K}$  is said to be *Cauchy complete* if, for every  $M$ , every left continuous  $\Phi: M \rightarrow B$  is isomorphic to one of the form  $b_*: M \rightarrow B$ . An object  $B$  in  $\mathcal{K}$  is said to be *very total* if, for every  $M$ ,  $\Phi \cdot l_B$  exists for every  $\Phi: M \rightarrow B$ . So, immediately from Proposition 9, we have «very total implies Cauchy complete». Suitably tempering the notion of very total with size requirements yields the notion of «total» in [S&W]. For many  $\mathcal{K}$  this is a much more important concept and will be dealt with in a later paper.

### 3. RELATIVE ADJUNCTIONS, POINTWISE EXTENSIONS.

A transformation

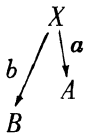


in  $\mathcal{K}$  is said to be a *relative unit for  $s$  as a left adjoint of  $t$  relative to  $j$*  if  $( )^*$  applied to it yields a right extension diagram in  $\mathfrak{M}$ . I. e., iff  $s^* \approx j^* \leftarrow t^*$ . Using Proposition 3 we can write this as  $B[s, l] \approx C[j, t]$ .

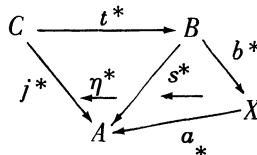
Relative counits are defined similarly via  $( )_*$ .

PROPOSITION 11. *Relative adjunctions are absolute liftings.*

PROOF. Consider the diagram above,



and

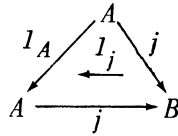


$$\begin{array}{ccc}
 a s & \longrightarrow & b \\
 \hline
 b^* & \longrightarrow & s^* \otimes a^*
 \end{array}$$

$$\frac{\frac{b^* \otimes a_* \longrightarrow s^*}{t^* \otimes b^* \otimes a_* \longrightarrow j^*}}{t^* \otimes b^* \longrightarrow j^* \otimes a^*}$$

$$aj \longrightarrow bt \quad \blacksquare$$

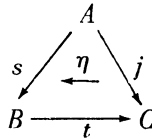
An arrow  $j: A \rightarrow B$  in  $\mathcal{K}$  is said to be *fully faithful* if the transformation  $\bar{j}: I_A \rightarrow j_* \otimes j^*$  in  $\mathfrak{M}$  is an isomorphism. In other words,  $j$  is fully faithful iff



is the unit for a relative adjunction ( $A[ I, I ] \approx B[ j, j ]$ ).

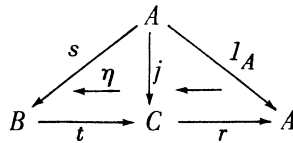
For  $\mathcal{K} = \mathbf{V}\text{-CAT}$  (equipped with the usual  $( )_*: \mathcal{K} \rightarrow \mathfrak{M}$ ) it is easy to see that the above definition coincides with that of  $\mathbf{V}$ -fully faithful. On the other hand, to say that the diagram above is an absolute lifting diagram is to say that the underlying functor of the  $\mathbf{V}$ -functor  $j$  is fully faithful. So for general  $\mathcal{K}$  the converse of Proposition 11 does not hold.

PROPOSITION 12. *If*



*is an absolute left lifting and either  $j$  is a left adjoint or  $t$  is a right adjoint, then the diagram is a relative adjunction.*

PROOF. If  $j \dashv r$  then by Proposition 1 of [S&W] the following diagram is an absolute left lifting:



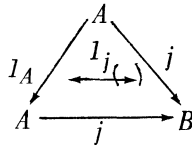
It follows that  $s \dashv tr$  and  $B[s, I] \approx A[ I, tr ] \approx C[ j, t ]$ . If  $f \dashv t$  we have  $s \approx jf$  and  $B[s, I] \approx B[ jf, I ] \approx C[ j, t ]$ .  $\blacksquare$

PROPOSITION 13. *For  $j: A \rightarrow B$  in  $\mathcal{K}$ , if  $j$  has a right adjoint (resp. left*

adjoint) with unit  $\eta$  (resp. counit  $\epsilon$ ), then the following are equivalent:

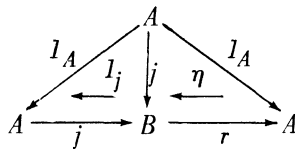
- (i)  $j$  is fully faithful.
- (ii) For all  $X$  the functor  $\mathcal{K}(X, j)$  is fully faithful.
- (iii)  $\eta$  (resp.  $\epsilon$ ) is an isomorphism.

PROOF. (ii) just says that



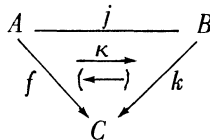
is an absolute left (and right) lifting, so the equivalence of (i) and (ii) follows from Propositions 11 and 12.

(ii)  $\iff$  (iii). In case  $j \dashv r$  with unit  $\eta$ , consider



By Proposition 1 of [S&W] the left triangle is an absolute left lifting ((ii)) iff the composite triangle is an absolute left lifting. Since  $l_A \dashv l_A$  the latter is the case iff  $\eta$  is an isomorphism. ■

A transformation



in  $\mathcal{K}$  is said to exhibit  $k$  as a *pointwise left* (resp. *right*) *extension* of  $f$  along  $j$  if  $(\ )^*$  (resp.  $(\ )_*$ ) applied to it yields a right lifting (resp. extension) diagram in  $\mathfrak{M}$ , i. e. iff  $k \approx j^* \cdot f$  (resp.  $k \approx \{j_*, f\}$ ).

In fact it is clear that ordinary extensions are just weak indexed colimits and limits of the types above, so pointwise extensions are extensions.

PROPOSITION 14. For  $\kappa$  in either case as above, if  $j$  is fully faithful, then  $\kappa$  is an isomorphism.

PROOF. For  $k \approx j^* \cdot f$ :

$$\begin{aligned}
 (jk)^* &\approx k^* \otimes j^* \approx (j^* \Rightarrow f^*) \otimes j^* \\
 &\approx j_* \Rightarrow (j^* \Rightarrow f^*) \quad (\text{Yoneda}) \\
 &\approx (j_* \otimes j^*) \Rightarrow f^* \approx 1_A \Rightarrow f^* \approx f^*. \quad \blacksquare
 \end{aligned}$$

**REFERENCES.**

- B. J. BENABOU, Introduction to bicategories, *Lecture Notes in Math.* 47, Springer (1967), 1-77.
- B&K. F. BORCEUX & G.M. KELLY, A notion of limit for enriched categories, *Bull. Australian Math. Soc.* 12 (1975), 49-72.
- L. F.W. LAWVERE, Metric spaces, generalized logic and closed categories, Preprint, Inst. Matem., Universita di Perugia, 1973.
- P & S. R. PARE & D. SCHUMACHER, Abstract families and the adjoint functor theorems, *Lecture Notes in Math.* 661, Springer (1978), 1-125.
- S<sub>1</sub>. R. STREET, The calculus of modules, Preprint.
- S<sub>2</sub>. R. STREET, Elementary cosmoi I, *Lecture Notes in Math.* 420, Springer (1974), 134-180.
- S&W. R. STREET & R.F.C. WALTERS, Yoneda structures on 2-categories, *J. Algebra* 50 (1978), 350-379.

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