CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

R. BETTI
A. CARBONI

# Cauchy-completion and the associated sheaf

*Cahiers de topologie et géométrie différentielle catégoriques*, tome 23, nº 3 (1982), p. 243-256
<http://www.numdam.org/item?id=CTGDC_1982__23_3_243_0>

CAHIERS DE TOPOLOGIE
ET GÉOMÉTRIE DIFFÉRENTIELLE

Vol. XXIII-3 (1982)

# CAUCHY-COMPLETION AND THE ASSOCIATED SHEAF

*by R. BETTI and A. CARBONI* *)

## INTRODUCTION.

We will follow the point of view that categories based on a bicategory B (briefly B-categories) should be thought as general spaces. Such categories arose by considering a variable base for homs and the suggestion for regarding them as spaces comes directly from a paper (Walters [4]) where it is shown that sheaves for a general site are equivalent to symmetric, skeletal, Cauchy-complete categories based on a bicategory constructed out of the site.

For a category, Cauchy-completeness means that any adjoint pair of bimodules can be represented by one functor and, in order to express fundamental constructions of sheaf theory by means of B-category theory, we need to show the existence of the general process of «Cauchy-completion». The experience of metric spaces developed in [3] suggests that this construction should be done by taking adjoint pairs of bimodules. We will prove that in fact we get in this way the general process of Cauchy-completion and that it particularizes to the associated sheaf. This last result will be obtained by showing that such completion is left adjoint to the embedding of a particular kind of symmetric B-categories (called adjoint-inverse, briefly a. i.), and by constructing a «comparison functor». This functor also leads to compare the B-categorical one with an already known one-step construction of the associated sheaf [2].

The following diagram summarizes the whole subject:

$$\begin{array}{ccc} Rel_J(\mathrm{C})\text{-}Cat\,a.i. & \xrightarrow{\ comp\ } & S^{\mathrm{C}^{op}} \\ & {\scriptstyle Cauchy}\searrow \quad \swarrow{\scriptstyle a} & \\ & sh_J(\mathrm{C}) & \end{array}$$

*) Work partially supported by the Italian C.N.R.

where $Rel_J(\mathrm{C})$ is the base bicategory associated to the site $(\mathrm{C}, J)$.

We will use a different (but equivalent to that of [4]) construction of the base bicategory and because the absence of papers on B-categories we feel the need to give the main definitions, though they are simply translations (from one to many objects) of classical V-category ones.

1. We introduce now a notion which corresponds to that of «polyad» in the terminology of [1].

DEFINITION. When B is a bicategory, a B-*category* $X$ is defined by assigning:

i) objects $x, y, \ldots$ ;

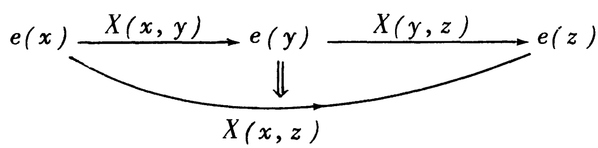
ii) to every object $x$ an «underlying» object $e(x)$ of B ;

iii) to every ordered pair $<x, y>$ of objects an «object of morphisms»

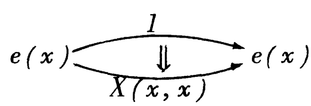
$$e(x) \xrightarrow{X(x,y)} e(y)$$

in the category $\mathrm{B}(e(x), e(y))$ ;

iv) to every ordered triple $<x, y, z>$ a «composition» in the category $\mathrm{B}(e(x), e(z))$ :




v) to every object $x$ an «identity» in the category $\mathrm{B}(e(x), e(x))$




The above data have to be subjected to the associativity and unity laws, which can be expressed by commutative diagrams of 2-cells in B. The base bicategories involved in the following are locally partially-ordered, so that these conditions hold trivially.

DEFINITION. If $X$ and $Y$ are two B-categories, a B-*functor* $f: X \to Y$ is a function on objects which preserves underlyings : $e(fx) = e(x)$ ; moreover, for each ordered pair $<x, x'>$ of $X$-objects, a 2-cell must be assigned :

$$e(x) \;\underset{Y(fx,\,fx')}{\overset{X(x,\,x')}{\Downarrow}}\; e(x')$$

which preserves identity and composition (always satisfied in the partially-ordered case).

DEFINITION. If $X$ and $Y$ are B-categories, a *bimodule* $X \nrightarrow Y$ assigns to every ordered pair of objects $x$ in $X$ and $y$ in $Y$ a 1-cell

$$e(y) \xrightarrow{\;\phi(y,x)\;} e(x)$$

subject to actions

$$e(y) \xrightarrow{\;\phi(y,x)\;} e(x) \xrightarrow{\;X(x,x')\;} e(x') \quad \Downarrow \quad e(y) \xrightarrow{\;\phi(y,x')\;} e(x')$$

$$e(y') \xrightarrow{\;Y(y',y)\;} e(y) \xrightarrow{\;\phi(y,x)\;} e(x) \quad \Downarrow \quad e(y') \xrightarrow{\;\phi(y',x)\;} e(x)$$

satisfying unity, associativity and mixed associativity (always true in the partially-ordered case).

If any hom-category of B allows arbitrary sups preserved by compositions, then bimodules $\phi : X \nrightarrow Y$ and $\psi : Y \nrightarrow Z$ can be *composed* by

$$(\psi \circ \phi)(z,x) = \bigvee_y \psi(z,y)\, \phi(y,x)\,.$$

Any B-functor $f: X \to Y$ becomes a bimodule

$$f_*: X \nrightarrow Y \quad \text{by} \quad f_*(y,x) = Y(y,fx),$$

and the essential feature of such bimodules is that there exists an adjoint bimodule

$$f^*: Y \nrightarrow X \ \text{ defined by } f^*(x,y) = Y(fx,y),$$

where *adjointness* $\phi \dashv \psi$ means

$$X(x,x') \le (\psi \circ \phi)(x,x') \ \text{ for each } x, x' \ \text{ and}$$
$$(\phi \circ \psi)(y,y') \le Y(y,y') \ \text{ for each } y,\ y'.$$

If $u$ is an object of B, let us denote by $\hat{u}$ the trivial B-category with just one object over $u$.

DEFINITION (Lawvere [3]). A B-category $Y$ is said to be *Cauchy-complete* (shortly C.c.) if for each $u$ and each pair of adjoint bimodules

$$\hat{u} \underset{\psi}{\overset{\phi}{\rightleftarrows}} Y \qquad (\phi \dashv \psi)$$

is representable by a functor $y: \hat{u} \to Y$, i.e., $\phi = y_*$ and $\psi = y^*$.

For this reason, in the following, dealing with adjoint pairs of bimodules, we will simply use $\phi$ to mean the pair, and $\phi_*$, $\phi^*$ to denote the left and right adjoint parts.

We now prove that Cauchy completion still exists in the B-categorical framework.

THEOREM. *The embedding of Cauchy complete* B-*categories in* B-*Cat has a left adjoint in the appropriate two-dimensional sense.*

PROOF. When $X$ is a B-category, define its Cauchy-completion $\tilde{X}$, by taking as objects all pairs of adjoint bimodules $\phi: \hat{u} \to\!\!\!\!\!\!\!+\;\; X$. The underlying object of $\phi$ is $u$ and the hom is defined by $\tilde{X}(\phi, \psi) = \phi^* \circ \psi_*$ (observe that $\phi^* \circ \psi_*$ has just one component $u \to v$ if $e(\psi) = v$).

Easily the adjointness conditions provide the necessary B-category operations in $\tilde{X}$.

There exists a B-functor $c: X \to \tilde{X}$ sending objects $x$ of $X$ into the adjoint pair it represents; $c$ is fully faithful:

$$\tilde{X}(c(x), c(x')) = \bigvee_{x''} X(x, x'') X(x'', x') \approx X(x, x').$$

Therefore we can identify without any ambiguity each $X$-object with its image in $\tilde{X}$. A direct calculation shows that:

$$(*) \qquad \tilde{X}(\phi, x) \approx \phi^*(x), \quad \tilde{X}(x, \psi) \approx \psi_*(x).$$

With these identities we can prove that $c_*$ and $c^*$ are inverse bimodules:

$$(c_* \circ c^*)(\phi, \psi) = \bigvee_x c_*(\phi, x) c^*(x, \psi) = \bigvee_x \tilde{X}(\phi, x) \tilde{X}(x, \psi) \approx$$

$$\approx \bigvee_x \phi^*(x) \psi_*(x) = \tilde{X}(\phi, \psi),$$

$$(c^* \circ c_*)(x, x') = \bigvee_\phi c^*(x, \phi) c_*(\phi, x') = \bigvee_\phi \tilde{X}(x, \phi) \tilde{X}(\phi, x') \approx$$
$$\approx \bigvee_\phi \phi_*(x)\, \phi^*(x') \approx X(x, x'),$$

because

$$\bigvee_\phi \phi_*(x) \phi^*(x') \leq X(x, x')$$

follows by adjointness, and in the other direction it suffices to take

$$\phi_* = X(-, x) \quad \text{and} \quad \phi^* = X(x, -).$$

$\tilde{X}$ is Cauchy complete: If $\phi : \hat{u} \longrightarrow\!\!\!\!\!\!\!+\;\; \tilde{X}$ is an adjoint pair of bimodules, the composites $c^* \circ \phi_*$ and $\phi^* \circ c_*$ are still adjoint because $c_*$ and $c^*$ are inverses each other, so give rise to a point $\psi$ of $\tilde{X}$ which lies over $u$. Consider the B-functor which takes the only object of $\hat{u}$ to $\psi$. The adjoint pair $\phi$ is represented by $\psi$ :

$$\tilde{X}(\psi, \theta) = \phi^* \circ c_* \circ \theta_* = \bigvee_{\psi'} \phi^*(\psi')(c_* \circ \theta_*)(\psi') =$$
$$= \bigvee_{\psi'} \phi^*(\psi') \bigvee_x c_*(\psi', x) \theta_*(x) = \bigvee_{\psi'} \phi^*(\psi') \bigvee_x \tilde{X}(\psi', x) \theta_*(x) \approx$$
$$\approx \bigvee_{\psi'} \phi^*(\psi') \bigvee_x \psi'^*(x) \theta_*(x) = \bigvee_{\psi'} \phi^*(\psi')\, \tilde{X}(\psi', \theta) \approx \phi^*(\theta).$$

In the same way it can be shown that $\tilde{X}(\theta, \psi) \approx \phi_*(\theta)$.

To show the universal property of the Cauchy-completion, extend any B-functor $g : X \to Y$ ($Y$ Cauchy complete) along $c$ to a B-functor

$\tilde{g} : \tilde{X} \to Y$ by $\tilde{g}(\phi)$ = the object of $Y$ which represents $g \circ \phi$.

The functor $\tilde{g}$ is determined up to invertible 2-cells in B-*Cat* ; in the partially ordered case $\tilde{g}$ equivalent to $\tilde{g}'$ just means that for each $\phi$ the objects $\tilde{g}(\phi)$ and $\tilde{g}'(\phi)$ are B-isomorphic, i.e.,

$$1 \leq Y(\tilde{g}(\phi), \tilde{g}'(\phi)) \quad \text{and} \quad 1 \leq Y(\tilde{g}'(\phi), \tilde{g}(\phi)).$$

2. Following the line of Lawvere's «Metric spaces» [3], where the pursued aim is that «*fundamental* structures are themselves categories ... by taking account of a certain natural generalization of category theory within itself» (namely V-category theory), the further generalization from V to B leads to consider sheaves also as categories.

If $(\mathrm{C}, J)$ is a site, we construct a bicategory $Rel_J(\mathrm{C})$ as follows: objects of $Rel_J(\mathrm{C})$ are those of C, 1-cells $R: u \to v$ are families of spans

$$u \xleftarrow{\ h\ } w \xrightarrow{\ k\ } v$$

which are saturated by composition, i. e. if $\langle h, k\rangle \in R$, then $\langle fh, fk\rangle \in R$ for all $f: w' \to w$. We write $\{\langle h, k\rangle\}$ for the 1-cell in $Rel_J(\mathrm{C})$ generated by $\langle h, k\rangle$. Composition $RS$ is defined as the family of spans $\langle h, k\rangle$ for which there exists a $g$ with

$$\langle h, g\rangle \in R \quad \text{and} \quad \langle g, k\rangle \in S.$$

Identities are given by $\{\langle 1, 1\rangle\}$. It is straightforward to verify that in this way we get a category which defines the 1-dimensional part of $Rel_J(\mathrm{C})$. The 2-cells of $Rel_J(\mathrm{C})$ are essentially depending upon the topology:

$R \underset{J}{<} S$ iff for all $\langle h, k\rangle \in R$ there exists a covering family $\mathfrak{U} = \{w_i \xrightarrow[g_i]{} w\}_{i \in I} \in J(w)$ such that $\langle g_i h, g_i k\rangle \in S$ for all $i \in I$.

The proof that $Rel_J(\mathrm{C})$ is a bicategory (in fact, a 2-category) involves directly the axioms of the topology $J$. Moreover $Rel_J(\mathrm{C})$ is locally a lattice and each $Rel_J(\mathrm{C})(u, v)$ is sup-complete: the sup is simply set-theoretical union of families, and it is easy to verify its strict preservation by composition. Observe that $Rel_J(\mathrm{C})$ is a *symmetric* bicategory, in the sense that there exists a natural isomorphism of categories

$$(-)^o : Rel_J(\mathrm{C})(u, v) \to Rel_J(\mathrm{C})(v, u)$$

such that $(R^o)^o = R$ and $(RS)^o = S^o R^o$.

We have a faithful functor

$$\mathrm{C} \to Rel_J(\mathrm{C}) \qquad \text{given by } \ h \mapsto \{\langle 1, h\rangle\}.$$

This functor allows to identify arrows in C with corresponding ones in $Rel_J(\mathrm{C})$. By this identification, arrows $h$ of C satisfy

$$hh^o \underset{J}{>} 1 \quad \text{and} \quad h^o h \underset{J}{<} 1.$$

The symmetry of the base allows to define *symmetric* $Rel_J(\mathrm{C})$*-categories* as those for which $X(x, x')^o = X(x', x)$.

DEFINITION. $L : S^{\mathrm{C}^{op}} \to Rel(\mathrm{C})\text{-}Cat$ ($Rel(\mathrm{C})$ denotes the bicategory associated to the minimal topology) is a functor defined as follows: $LF$ has objects the sections $x \in Fu$ whose underlying object is $u$. If $x \in Fu$ and $y \in FV$, then $LF(x, y)$ is the family of spans

$$\langle h, k \rangle \text{ such that } x/h = y/k.$$

Let us observe that the functor $L$ takes its images in the full subcategory of symmetric and skeletal $Rel(\mathrm{C})$-categories, where *skeletal*, for a B-category $X$, means:

$$1 \leq X(x, y) \text{ and } 1 \leq X(y, x) \text{ implies } x = y.$$

It is easy to check that the property to be skeletal is equivalent to the uniqueness of representability of bimodules $\hat{u} \to X$, and observe that skeletality destroys the 2-dimensional part of B-*Cat*.

Finally, observe that by construction the partial order of topologies is preserved, i. e., if $J < J'$, then there is a canonical embedding

$$Rel_J(\mathrm{C})\text{-}Cat \to Rel_{J'}(\mathrm{C})\text{-}Cat$$

which does not preserve skeletality.

We now need some remarks about symmetry and Cauchy-completion. First observe it is not always true that the Cauchy-completion of a symmetric B-category still is symmetric: consider the monoid $M = Set(A, A)$ as a symmetric *Set*-category with just one object. It is known that Cauchy-completion for ordinary categories is the universal process of splitting idempotents [ 3, page 164] ; this means that $\tilde{M}$ is not symmetric but in trivial cases. However, in particular cases (e. g. metric spaces) the Cauchy-completion of a symmetric B-category is symmetric. So far we don't know whether the same property holds for all $Rel_J(\mathrm{C})$-categories. The following lemma provides a characterization for the general case.

LEMMA 1. *Let $X$ be a* B-*category. The Cauchy-completion $\tilde{X}$ is symmetric iff each adjoint pair $\phi : \hat{u} \to X$ is an inverse pair (i. e., $\phi_*(x)^o = \phi^*(x)$ ).*

PROOF. In one direction the proof comes directly by the definition of $\tilde{X}$. In the other one, just consider the formulas (*) in the proof of the theorem on Cauchy-completion and take into account the symmetry of $X$.

Observe that the a. i. property implies the symmetry of $X$ ; it suffices to particularize the a. i. property to representable bimodules. As we have already remarked, we don't know if the a. i. property is equivalent to the symmetry of $X$ in the $Rel_J(\mathrm{C})$ case.

The previous lemma implies that the Cauchy-completion restricts:

$$\begin{array}{ccc} \text{B-}Cat & \underset{e}{\overset{\sim}{\rightleftarrows}} & \text{B-}Cat\ C.\ c. \\ \uparrow & & \uparrow \\ \text{B-}Cat\ a.\ i. & \underset{e}{\overset{\sim}{\rightleftarrows}} & \text{B-}Cat\ sym.\ C.\ c. \end{array}$$

(C.c. = Cauchy-complete) and that B-*Cat a. i.* is the biggest full subcategory through which the adjunction restricts.

In view of Walters result [4], in the case $\mathrm{B} = Rel_J(\mathrm{C})$ let us define a functor $\Gamma_J$ which will provide a useful description of the $\tilde{}$-process.

$$Rel_J(\mathrm{C})\text{-}Cat \xrightarrow{\Gamma_J} sh_J(\mathrm{C})$$

is defined in the following way:

$\Gamma_J X(u)$ = isomorphism classes of adjoint pairs of bimodules $\phi : \hat{u} \dashrightarrow X$.

When $h : v \to u$ is an arrow in C, the restriction is defined by the adjoint pair over $\hat{v}$: $\phi_*/h(x) = \phi_*(x)h^o$ and $\phi^*/h(x) = h\phi^*(x)$.

Functoriality of $\Gamma_J X$ is an easy matter. For sheaf conditions, let

$$\mathfrak{U} = \{ u_i \xrightarrow{h_i} u \}$$

be a $J$-covering family, and $\phi_i : \hat{u}_i \dashrightarrow X$ be a compatible family. Define $\phi : \hat{u} \dashrightarrow X$ by:

$$\phi_*(x) = \bigvee_i \phi_{i*}(x)h_i, \quad \phi^*(x) = \bigvee_i h_i^o \phi_i^*(x).$$

They are adjoint: to check

$$1 \leq_J \bigvee_x [\bigvee_i h_i^o\ \phi_i^*(x) \bigvee_j \phi_{j*}(x)h_j]$$

it is sufficient to take $i = j$, and so to check

$$1 \underset{J}{<} \bigvee_i [\, h_i^o ( \bigvee_x \phi_i^*(x) \phi_{i*}(x) h_i \,] .$$

But $\phi_{i*} \dashv \phi_i^*$, so it is enough to check $1 \underset{J}{<} \bigvee_i h_i^o h_i$, which is true because $\{h_i\}$ is a covering family. To verify the other adjointness condition, first observe that compatibility means that, for each commutative square $k_i h_i = k_j h_j$, it holds

$$\phi_{i*} k_i^o \approx \phi_{j*} k_j^o \quad \text{and} \quad k_i \phi_i^* \approx k_j \phi_j^* \quad \text{for each } i,\, j.$$

Hence:

$$\phi_{i*} k_i^o k_j \approx \phi_{j*} k_j^o k_j \underset{J}{<} \phi_{j*}.$$

So, for each $<k_i, k_j>$ in $h_i h_j^o$, it holds

$$\phi_{i*}(x) k_i^o k_j \phi_j^*(x') \underset{J}{<} \phi_{j*}(x)\, \phi_j^*(x') \underset{J}{<} X(x, x'),$$

hence $\phi_*(x) \circ \phi^*(x') \underset{J}{<} X(x, x')$.

LEMMA 2. *There exists a functor* $L_J$

$$\begin{array}{ccc} Rel_J(\mathrm{C})\text{-}Cat\ a.\ i. & \underset{e}{\overset{\sim}{\rightleftarrows}} & Rel_J(\mathrm{C})\text{-}Cat\ sym.\ C.\ c. \\ & {}_{\Gamma_J}\searrow \quad \nearrow_{L_J} & \\ & sh_J(\mathrm{C}) & \end{array}$$

*such that* $\Gamma_J \dashv L_J\, e$, *in the appropriate 2-dimensional sense.*

PROOF. $L_J$ is defined as in Walters [4], Proposition 1, by the composition:

$$sh_J(\mathrm{C}) \longrightarrow S^{\mathrm{C}^{op}} \xrightarrow{\ L\ } Rel(\mathrm{C})\text{-}Cat \longrightarrow Rel_J(\mathrm{C})\text{-}Cat$$

where it is shown that it factorizes through $Rel_J(\mathrm{C})$ *-Cat sym. c. c.* and that it is fully-faithful. Observe now that for each $X$ in $Rel_J(\mathrm{C})$ *-Cat a. i.*, it holds $\tilde{X} \approx L_J(\Gamma_J X)$. Clearly both categories agree on elements (up to isomorphisms); for homs:

$$\tilde{X}(\phi, \psi) = \phi^* \circ \psi_* = \{ <h, k> \mid \phi_* \circ h^o \approx \psi_* \circ k^o \} = L_J(\Gamma_J X(\phi, \psi)).$$

Indeed, let $<h, k>$ be such that $\phi_* \circ h^o \approx \psi_* \circ k^o$; then

$$\phi^* \circ \phi_* \circ h^o \circ k \approx \phi^* \circ \psi_* \circ k^o \circ k \underset{J}{<} \phi^* \circ \psi_* ;$$

but $1 \underset{J}{\leq} \phi^* \circ \phi_*$ and $<h, k> \in h^o k$, thus $<h, k> \in \phi^* \circ \psi_*$ . Conversely, let $<h, k> \in \phi^* \circ \psi_*$ ; then $h^o k \underset{J}{\leq} \phi^* \circ \psi_*$, therefore

$$\phi_* \circ h^o \underset{J}{\leq} \phi_* \circ h^o \circ k \circ k^o \underset{J}{\leq} \phi_* \circ \phi^* \circ \psi_* \circ k^o \underset{J}{\leq} \psi_* \circ k^o .$$

Now

$$\psi_* \circ k^o \underset{J}{\leq} \psi_* \circ k^o \circ h \circ h^o \underset{J}{\leq} \psi_* \circ \psi^* \circ \phi_* \circ h^o \underset{J}{\leq} \phi_* \circ h^o ,$$

because $h^o k \underset{J}{\leq} \phi^* \circ \psi_*$ and the a. i. property implies

$$k^o h \underset{J}{\leq} \psi_*^o \circ \phi^{*o} = \psi^* \circ \phi_* .$$

Now the following chain of equivalences proves the adjunction :

| | |
|---|---|
| $X \longrightarrow e(L_J F)$ | by the theorem on $\tilde{}$ |
| $\tilde{X} \longrightarrow L_J F$ | by the previous remark |
| $L_J(\Gamma_J X) \longrightarrow L_J F$ | $L_J$ is 2-fully-faithful |
| $\Gamma_J X \longrightarrow F$ . | |

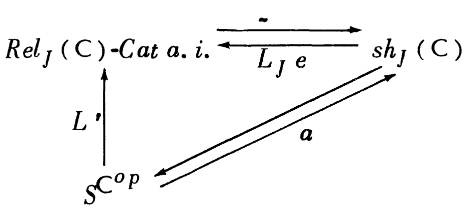
THEOREM *(Walters [4] Proposition 2). The functor $L_J$ is a 2-equivalence.*

A direct proof may be obtained by considering the adjunction: $\Gamma_J \dashv L_J e$. Because $L_J e$ is 2-fully-faithful, it is enough to prove that $\eta_X : X \to e(L_J(\Gamma_J X))$ is an equivalence iff $X$ is Cauchy-complete.

By this theorem, we will call $\Gamma_J$ simply $\tilde{}$.

3. We want now to compare the previous adjunction with the associated sheaf functor.

THEOREM. *There exists a comparison functor $L'$:*




To prove the theorem we need a suitable description of the associated sheaf functor $a$. For our purpose we found the best one to be that in [2], where $aF(u)$ is given by «$u$-locally compatible families of elements of $F$, with covering support and closed», which means:

DEFINITION 2. If $F$ is a presheaf and $u$ an object of C, a *$u$-locally compatible family with covering support* is the assignment for each arrow $i: v \to u$ of a family $\mathcal{F}_i \subset F v$ such that:

1° if $x \in \mathcal{F}_i$, for each $h: w \to v$, $x/h \in \mathcal{F}_{hi}$;

2° the crible $\{ i: v \to u \mid \mathcal{F}_i \neq \emptyset \}$ is $J$-covering («covering support»);

3° if $x, y \in \mathcal{F}_i$, $\{ k: w \to v \mid x/k = y/k \}$ is $J$-covering («local compatibility»).

Such a family is *closed* if moreover:

4° if $x \in Fv$ and $\{ k: w \to v \mid x/k \in \mathcal{F}_{ki} \}$ is a $J$-covering family, then $x \in \mathcal{F}_i$.

Define $L'F$ as $LF$ but thought in $Rel_J(\mathrm{C})$*-Cat sym.*

LEMMA 1. *If $\phi: \hat{u} \dashrightarrow L'F$ is a bimodule, and $\{<h, k>\} \underset{J}{<} \phi(x)$ then $k \underset{J}{<} \phi(x/h)$.*

PROOF. Directly we have $k \underset{J}{<} h\{<h, k>\}$.

By the bimodule property and the assumption:

$$h\{<h, k>\} \underset{J}{<} L'F(x/h, x)\, \phi(x) \underset{J}{<} \phi(x/h).$$

LEMMA 2. *Isomorphism classes of pairs of adjoint bimodules $\phi: \hat{u} \dashrightarrow L'F$ are in 1-1 correspondance with u-locally compatible, with covering support, closed families of parts of $F$.*

PROOF. Let us consider such a $u$-family $\mathcal{F}$. Define a pair of adjoint bimodules $\phi = \phi(\mathcal{F})$ by

$$\phi_*(x) = \bigvee_{x' \in \mathcal{F}_i} L'F(x, x')\, i, \quad \phi^*(x) = \phi_*(x)^o.$$

The proof of the adjointness condition $1 \underset{J}{<} \bigvee_x \phi^*(x)\phi_*(x)$ is the same as that in the proof of sheaf conditions for $\Gamma_J$, by using condition 2 on $\mathcal{F}$. The other adjointness condition $\phi_*(x)\,\phi^*(y) \underset{J}{<} L'F(x,y)$ holds because for each $x' \in \mathcal{F}_i$ and $x'' \in \mathcal{F}_j$ we have

$$L'F(x,x')ij^o \underset{J}{<} L'F(x,x'') :$$

$<r,s> \in L'F(x,x')ij^o$ means that there exists $t$ such that $x/r = x'/t$, $ti = sj$ ; by 1,

$$x/r = x'/t \in \mathfrak{F}_{ti} \quad \text{and} \quad x''/s \in \mathfrak{F}_{sj} = \mathfrak{F}_{ti} ;$$

by 3, $x/r$ and $x''/s$ agree on a covering.

Conversely, given an adjoint pair of bimodules $\phi: \hat{u} \nrightarrow L'F$, define $\mathfrak{F} = \mathfrak{F}(\phi)$ by

$$\mathfrak{F}_i = \{ y \in Fv \mid i \underset{J}{<} \phi_*(y) \text{ and } i^o \underset{J}{<} \phi^*(y) \}$$

for each $i: v \to u$.

Condition 1: if $i \underset{J}{<} \phi_*(y)$, then for each $k: w \to v$ also $<k, ki> \underset{J}{<} \phi_*(y)$ which by Lemma 1 implies $ki \underset{J}{<} \phi_*(y/k)$ ; $i^o k^o \underset{J}{<} \phi^*(y/k)$ follows in a similar way from $i^o \underset{J}{<} \phi^*(y)$ and a «dual» of Lemma 1.

Condition 2: by the adjointness condition $1 \underset{J}{<} \phi_* \circ \phi^*$, it follows that there exists a covering

$$\mathfrak{U} = \{ u_\alpha \xrightarrow{k_\alpha} u \}$$

such that for each $\alpha$ there exists $x_\alpha$ with $<k_\alpha, k_\alpha> \in \phi^*(x_\alpha)\phi_*(x_\alpha)$. This means that there exists $m_\alpha$ such that

$$<k_\alpha, m_\alpha> \in \phi^*(x_\alpha) \quad \text{and} \quad <m_\alpha, k_\alpha> \in \phi_*(x_\alpha) .$$

By Lemma 1 we have $k_\alpha \underset{J}{<} \phi_*(x_\alpha/m)$. So the family

$$\{ i: v \to u \mid \mathfrak{F}_i \neq \emptyset \}$$

contains a covering family, namely $\mathfrak{U}$.

Condition 3: if $x, y \in \mathfrak{F}_i$, then $ii^o \underset{J}{<} \phi_*(x)\phi^*(y)$. By the adjointness condition

$$\phi_*(x)\phi^*(y) \underset{J}{<} L'F(x,y)$$

and because $1 \underset{J}{<} ii^o$, we have $1 \underset{J}{<} L'F(x,y)$, which proves condition 3.

Condition 4: we have to show that for each $i: v \to u$ and each $y \in Fv$, if

$$\mathfrak{U} = \{ k: w \to v \mid y/k \in \mathfrak{F}_{ki} \}$$

is a covering family, then $i \underset{J}{<} \phi_*(y)$ and $i^o \underset{J}{<} \phi^*(y)$. So

$$k i \underset{J}{\leq} \phi_*(y/k) \quad \text{and} \quad i^o k^o \underset{J}{\leq} \phi^*(y/k).$$

Because $k^o \in L'F(y, y/k)$, then

$$\{<k, ki>\} = k^o k i \underset{J}{\leq} L'F(y, y/k)\ \phi_*(y/k) \underset{J}{\leq} \phi_*(y)$$

holds for each $k$ in $\mathfrak{U}$. It follows $i \underset{J}{\leq} \phi_*(y)$. Analogously $i^o \underset{J}{\leq} \phi^*(y)$.

Let us check that the two correspondances are inverse each other, when the first one is restricted to closed families. If $\mathfrak{F}'$ is the family associated to $\phi(\mathfrak{F})$, then $\mathfrak{F}_i \subset \mathfrak{F}'_i$ for each $i: v \to u$ : $\mathfrak{F}'_i$ being

$$\{ y \in Fv \mid i \underset{J}{\leq} \bigvee_{x'} L'F(y, x')k \})$$

it is enough to take $k = i$ and $x' = y$. Suppose now $\mathfrak{F}$ closed; let

$$y \in \mathfrak{F}'_i \ , \ \text{ i.e. } \ i \underset{J}{\leq} \bigvee_{x'} L'F(y, x')k,$$

which means there exists a covering $\mathfrak{U} = \{ h_a : w_a \to v\}$ such that for each $a$ there exist $x_a$ and $k_a$ with

$$x_a \in \mathfrak{F}_{k_a} \quad \text{and} \quad <h_a, h_a i> \in L'F(y, x_a)\, k_a .$$

It follows there exists $t$ with

$$y/h_a = x_a/t \quad \text{and} \quad h_a i = t k_a .$$

So by condition 1, $y/h_a \in \mathfrak{F}_{t k_a} = \mathfrak{F}_{h_a i}$. Therefore the family

$$\{ h : w \to v \mid y/h \in \mathfrak{F}_{h i} \}$$

contains a covering family. Because $\mathfrak{F}$ is closed, $y \in \mathfrak{F}_i$.

In the other direction, if $\phi'$ is the adjoint pair of bimodules corresponding to $\mathfrak{F}(\phi)$, it is easy to check $\phi \approx \phi'$ : for each $y \in \mathfrak{F}_i$ it holds

$$L'F(x, y)\, i \underset{J}{\leq} L'F(x, y)\, \phi_*(y) \underset{J}{\leq} \phi_*(x),$$

thus

$$\phi'_*(x) = \bigvee_{y \in \mathfrak{F}_i} L'F(x, y)\, i \underset{J}{\leq} \phi_*(x) \ ;$$

conversely, if $\{<h, k>\} \underset{J}{\leq} \phi_*(x)$, then by Lemma 1 we get $k \underset{J}{\leq} \phi_*(x/h)$, i.e. $x/h \in \mathfrak{F}_k$ ; so $<h, k> \in \phi'_*(x)$, because it belongs to $L'F(x, x/h)\, k$ .

PROOF OF THE THEOREM. The proof of Lemma 2 shows that $L'F$ is an a.i. category. The stated bijectivity proves one commutativity, namely $aF = L'F$. The other one is trivial.

## REFERENCES.


1. J. BENABOU, Introduction to bicategories, *Lecture Notes in Math.* 47, Springer (1967), 1-77.
2. M. F. FARINA & G.C. MELONI, The associated sheaf functor, *Rend. Ist. Lomb.* Milano (to be published).
3. F. W. LAWVERE, Metric spaces, generalized logic and closed categories, *Rend. Sem. Mat. e Fis. di Milano* XLIII (1973), 135-166.
4. R. F. C. WALTERS, Sheaves and Cauchy-completeness, *J. Pure and Appl. Alg.* 24 (1982), to appear.
5. R.F.C. WALTERS, Sheaves and Cauchy-complete categories, *Cahiers Topo. et Géom. Diff.* XXII-3 (1981), 283-286.


ADDENDUM IN PROOFS.

After this work was submitted, it has been shown that the a. i. hypothesis of Lemma 2 and of the Theorem of Section 3 is not necessary, because from a result by Betti and Walters (The symmetry of the Cauchy-completion of a category, to appear on the Proc. of 1981 Hagen Conference) it follows that in the $Rel_J(C)$ case the Cauchy-completion preserves symmetry (see the remark after Lemma 1, where this problem was posed).


R. BETTI: Istituto Matematica « F. Enriques »
Via Saldini 50
MILANO. ITALIE

A. CARBONI: Dipartimento di Matematica
Universita degli Studi della Calabria
87036 Arcavacata di Rende
COSENZA. ITALIE