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JOHN DUSKIN

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AN OUTLINE OF NON-ABELIAN COHOMOLOGY IN A TOPOS: (I) THE THEORY OF BOUQUETS AND GERBES

by John DUSKIN

In this paper we will outline an alternative approach to the non-abelian cohomology theory of a topos developed by Grothendieck and Giraud given in Giraud (1971). We will show that the study of this somewhat complicated theory is entirely equivalent to the study of a certain simple and, fortunately, quite manageable subcategory of the category of groupoid objects of the topos which we have chosen to call *bouquets*. The result is an «internalization» of the theory much as has classically been possible for H^1. Full details of the proofs will appear elsewhere as will a separate discussion of H^2 for group coefficients. (Cf. Johnstone (1977) for an introduction as well as a detailed presentation of the «yoga of internal category theory» which our development uses.) As is fitting in this memorial series dedicated to the work of Ehresmann, we note that it was he who first emphasised the crucial role of groupoids in the definition of non-abelian cohomology [Ehresmann (1964)].

In all that follows we will assume that the ambient category E is a Grothendieck topos, i.e. the category of sheaves on some U-small site. It will be quite evident, however, that a considerable portion of the theory is definable in any Barr-exact category [Barr (1971)] provided that the term «epimorphism» is always understood to mean «(universal) effective epimorphism».

THE CATEGORY OF BOUQUETS OF E.

DEFINITION 1. By a *groupoid object* of E we shall mean, as usual, a diagram $\mathfrak{G}$:

$$\begin{tikzcd} Ar(\mathfrak{G}) \arrow[r, shift left, "S"] \arrow[r, shift right, "T"'] & Ob(\mathfrak{G}) \arrow[l, bend right=50, "Id"'] \end{tikzcd}$$

in E such that for any object T in E, the diagram

$$\begin{tikzcd} Hom_{\mathrm{E}}(T, \mathfrak{G}) : Hom_{\mathrm{E}}(T, Ar(\mathfrak{G})) \arrow[r, shift left] \arrow[r, shift right] & Hom_{\mathrm{E}}(T, Ob(\mathfrak{G})) \arrow[l, bend right=50, "Id"'] \end{tikzcd}$$

possesses a groupoid structure in *ENS* (i.e., a category structure in which every arrow is invertible) for which any arrow $f: T \to U$ in E, by restriction, defines a functor

$$Hom_{\mathrm{E}}(f, \mathfrak{G}) : Hom(T, \mathfrak{G}) \to Hom(U, \mathfrak{G}).$$

By a *functor* $\mathfrak{F} : \mathfrak{G}_1 \to \mathfrak{G}_2$ *of groupoid objects* we shall mean a commutative diagram

$$\begin{tikzcd} A_1 \arrow[r, "Ar(\mathfrak{F})"] \arrow[d, shift right, "T"'] \arrow[d, shift left, "S"] & A_2 \arrow[d, shift right, "T"'] \arrow[d, shift left, "S"] \\ O_1 \arrow[r, "Ob(\mathfrak{F})"'] \arrow[u, bend left=60] & O_2 \arrow[u, bend left=60] \end{tikzcd}$$

$$\overline{\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$\mathfrak{G}_1 \xrightarrow{\quad\mathfrak{F}\quad} \mathfrak{G}_2$$

such that

$$Hom_{\mathrm{E}}(T, \mathfrak{F}) : Hom_{\mathrm{E}}(T, \mathfrak{G}_1) \to Hom_{\mathrm{E}}(T, \mathfrak{G}_2)$$

defines a functor of the corresponding groupoids in *ENS*. By an *equivalence of* $\mathfrak{G}_1$ *with* $\mathfrak{G}_2$ we shall mean functors

$$\mathfrak{F} : \mathfrak{G}_1 \to \mathfrak{G}_2 \quad \text{and} \quad \boldsymbol{\mathfrak{G}} : \mathfrak{G}_2 \to \mathfrak{G}_1$$

such that $Hom_{\mathrm{E}}(T, \mathfrak{F})$ has $Hom_{\mathrm{E}}(T, \boldsymbol{\mathfrak{G}})$ as a quasi-inverse.

DEFINITION 2. By an *essential equivalence of* $\mathfrak{G}_1$ *with* $\mathfrak{G}_2$, we shall mean a functor $\mathfrak{F} : \mathfrak{G}_1 \to \mathfrak{G}_2$ which satisfies the following two conditions:

(a) $\mathfrak{F}$ is *fully faithful* (i.e. the commutative diagram

$$\begin{tikzcd} A_1 \arrow[r, "Ar(\mathfrak{F})"] \arrow[d, "<T, S>"] & A_2 \arrow[d, "<T, S>"'] \\ O_1 \times O_1 \arrow[r, "Ob(\mathfrak{F}) \times Ob(\mathfrak{F})"'] & O_2 \times O_2 \end{tikzcd}$$

is cartesian); and

(b) $\mathfrak{F}$ is *essentially epimorphic* (i.e., the canonical map $T.pr_{A_2}$: $O_1 \times_S A_2 \to O_2$ obtained by composition from the cartesian square

$$\begin{tikzcd} O_1 \times_S A_2 \arrow[r, "pr_{A_2}"] \arrow[d, "pr_{O_1}"'] & A_2 \arrow[r, "T"] \arrow[d, "S"] & O_2 \\ O_1 \arrow[r, "Ob(\mathfrak{F})"'] & O_2 & \end{tikzcd}$$

is an epimorphism).

Since this epimorphism does not necessarily split, an essential equivalence does not necessarily admit a quasi-inverse.

DEFINITION 3. A groupoid object

$$\mathfrak{G}: A \underset{T}{\overset{S}{\rightrightarrows}} O$$

will be called a *bouquet* of E provided it satisfies the additional two conditions:

(a) $\mathfrak{G}$ is *(locally) non empty* (i.e., the canonical map $Ob(\mathfrak{G}) \to 1$ into the terminal object of E is an epimorphism); and

(b) $\mathfrak{G}$ is *connected* (i.e., the canonical map

$$\langle T, S\rangle : Ar(\mathfrak{G}) \to Ob(\mathfrak{G}) \times Ob(\mathfrak{G})$$

is an epimorphism).

EXAMPLES OF BOUQUETS. Clearly every group object G of E (considered as a groupoid object for which $Ob(\mathfrak{G}) \to 1$ is an isomorphism) is a bouquet of E, as is any group object of the category E/X provided $X \to 1$ is an epimorphism (such a group object is considered as a groupoid of E for which the source and target arrows $A \underset{T}{\overset{S}{\rightrightarrows}} X$ coincide). The corresponding groupoid will be called a *locally given group*. Slightly less trivially, any *homogeneous space* O under a group action $a: O \times G \to O$ may be considered as defining a bouquet using the groupoid

$$O \times G \underset{pr}{\overset{a}{\rightrightarrows}} G \longrightarrow 1$$

defined by the group action. In particular, if $p: G_1 \to G_2$ is an epimorphism of group objects of E and E is a torsor under G_2 (i.e., principal homogeneous space), then the homogeneous space defined by restricting the group action to G_1 defines a bouquet of E which will be called the *coboundary bouquet of the torsor* E; it will play a crucial rôle in extending the classical exact sequence of pointed sets associated with any short exact sequence of groups of E.

As a moments reflection will show, the concept of a bouquet is closely related to that of a group, for if both the epimorphisms

$$A \twoheadrightarrow O \times O \quad \text{and} \quad O \twoheadrightarrow 1$$

split (as in ENS for example), then the bouquet is simply a category object which is equivalent to a group, which may then be taken as the group of automorphisms $Aut(s)$ of any internal object defined by a splitting s of $O \twoheadrightarrow 1$ (any two such groups are isomorphic). Of course, in general $O \twoheadrightarrow 1$ may be split without $A \twoheadrightarrow O \times O$ being split. In this case, what we obtain is only an *essential equivalence* of $\mathfrak{G}$ *with a group object*, via $i_s : Aut(s) \hookrightarrow \mathfrak{G}$ and any two such groups are only *locally isomorphic*.

In any case the epimorphism $O \twoheadrightarrow 1$ is *locally split* since if we pullback $\mathfrak{G}$ along the epimorphism $O \twoheadrightarrow 1$ we obtain a bouquet $\mathfrak{G}_O$ in E$/O$ for which the diagonal $\Delta : O \to O \times O$ defines a splitting (in E$/O$) of the canonical map $pr_O : Ob(\mathfrak{G}_O) \to O$ into the terminal object O of E$/O$. Thus, in E$/O$, $\mathfrak{G}_O$ is essentially equivalent to the group $Aut(\Delta) \hookrightarrow \mathfrak{G}_O$. But $Aut(\Delta)$ in E is isomorphic to the subgroupoid $\mathfrak{S} \to O$ of automorphisms of $\mathfrak{G}$, defined by the cartesian square

$$\begin{tikzcd}
\mathfrak{S} \arrow[r, hook] \arrow[d] & A \arrow[d, "<T, S>"] \\
O \arrow[r, "\Delta"'] & O \times O
\end{tikzcd}$$

Consequently, we have the following characterization:

THEOREM 1. *A bouquet of* E *is a groupoid object of* E *which is locally essentially equivalent to a locally given group (which may be taken to be the subgroupoid* $\mathfrak{S}$ *of internal automorphisms of* $\mathfrak{G}$ *considered as a group* *object in* E/O*).*

Note that every functor $\mathfrak{F}: \mathfrak{S}_1 \to \mathfrak{S}_2$ of bouquets is essentially epimorphic, thus $\mathfrak{F}$ is an essential equivalence of bouquets iff it is fully faithful. We will designate by $BOUQ(\mathrm{E})$ the subcategory of $CAT(\mathrm{E})$ whose objects are the bouquets of E and whose morphisms are essential equivalences of bouquets.

THE LIEN OF A BOUQUET.

Since every bouquet is locally essentially equivalent to the locally given group of its internal automorphisms, the question immediately arises: is there a *globally* given group with which the bouquet is locally essentially equivalent? If $\mathfrak{S}'$ is such a bouquet and G is the given global group, then there exists a bouquet $\mathfrak{S}: A \rightrightarrows O$ which is essentially equivalent to $\mathfrak{S}'$ and which is supplied with an isomorphism $p: \mathfrak{G} \xrightarrow{\sim} G_O$ of group objects in E/O. Since this means that we have a cartesian square

$$\begin{tikzcd}
\mathfrak{G} \arrow[r] \arrow[d] & G \arrow[d] \\
O \arrow[r, two heads] & 1
\end{tikzcd}$$

in E which makes $\mathfrak{G}$ the localization of G over O, it should follow that we could recover G from the group $\mathfrak{G}$ by means of a descent datum somehow supplied intrinsically by the bouquet $\mathfrak{S}$. Now there is indeed a naturally occuring candidate for the provision of such a descent datum: the canonical action of $\mathfrak{S}$ on $\mathfrak{G}$ by inner isomorphisms. Thus consider the diagram

$$\begin{tikzcd}
O\times O\times O \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & O\times O \arrow[r, shift left, "pr_2"] \arrow[r, shift right, "pr_1"'] & O \arrow[r, two heads] & 1 \\
A\times_O A \arrow[u] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & A \arrow[u] \arrow[r, shift left, "\mathrm{S}"] \arrow[r, shift right, "\mathrm{T}"'] & O \arrow[u, "Id"'] &
\end{tikzcd}$$

In the category E/A, we have the group isomorphism

$$\begin{array}{rcl}
int(\mathfrak{S}): A\times_{\mathrm{S}}\mathfrak{G} & \xrightarrow{\sim} & A\times_{\mathrm{T}}\mathfrak{G} \\
\| & & \| \\
\mathrm{S}^*(\mathfrak{G}) & \xrightarrow{\sim} & \mathrm{T}^*(\mathfrak{G})
\end{array}$$

given by the assignment

$$(f: X \to Y,\ a: X \to X) \mapsto (f: X \to Y,\ faf^{-1}: Y \to Y).$$

As a «glueing» it is easily seen to satisfy the «cocycle condition» when restricted to the category $\mathrm{E}/A \times_O A$. Thus in order that it define a true descent datum $d: pr_1^*(\mathfrak{G}) \to pr_2^*(\mathfrak{G})$ in $\mathrm{E}/O \times O$ it is necessary and sufficient that it have the same restriction when pulled back along the two projections of the graph of the equivalence relation associated with the epimorphism $<\mathrm{T}, \mathrm{S}>: A \twoheadrightarrow O \times O$. Since this equivalence relation consists of the object of internal ordered pairs of arrows $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ which have the same source and target, $int(\mathfrak{G})$ defines a descent datum on $\mathfrak{G}$ iff for all $a: X \to X$ and all pairs (f, g), $faf^{-1} = gag^{-1}$, i.e. «inner isomorphism» be independent of choice of representative. But since

$$faf^{-1} = (fg^{-1})gag^{-1}(fg^{-1})^{-1}$$

and $fg^{-1}: Y \to Y$ is an automorphism this can occur only if $Aut(Y)$ is abelian for all Y, i.e. iff $\mathfrak{G} \to O$ is an abelian group object on E/O, a clearly untenable assumption if we wish to consider bouquets which are locally essentially equivalent to the localization of a given non-abelian group (where $\mathfrak{G} \to O$ cannot be abelian).

What does survive here even if $\mathfrak{G} \to O$ is not abelian is based on the observation that $int(f)$ and $int(g)$ while not identical for all f and g do differ by an inner automorphism of Y (that defined by $fg^{-1}: Y \to Y$) and thus are equal modulo an inner automorphism. This necessitates the replacement of the fibered category of locally given groups with a new fibered category called the *liens of* E.

This new fibered category is defined as follows: First we define the fibered category $LI(\mathrm{E})$ of *pre-liens* of E. Its fiber at any object X of E has as objects the group objects of E/X. Its morphisms, however, consist of the global sections (over X) of the coequalizer (i.e. orbit space)

$$Hom_X(G_1, G_2) \times G_2 \rightrightarrows Hom_X(G_1, G_2) \longrightarrow Hex_X(G_1, G_2)$$

of the sheaf of group homomorphisms of G_1 into G_2 under the action of G_2 by composition with inner automorphisms of G_2. Under pullbacks this defines a fibered category over E in which morphisms still glue along a covering even though their «true existence» may be only local over some covering $C \twoheadrightarrow X$ of X. We now define the fibered category $LIEN(\mathrm{E})$ of *liens* (or *ties*) of E by completing it to a stack so that every descent datum in $LI(\mathrm{E})$ over a covering $X \twoheadrightarrow Y$ is effective. An object of the fiber of $LIEN(\mathrm{E})$ over the terminal object 1 will be called a (global) *lien of* E. As we shall see when we discuss the Grothendieck-Giraud theory later in this paper it will be convenient to regard a lien of E as represented by an equivalence class under refinement of a descent datum in $LI(\mathrm{E})$ on some locally given group $\mathfrak{G} \to O$ over a covering $O \twoheadrightarrow 1$ of E, i.e., by some given global section of $Hex(pr_1^*(\mathfrak{G}), pr_2^*(\mathfrak{G}))$ over $O \times O$.

Clearly, from our preceding analysis, for any bouquet $\mathfrak{G}$, the canonical action by inner isomorphisms supplies $\mathfrak{G} \to O$ with such a descent datum and hence defines not a global group but rather a global lien which is unique up to a unique isomorphism. It will be called the *lien of the bouquet* $\mathfrak{G}$ and will be denoted by $lien(\mathfrak{G})$. If $\mathfrak{F} : \mathfrak{G}_1 \to \mathfrak{G}_2$ is an essential equivalence of bouquets, then $lien(\mathfrak{G}_1) \xrightarrow{\sim} lien(\mathfrak{G}_2)$.

We now give the following

DEFINITION 4. Let L be a given global lien. We define the category $BOUQ(\mathrm{E}; L)$ as the subcategory of $BOUQ(\mathrm{E})$ consisting of those bouquets of E which have lien isomorphic to L (together with essential equivalences as morphisms). We will designate the class of connected components of this category by $\mathrm{H}^2(\mathrm{E}; L)$ and call it the *second cohomology class of* E *with coefficients in the lien* L. If E is a site, we define $\mathrm{H}^2(\mathrm{E}; L)$ as $\mathrm{H}^2(\mathrm{E}^{\sim}; L)$ where $\mathrm{E}^{\sim}$ is the associated topos of sheaves over the site E.

If G is a sheaf of groups, then we define the *(unrestricted) second cohomology class of* E *with coefficients in the group* G as $\mathrm{H}^2(\mathrm{E}; lien(G))$ and point this class by the class of the group G, considered as a bouquet whose lien is that of G.

In general $\mathrm{H}^2(\mathrm{E}; L)$ may be empty. However, if $\mathfrak{G}$ is a bouquet with lien L which admits a global section $s : 1 \to Ob(\mathfrak{G})$, then, since

$aut_{\mathfrak{G}}(s) \hookrightarrow \mathfrak{G}$ is an essential equivalence, $L \xrightarrow{\sim} lien(aut_{\mathfrak{G}}(s))$, and L is represented by a global group. Such a bouquet will be said to be *neutral.* A class in $H^2(E;L)$ will be said to be neutral if at least one of its representatives is neutral. If $L \xrightarrow{\sim} lien(G)$, then the (neutral) class of G will be called the *trivial* or *unit class of* $H^2(E;lien(G))$.

$H^2(E;L)$ is not, in general, functorial on morphisms of liens and $H^2(E;lien(G))$ does not, in general, lead to a continuation of the cohomology exact sequence of groups and pointed sets associated with a short exact sequence of groups in E. We will rectify this at a later point by considering a smaller class of members of $BOUQ(E)$ associated with a given group G.

We now will define a neutral element preserving bijection of the above defined $H^2(E;L)$ with the set of the same name defined in Giraud (1971).

BOUQUETS AND GERBES.

Recall the following

DEFINITION 5. If E is a U-small site for some universe U, then a *gerbe* (over E) is a fibered category F over E which satisfies the following conditions:

(a) F is a *stack* (fr. *champ*), i.e., both objects and arrows in the fibers over any covering glue;

(b) F is *fibered in* U-small *groupoids*, i.e. for each object X in E, the category fiber F_X is a U-small groupoid;

(c) there is a covering of E such that each of the fibers over that covering are non empty; and

(d) any objects x and y of a fiber F_X are locally isomorphic.

From (a) it follows that for any object x in F_X, the presheaf $Aut_X(x)$ on E/X is, in fact, a sheaf and that for any isomorphism $f: x \to y$ in F_X, the induced group isomorphism of $Aut_X(x)$ with $Aut_X(y)$ is unique up to an inner automorphism. Thus conditions (c) and (d) define the existence of a global lien over E (which is unique up to a unique isomorphism) and is called the *lien of the gerbe* F. For any lien L, Grothendieck and Giraud make the following

DEFINITION 6. The *second cohomology set of* E *with coefficients in the lien* L, $H^2_{Gir}(L)$, is the set of equivalence classes under cartesian equivalence of fibered categories of those gerbes of E which have lien L. If G is a sheaf of groups, then $H^2_{Gir}(lien(G))$ is pointed by the *gerbe* $TORS_{\mathrm{E}}(G)$ *of* G*-torsors* over E, whose fiber at any X in E is the groupoid of torsors (i.e., principal homogeneous spaces) in E/X under the group G_X.

We now intend to establish a mapping of our $\mathrm{H}^2(\mathrm{E}; L)$ into $H^2_{Gir}(\mathrm{E}; L)$ which will turn out to be a bijection. To do this we must extend the standard definition of torsor under a sheaf of groups to that of a torsor under a sheaf of groupoids. Recall that this is done as follows:

Let $\mathfrak{S}: A \rightrightarrows O$ be a groupoid in a topos E. By an *internal (contravariant) functor from* $\mathfrak{S}$ *into* E (or a (right-) *operation* of $\mathfrak{S}$ on an object E of E, or more simply, an $\mathfrak{S}$*-object of* E) we shall mean an object E of E supplied with an arrow $p_0 : E \to O$ together with an action

$$\xi : E \;{}_{p_0}\!\times_{\mathrm{T}} A \to E$$

such that for all $T \in Ob(\mathrm{E})$, $Hom_{\mathrm{E}}(T, \xi)$ defines a right action of the groupoid $Hom_{\mathrm{E}}(T, \mathfrak{S})$ on the set $Hom_{\mathrm{E}}(T, E) \to Hom_{\mathrm{E}}(T, O)$ in the usual equivariant set theoretic sense.

$$\begin{array}{ccc} E \times_{\mathrm{T}} A & \xrightarrow{pr} & A \\ {\scriptstyle pr}\downarrow\;\downarrow{\scriptstyle \xi} & & \downarrow\downarrow \\ E & \xrightarrow{p_0} & O \\ \downarrow & & \\ 1 & & \end{array}$$

Using the obvious definition of $\mathfrak{S}$*-equivariant map of such* $\mathfrak{S}$*-objects* (or *internal natural transformation*) we obtain the category $OPER(\mathrm{E}; \mathfrak{S})$ (also denoted by $\mathrm{E}^{\mathfrak{S}^{op}}$) of $\mathfrak{S}$-objects of E and equivariant maps of $\mathfrak{S}$-objects. For each X of E we also have the corresponding category of $\mathfrak{S}$*-objects* of E *above* X, defined by $OPER(\mathrm{E}/X; \mathfrak{S}_X)$. Under pullbacks this defines the corresponding *fibered category* $OPER_{\mathrm{E}}(\mathfrak{S})$ of $\mathfrak{S}$-*objects over* E.

For each global section $[x]: 1 \to O$ of O, we have the corresponding *internal representable* functor defined by $[x]$ which is defined using the fibered product $1 \times_{\mathrm{T}} A \xrightarrow{\mathrm{S} pr_A} O$ for «total space» and the composition in $\mathfrak{S}$ to define the action. In sets this just gives the category $\mathfrak{S}/[x]$ of $\mathfrak{S}$-objects above the object $[x]$.

We have the following immediate result:

THEOREM 2. *In order that an internal functor ($\mathfrak{S}$-object) be representable,* i.e., isomorphic to $\mathfrak{S}/[x]$ for some global section $[x]: 1 \to O$, *it is necessary and sufficient that it satisfy the following two conditions:*

(a) the canonical map $\langle pr_E, \xi\rangle : E \times_{\mathrm{T}} A \to E \times E$ *is an isomorphism (i.e., the action* ξ *is a principal action); and*

(b) the canonical mapping $E \to 1$ *admits a splitting* $s: 1 \to E$ *(i.e.,* E *is globally non empty).*

We now make the following

DEFINITION 7. By an $\mathfrak{S}$-*torsor of* E *above* X we shall mean an $\mathfrak{S}_X$-object of E/X which is *locally representable*, i.e., becomes representable when restricted to some covering of E/X.

For the canonical topology on E, this is entirely equivalent to the following two conditions on the defining diagram in E:

$$\begin{tikzcd}
E \times_{\mathrm{T}} A \arrow[r] \arrow[d, shift right, "pr"'] \arrow[d, shift left, "\xi"] & A \arrow[d, shift right, "\mathrm{T}"'] \arrow[d, shift left, "\mathrm{S}"] \\
E \arrow[r] \arrow[d, "p"'] & O \\
X &
\end{tikzcd}$$

(a) the canonical map $E \times_{\mathrm{T}} A \to E \times_X E$ *is an isomorphism;* and

(b) the canonical map $p: E \to X$ *is an epimorphism.*

A torsor is thus *representable iff it is split, i.e.,* p *admits a section* $s: X \to E$.

Again, since this definition is stable under change of base we have the corresponding fibered category $TORS_{\mathrm{E}}(\mathfrak{G})$ of $\mathfrak{G}$*-torsors of* E whose fiber at any X is just $TORS(\mathrm{E}/X; \mathfrak{G}_X)$. If $\mathfrak{G}$ is a group object in E , we immediately recover the usual definition of $\mathfrak{G}$-objects of E and $\mathfrak{G}$-torsors of E. Note also that a group, as a category, has up to isomorphism only one representable functor, which is just $\mathfrak{G}_d$, i.e. $\mathfrak{G}$ acting on itself on the right by multiplication.

As with the case of group objects, $OPER(\mathrm{E}; \mathfrak{G})$ is functorial on functors $\mathfrak{F}: \mathfrak{G}_1 \to \mathfrak{G}_2$ of groupoids:

$$OPER(\mathrm{E}; \mathfrak{F}): OPER(\mathrm{E}; \mathfrak{G}_2) \to OPER(\mathrm{E}; \mathfrak{G}_1)$$

is just defined by «restricting the $\mathfrak{G}_2$-action» on the object $p_0: E \to O_2$ to that of $\mathfrak{G}_1$ on $E \times_{p_0} O_1 \xrightarrow{pr} O_1$. Also, as with groups, $OPER(\mathrm{E}; \mathfrak{F})$ has a left exact left adjoint which carries torsors under $\mathfrak{G}_1$ to torsors (above the same base) under $\mathfrak{G}_2$. Its restriction to the corresponding categories of torsors will be denoted by

$$TORS_X(\mathfrak{F}): TORS(\mathrm{E}/X; \mathfrak{G}_1) \to TORS(\mathrm{E}/X; \mathfrak{G}_2).$$

We now may state the principal result of this section:

THEOREM 3. *Let* E *be a Grothendieck topos (over some* U*-small site) and* $\mathfrak{G}$ *a groupoid object of* E. *Then:*

1° The fibered category $TORS_{\mathrm{E}}(\mathfrak{G})$ *is a* U*-small stack of groupoids over* E (i.e., each of the fibers is a U-small groupoid and descent data on $\mathfrak{G}$-torsors defined over a covering is always effective (on both objects and arrows).

2° If $\mathfrak{G}$ *is a bouquet, then* $TORS_{\mathrm{E}}(\mathfrak{G})$ *is a gerbe whose lien is isomorphic to the lien of* $\mathfrak{G}$.

3° If $\mathfrak{F}: \mathfrak{G}_1 \to \mathfrak{G}_2$ *is an essential equivalence, then*

$$TORS(\mathfrak{F}): TORS(\mathfrak{G}_1) \to TORS(\mathfrak{G}_2)$$

is a (full) cartesian equivalence of fibered categories.

That $\mathfrak{G}$-torsors over a covering glue is diagram chasing exercise from the theory of Barr-exact categories. That every morphism of $\mathfrak{G}$-torsors above X is an isomorphism is a consequence of the well known theorem which appears on Grothendieck (1962). That each of the categories $TORS(\mathrm{E}/X; \mathfrak{G})$ is equivalent to a small U-category follows from the fact that the groupoid of torsors which are trivialized when restricted to a given covering is equivalent to the groupoid of simplicial maps («Čech cocycles») and homotopies of simplicial maps from the nerve of the covering into the nerve of the groupoid $\mathfrak{G}$. If $\mathfrak{G}$ is a bouquet, then the epimorphism $O \twoheadrightarrow 1$ furnishes a covering of E for which the fiber $TORS(\mathrm{E}/O; \mathfrak{G})$ contains the split torsor defined by the identity map $Id: O \to O$. The epimorphism $A \twoheadrightarrow O \times O$ furnishes a covering which makes any two torsors in a given fiber locally isomorphic. That the lien of $\mathfrak{G}$ is isomorphic to the lien of $TORS(\mathfrak{G})$ follows from the fact that the sheaf $\mathfrak{E} \to O$ of internal automorphisms of $\mathfrak{G}$ is isomorphic to the sheaf of automorphisms of the split tensor defined by $Id: O \to O$. That essential equivalences define cartesian equivalences of the corresponding fibered groupoids of torsors is based on an analysis of the construction of this functor which is through «twisting» the given torsor by the $\mathfrak{G}_1$-object above O_2 defined through

$$\begin{array}{ccc} O_1 \times_{\mathrm{T}} A_2 & \xrightarrow{pr} & O_1 \\ \Big\downarrow{\scriptstyle Spr} & & \\ O_2 & & \end{array}$$

This $\mathfrak{G}_1$-object is an $\mathfrak{G}_1$-torsor above O_2 iff $\mathfrak{F}$ is an essential equivalence. Note that $\mathfrak{F}$ has a quasi-inverse iff this torsor is split.

EXAMPLE. If $p: A \to B$ is an epimorphism of group objects and T a given B-torsor with $\partial(T)$ the associated coboundary bouquet, $TORS(\partial(T))$ is cartesian equivalent to the gerbe of liftings of T to A, as defined in Giraud (1971).

Now let E be a U-small site and $\mathrm{E}^{\sim}$ its associated category of sheaves. From Theorem 3 it follows that for any bouquet $\mathfrak{G}$ with lien L, the restriction $TORS(\mathrm{E}; \mathfrak{G})$ of the gerbe $TORS(\mathrm{E}^{\sim}; \mathfrak{G})$ to E (whose fiber for any representable X is just $TORS(\mathrm{E}^{\sim}/a(X); \mathfrak{G})$ where $a(X)$ is the associated sheaf) is a gerbe over E whose lien is also L. Moreover, if $\mathfrak{G}_1$ and $\mathfrak{G}_2$ lie in the same connected component of $BOUQ(\mathrm{E}^{\sim}, L)$ then $TORS(\mathrm{E}, \mathfrak{G}_1)$ is cartesian equivalent to $TORS(\mathrm{E}; \mathfrak{G}_2)$. Thus we have

THEOREM 4. *The assignment* $\mathfrak{G} \mapsto TORS(\mathrm{E}; \mathfrak{G})$ *defines a functor*

$$TORS_{\mathrm{E}}: BOUQ(\mathrm{E}^{\sim}; L) \to GERB(\mathrm{E}; L)$$

which induces a mapping

$$T: \mathrm{H}^2(\mathrm{E}; L) \to H^2_{Gir}(\mathrm{E}; L)$$

which clearly preserves base points if $L \xrightarrow{\sim} lien(G)$ *and also neutral classes* (a gerbe is said to be *neutral* iff it admits a cartesian section).

We now may state the main result of this paper:

THEOREM 5. *The mapping*

$$T: \mathrm{H}^2(\mathrm{E}; L) \to H^2_{Gir}(\mathrm{E}; L)$$

is a bijection which carries the equivalence class of neutral elements: $\mathrm{H}^2(\mathrm{E}; L)'$ *bijectively onto* $H^2_{Gir}(\mathrm{E}; L)'$, *the equivalence class of neutral elements of* $H^2_{Gir}(\mathrm{E}; L)$.

We will establish this theorem by defining a functor

$$GERB(\mathrm{E}; L) \to BOUQ(\mathrm{E}; L)$$

which will induce an inverse for T.

To this end recall [Giraud (1962)] that if F is a category over E then for each object X in E we may consider the category $Cart_{\mathrm{E}}(\mathrm{E}/X, \mathrm{F})$ whose objects are cartesian E-functors from the category E/X of objects of E above X into F and whose morphisms are E-natural transformations of such E-functors. If F_X denotes the category fiber of F at X, then evaluation of such a cartesian functor at the terminal object $Id: X \to X$ of E/X defines a functor

$$ev_X: Cart(\mathrm{E}/X, \mathrm{F}) \to \mathrm{F}_X$$

which is an equivalence of categories provided F is a fibration. If E is U-small and F is fibered in U-small categories, then the assignment $X \mapsto Cart_{\mathrm{E}}(\mathrm{E}/X, \mathrm{F})$ defines a presheaf of U-small categories and thus a category object of $\mathrm{E}^{\wedge}$ which will be denoted by $Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F})$. The split fibration over E which it determines is denoted by SF and is called the (right) *split fibration* E-*equivalent to* F.

Now suppose that E is a U-small site. Since the associated sheaf functor $a : \mathrm{E}^{\wedge} \to \mathrm{E}^{\sim}$ is left exact we may apply the functor to the nerve of category object $Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F})$ and obtain a sheaf of categories together with a canonical *functor* (in $CAT(\mathrm{E}^{\wedge})$)

$$\begin{tikzcd}
Ar(Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F})) \arrow[r, "a_1"'] \arrow[d, shift right, "\mathrm{T}"'] \arrow[d, shift left, "\mathrm{S}"] & a\,Ar(Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F})) \arrow[d, shift right, "a(\mathrm{T})"'] \arrow[d, shift left, "a(\mathrm{S})"] \\
Ob(Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F})) \arrow[r, "a_0"'] & a\,Ob(Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F}))
\end{tikzcd}$$

$$\underline{a} : Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F}) \longrightarrow a\,Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F})$$

(The split fibration over E determined by $a\,Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F})$ is denoted by KSF.) We may now state:

THEOREM 6. *1° If* F *is fibered in groupoids, then* $Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F})$ *(as well as* $a\,Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F})$*) is a groupoid (and conversely).*

2° If F *is a stack then* $\underline{a} : Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F}) \to a\,Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F})$ *is an equivalence of categories in* $\mathrm{E}^{\wedge}$ *(and conversely).*

3° If F *is a gerbe with lien* L*, then* $a\,Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F})$ *is a bouquet with lien* L *in* $\mathrm{E}^{\sim}$ *(and conversely).*

On the basis of Theorem 6 we see that the assignment

$$\mathrm{F} \mapsto a\,Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F})$$

defines a functor

$$a\,Cart_{\mathrm{E}}(\mathrm{E}/-,-) : GERB(\mathrm{E};L) \to BOUQ(\mathrm{E};L)$$

and this a mapping

$$a : H^2_{Gir}(\mathrm{E};L) \to \mathrm{H}^2(\mathrm{E};L)$$

which we claim provides an inverse for $T : \mathrm{H}^2(\mathrm{E};L) \to H^2_{Gir}(\mathrm{E};L)$.

In order to prove Theorem 6 and complete the proof of Theorem 5, we shall now discuss certain relations between «internal and external completeness» for fibrations and category objects.

REMARK. We note that a portion of the «internal version» of what follows (specifically Theorem 7...) was discovered by Joyal (1974) and later, but independantly, by Penon (1979). It is also closely related to work of Bunge (1979) establishing a conjecture of Lawvere (1974). Our principal addition is the relation between internal and external completeness (Theorem 6, 2°).

EXTERNAL AND INTERNAL COMPLETENESS.

The informal and intuitive definition of a stack which we have used so far («every descent datum on objects or arrows is effective» or «objects and arrows from the fibers over a covering glue») [Grothendieck (1959)] implicitly uses the respective notions of *fibered category defined via a pseudo-functor* $F_{(\ )}: E^o \to CAT$ as formalized in 1960 [Grothendieck (1971)] and *covering* as defined in the original description of Grothendieck topologies formalized in Artin (1962). For the formal development of the theory, however, both of those notions are cumbersome and were replaced by the intrinsic formulation of Giraud (1962, 1971) which replaces «*covering*» by «*covering sieve*» (which we shall view as a subfunctor $R \hookrightarrow X$ of a representable X in $E^\wedge$ as in Demazure (1970)) and notes that every descent datum over a covering of X corresponds to an E-*cartesian functor*

$$\begin{tikzcd} E/R \arrow[r, "d"] \arrow[d] & F \arrow[dl] \\ E & \end{tikzcd}$$

from the category E/R *of representables of* $E^\wedge$ *above the covering presheaf* R (isomorphic to the corresponding covering seive of X) *into the fibration* (E-category) defined by the pseudo-functor $F_{(\ )}$ [cf. Grothendieck (1971)].

Thus *every descent datum on arrows, respectively, on both objects and arrows is effective iff the canonical restriction functor*

$$i : \mathit{Cart}_{\mathrm{E}}(\mathrm{E}/X, \mathrm{F}) \to \mathit{Cart}_{\mathrm{E}}(\mathrm{E}/R, \mathrm{F})$$

is fully faithful, respectively, an equivalence of categories for every covering subfunctor $R \hookrightarrow X$ in the topology of the site.

The corresponding terminology for a fibration $\mathrm{F} \to \mathrm{E}$ over a site E is then

DEFINITION 8. A fibration F is said to be *precomplete*, respectively, *complete,* provided that for every covering sieve $R \hookrightarrow X$ of a representable X, the canonical functor

$$i : \mathit{Cart}_{\mathrm{E}}(\mathrm{E}/X, \mathrm{F}) \to \mathit{Cart}_{\mathrm{E}}(\mathrm{E}/R, \mathrm{F})$$

is fully faithful, respectively, an equivalence of categories.

Since every fibration is determined up to isomorphism by its associated pseudo-functor, we see that *a stack is just a complete fibration.*

We now look the «internal version» of this notion. Let E be a topos and $\mathfrak{S} : A \rightrightarrows O$ a category object in E. The presheaf of categories defined by the assignment

$$X \mapsto \mathit{Hom}_{\mathrm{E}}(X, \mathfrak{S}) \;\; (= \mathit{Hom}_{\mathrm{E}}(X, A) \rightrightarrows \mathit{Hom}_{\mathrm{E}}(X, O))$$

considered as a pseudo-functor defines a *split* fibration (its «*externalization*» $\mathrm{Ex}(\mathfrak{S}) \to \mathrm{E}$) which has as objects the arrows $x : X \to O$ of E and for which an arrow $a : x \to y$ of projection $f : X \to Y$ in E is just an arrow:

$$a : X \to A \quad \text{such that} \quad \mathrm{S}a = x \text{ and } \mathrm{T}a = yf.$$

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[dr, "x"'] \arrow[d, "f"'] & A \arrow[d, shift right, "\mathrm{T}"'] \arrow[d, shift left, "\mathrm{S}"] \\ Y \arrow[r, "y"'] & O \end{tikzcd}$$

Such an arrow is cartesian iff a is an isomorphism in the category

$$\mathit{Hom}_{\mathrm{E}}(X, \mathfrak{S}) = \mathrm{Ex}(\mathfrak{S})_X,$$

the fiber at X. Now let

$$C_{\cdot}/X : C \times_X C \times_X C \equiv\!\!\!\!\!\rightrightarrows C \times_X C \rightrightarrows C \twoheadrightarrow X$$

be the nerve of a covering $p : C \twoheadrightarrow X$. It is not difficult to see that *a des-*

cent datum on $Hom_{\mathrm{E}}(-,\mathfrak{G})$ over the covering $C \to X$ is nothing more than a simplicial map $d: C./X \to \mathfrak{G}$

$$\begin{tikzcd} C\times C\times C \arrow[r, "d_2"] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & A\times A \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\ C\times C \arrow[r, "d_1"] \arrow[d, shift left] \arrow[d, shift right] & A \arrow[d, shift left] \arrow[d, shift right] \\ C \arrow[r, "d_0"] \arrow[d, "p"'] & O \\ X & \end{tikzcd}$$

i.e., an internal functor from the groupoid $C\times_X C \rightrightarrows C$ into $\mathfrak{G}$. Similarly, a morphism of such descent data is nothing more than a homotopy of such simplicial maps, i.e., an internal natural transformation of such internal functors. Consequently, a datum is effective iff there exists an arrow $x: X \to O$ such that the trivial functor defined through $xp: C \to O$ is isomorphic to d. Thus $\mathrm{Ex}(\mathfrak{G}) \to \mathrm{E}$ *is complete* (for the canonical topology) *iff for every epimorphism* $C \twoheadrightarrow X$, *the canonical restriction functor*

$$p_{\mathfrak{G}}: Hom_{\mathrm{E}}(X,\mathfrak{G}) \to Simpl_{\mathrm{E}}(C./X,\mathfrak{G}).$$

is an equivalence of categories. But if $\mathfrak{G}$ is a groupoid, we have already noted that $Simpl_{\mathrm{E}}(C./X,\mathfrak{G})$ is equivalent to the category of $\mathfrak{G}$-torsors above X which are split when restricted along $C \twoheadrightarrow X$, while $Hom_{\mathrm{E}}(X,\mathfrak{G})$ is equivalent to the category of split $\mathfrak{G}$-torsors above X. Under refinement, we thus obtain that $\mathrm{Ex}(\mathfrak{G}) \to \mathrm{E}$ is complete iff the canonical functor

$$Spl\ TORS(X;\mathfrak{G}) \to TORS(X;\mathfrak{G})$$

is an equivalence of categories, i.e., *every torsor splits*. Finally note that the canonical functor defined by p:

$$\begin{tikzcd} C\times C \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & X \arrow[d, shift left] \arrow[d, shift right] \\ C \arrow[r, "p"] & X \end{tikzcd}$$

is an essential equivalence and thus this same property is linked to an «injectivity» property of $\mathfrak{G}$ with respect to essential equivalences. In summary, we have the following

THEOREM 7. *For any topos* E *and any groupoid object* $\mathfrak{G}$ *in* E, *the following statements are equivalent:*

1° $\mathrm{Ex}(\mathfrak{G}) \to \mathrm{E}$ *is a complete fibration, i.e., every descent datum is effective.*

2° For any covering

$$C \times_X C \rightrightarrows C \twoheadrightarrow X,$$

the fully faithful restriction functor

$$Hom_{\mathrm{E}}(X, \mathfrak{G}) \to Simpl_{\mathrm{E}}(C_. /X, \mathfrak{G})$$

is an equivalence of groupoids (i.e., Čech cohomology is neutral).

3° For any object X *in* E, *the fully faithful functor*

$$Spl\, TORS(X; \mathfrak{G}) \to TORS(X; \mathfrak{G})$$

is an equivalence of groupoids (i.e., every locally representable functor is representable).

4° For any essential equivalence of groupoids $H : \mathfrak{F}_1 \to \mathfrak{F}_2$ *the fully faithful functor*

$$CAT_{\mathrm{E}}(H, \mathfrak{G}) : CAT_{\mathrm{E}}(\mathfrak{F}_2, \mathfrak{G}) \to CAT_{\mathrm{E}}(\mathfrak{F}_1, \mathfrak{G})$$

is essentially surjective, i.e., given any functor $F : \mathfrak{F}_1 \to \mathfrak{G}$, *there exists a functor* $\bar{F} : \mathfrak{F}_2 \to \mathfrak{G}$ *such that* $\bar{F} H \xrightarrow{\sim} F$.

The linkage of 4 with the others uses the canonical torsor under $\mathfrak{F}_1$ defined by the essential equivalence H. A similar theorem holds for category objects, locally representable functors, and existence of adjoints.

DEFINITION 9. A groupoid which satisfies any one and hence all of the equivalent conditions of Theorem 7 will be said to be (internally) *complete*. A functor $c : \mathfrak{G} \to \bar{\mathfrak{G}}$ will be called a *completion of* $\mathfrak{G}$ provided any other functor $\mathfrak{G} \to \mathfrak{C}$ into a complete groupoid $\mathfrak{C}$ factors essentially uniquely through $\bar{\mathfrak{G}}$.

Such a completion is essentially unique and we have the following:

COROLLARY 1. *If* $\mathfrak{G}$ *is complete, then:*

(a) Any essential equivalence $H: \mathfrak{G} \to \mathfrak{T}$ *admits a quasi-inverse* $H': \mathfrak{T} \to \mathfrak{G}$.

(b) $c: \mathfrak{G} \to \bar{\mathfrak{G}}$ *is a completion of* $\mathfrak{G}$ *iff* $\bar{\mathfrak{G}}$ *is complete and* c *is an essential equivalence.*

We are now in a position to return to our original situation where E is a U-small site and $F \to E$ is a fibration fibered in U-small groupoids. We have the following

LEMMA 1. *(a) For any presheaf* P *in* $E^\wedge$, *one has a natural equivalence of groupoids*

$$Nat(P, Cart_E(E/-, F)) \xrightarrow{\approx} Cart_E(E/P, F).$$

(b) As a groupoid object in $E^\wedge$, $Cart_E(E/-, F)$ *is complete in the canonical topology on* $E^\wedge$.

Here $E/P \to E$ denotes the fibered category of representables above P. It is, in fact, the restriction to E of the fibration

$$\mathrm{Ex}\,(P \underset{id}{\overset{id}{\rightrightarrows}} P) \to E^\wedge$$

where $P \underset{id}{\overset{id}{\rightrightarrows}} P$ is the discrete groupoid object defined by the object P of $E^\wedge$. Of course

$$Nat(P, Q) = Hom_{E^\wedge}(P, Q).$$

As an immediate corollary, we have the following as a literal translation of the original external definition of completeness:

COROLLARY 2. *The fibration* $F \to E$ *is precomplete, respectively, complete in the topology of* E *iff the following two equivalent conditions hold:*

(a) For every covering subfunctor $R \hookrightarrow X$ *of a representable* X, *the canonical restriction functor*

$$Nat(X, Cart_E(E/-, F)) \to Nat(R, Cart_E(E/-, F))$$

is fully faithful, respectively, an equivalence of categories.

(b) For every covering $C: (X_\alpha \to X)_{\alpha \in I}$ *in the topology of* E, *the*

canonical restriction functor

$$Nat(X, Cart_{\mathrm{E}}(\mathrm{E}/-, \mathrm{F})) \to Simpl_{\mathrm{E}^{\wedge}}(C_{.}/X, Cart_{\mathrm{E}}(\mathrm{E}/-, \mathrm{F}))$$

defined by the projection of the nerve

$$C_{.} : \amalg X_{\alpha} \times_{X} X_{\beta} \rightrightarrows \amalg X_{\alpha} \xrightarrow{p} X$$

onto X *is fully faithful, respectively, an equivalence of categories.*

This is immediate since $(X_{\alpha} \to X)_{\alpha \in I}$ is a covering iff the image of $\amalg X_{\alpha} \xrightarrow{p} X$ is a covering subfunctor.

We may now return to the proof of Theorem 6, part 2. We shall break it into several parts.

LEMMA 2. *A fibration* $\mathrm{F} \to \mathrm{E}$ *is precomplete iff any one and hence all of the following equivalent conditions hold:*

(a) For any object X *in* E *and arrow*

$$X \xrightarrow{<x, y>} Ob(Cart_{\mathrm{E}}(\mathrm{E}/-, \mathrm{F})) \times Ob(Cart_{\mathrm{E}}(\mathrm{E}/-, \mathrm{F}))$$

in $\mathrm{E}^{\wedge}$, *the presheaf above* X *defined by the cartesian square*

$$\begin{array}{ccc} Hom_{X}(x, y) & \xrightarrow{\quad pr \quad} & Ar(Cart_{\mathrm{E}}(\mathrm{E}/-, \mathrm{F})) \\ {\scriptstyle pr}\downarrow & & \downarrow{\scriptstyle <T, S>} \\ X & \xrightarrow[<x, y>]{} & Ob(Cart_{\mathrm{E}}(\mathrm{E}/-, \mathrm{F})) \times Ob(Cart_{\mathrm{E}}(\mathrm{E}/-, \mathrm{F})) \end{array}$$

is a sheaf (above X*).*

(b) For any $X \in Ob(\mathrm{E})$ *and any pair of objects* x, y *in* F_{X}, *the presheaf* $Hom_{X}(x, y)$ *on* E/X *defined by*

$$Hom_{X}(x, y)(T \xrightarrow{f} X) = Hom_{\mathrm{F}_{T}}(\mathrm{F}_{f}(x), \mathrm{F}_{f}(y))$$

is a sheaf (in the induced topology on E/X*).*

(c) If for any presheaf P $\mathrm{L}(P)$ *designates the presheaf whose value at* X *is given by*

$$\mathrm{L}(X) = \varinjlim_{R \in Cov(X)} Nat(R, P)$$

(so that $\mathrm{LL}(P)$ *is the associated sheaf functor at* P*) and* $l : P \to \mathrm{L}P$ *is the canonical map, then the canonical functor*

$$l : Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F}) \to \mathrm{L}\,Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F})$$

is fully faithful.

(d) the canonical functor

$$\underline{a} : Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F}) \to a\,Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F})$$

is fully faithful.

LEMMA 3. *If* F → E *is precomplete and fibered in groupoids, then the following statements are equivalent:*

(a) $\quad l : Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F}) \to \mathrm{L}\,Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F})$

is essentially epimorphic.

(b) $\quad l : Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F}) \to \mathrm{L}\,Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F})$

admits a quasi-inverse in $CAT(\mathrm{E}^{\circ})$.

(c) $\quad \underline{a} : Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F}) \to a\,Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F})$

is an equivalence of categories in $CAT(\mathrm{E}^{\wedge})$.

The essential observation in the proof of this Lemma is that $Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F})$ is always complete in the canonical topology on $\mathrm{E}^{\wedge}$ and hence any essential equivalence out of it always admits a quasi-inverse (Theorem 7, 4°).

Combining these results we have

LEMMA 4. F → E *is complete iff any one (and hence all) of the following equivalent conditions hold:*

(a) $\quad l : Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F}) \to \mathrm{L}\,Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F})$

is an essential equivalence.

(b) $\quad l : Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F}) \to \mathrm{L}\,Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F})$

is an equivalence.

(c) $\quad \underline{a} : Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F}) \to a\,Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F})$

is an equivalence.

Thus Theorem 6, 2°, is established.

It only remains to do Part 3°. But this is almost obvious since the two additional defining conditions for a gerbe are immediately seen to be equivalent to the pair of assertions that the natural transformations

$$Ar(Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F})) \xrightarrow{<\mathrm{T},\,\mathrm{S}>} Ob(Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F}) \times Ob(Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F}))$$
$$\text{and} \quad Ob(Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F})) \xrightarrow{p} 1$$

are covering in the topology on $\mathrm{E}^{\wedge}$ induced by that of E, i.e., from a well-known theorem of Grothendieck topologies [Demazure (1970)] iff

$$a<\mathrm{T},\mathrm{S}> \ (\approx <a(\mathrm{T}), a(\mathrm{S})>) \quad \text{and} \quad a(p)$$

are epimorphisms in $\mathrm{E}^{\sim}$. The preceding lemmata combined with 7° now give two corollaries:

COROLLARY 3. *If* $\mathrm{F} \to \mathrm{E}$ *is externally complete, then* $a\,Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F})$ *is (internally) complete in* $\mathrm{E}^{\sim}$ *(i.e., every torsor under* $a\,Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F})$ *splits.*

COROLLARY 4. *If* $\mathrm{F} \to \mathrm{E}$ *is complete, then one has a chain of cartesian equivalences*

$$\mathrm{F} \xrightarrow{\approx} \mathrm{SF} \xrightarrow{\approx} \mathrm{LSF} \xrightarrow{\approx} \mathrm{KSF} \xrightarrow{\approx} TORS_{\mathrm{E}}(aCart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F})).$$

We can now restate and complete the proof of Theorem 5 with a refinement. For this, let $C\,BOUQ(\mathrm{E};L)$ be the subcategory of the category $BOUQ(\mathrm{E};L)$ consisting of these bouquets which are (internally) complete. Let $\pi_0 C\,BOUQ(\mathrm{E};L)$ be its class of connected components, which is just the same as its equivalence classes under actual equivalence since every essential equivalence here is necessarily an equivalence.

THEOREM 5'. *The 2-functor* $a\,Cart_{\mathrm{E}}(\mathrm{E}/-,-)$ *defines a weak 2-equivalence of the category* $GERB(\mathrm{E};L)$ *with* $BOUQ(\mathrm{E};L)$ *and a (full) 2-equivalence of the category* $GERB(\mathrm{E};L)$ *with* $C\,BOUQ(\mathrm{E};L)$.

From it one deduces bijections on the classes of connected components

$$H^2_{Gir}(\mathrm{E};L) \xrightarrow{\sim} \pi_0(C\,BOUQ(\mathrm{E};L)) \xrightarrow{\sim} H^2(\mathrm{E};L).$$

In effect, Corollary 4 establishes for each gerbe F a cartesian E-equivalence

$$\mathrm{F} \xrightarrow{\approx} \mathit{TORS}_{\mathrm{E}}(a\, \mathit{Cart}_{\mathrm{E}}(\mathrm{E}/\bullet, \mathrm{F})),$$

which takes care of one composition. For the other composition let $\mathfrak{G}$ be a bouquet of E and $\mathrm{Ex}_{\mathrm{E}}(\mathfrak{G}) \to \mathrm{E}$ be the restriction to E of the external fibration which it defines. The assignment to any object $x: X \to O$ of $\mathrm{Ex}_{\mathrm{E}}(\mathfrak{G})$ of the split torsor above X under $\mathfrak{G}$ defines a fully faithful cartesian functor

$$\mathit{Spl}: \mathrm{Ex}_{\mathrm{E}}(\mathfrak{G}) \to \mathit{TORS}_{\mathrm{E}}(\mathfrak{G})$$

which is covering since any torsor under $\mathfrak{G}$ is locally split. One thus obtains functorially a fully faithful covering functor in $\mathrm{E}^{\wedge}$

$$\mathit{Cart}_{\mathrm{E}}(\mathit{Spl}): \mathit{Cart}_{\mathrm{E}}(\mathrm{E}/\bullet, \mathrm{Ex}_{\mathrm{E}}(\mathfrak{G})) \to \mathit{Cart}_{\mathrm{E}}(\mathrm{E}/\bullet, \mathit{TORS}_{\mathrm{E}}(\mathfrak{G})).$$

Since for any presheaf of categories $\mathfrak{G}$ one has an ***essential equivalence*** (!)

$$\mathit{sub}: \mathfrak{G} \to \mathit{Cart}_{\mathrm{E}}(\mathrm{E}/\bullet, \mathrm{Ex}_{\mathrm{E}}(\mathfrak{G}))$$

(in $\mathrm{E}^{\wedge}$), one has by composition a fully faithful covering functor

$$\Sigma_{\mathfrak{G}}: \mathfrak{G} \to \mathit{Cart}_{\mathrm{E}}(\mathrm{E}/\bullet, \mathit{TORS}_{\mathrm{E}}(\mathfrak{G}))$$

and a commutative diagram in $\mathit{CAT}(\mathrm{E}^{\wedge})$

$$\begin{tikzcd}
\mathfrak{G} \arrow[r, "\Sigma_{\mathfrak{G}}"] \arrow[d, "a_{\mathfrak{G}}"'] & \mathit{Cart}_{\mathrm{E}}(\mathrm{E}/\bullet, \mathit{TORS}_{\mathrm{E}}(\mathfrak{G})) \arrow[d, "a_c"] \\
a\mathfrak{G} \arrow[r, "a\Sigma_{\mathfrak{G}}"] & a\, \mathit{Cart}_{\mathrm{E}}(\mathrm{E}/\bullet, \mathit{TORS}_{\mathrm{E}}(\mathfrak{G}))
\end{tikzcd}$$

in which $a_{\mathfrak{G}}$ is an isomorphism and a_c an equivalence. But $\Sigma_{\mathfrak{G}}$ is a covering iff $a\Sigma_{\mathfrak{G}}$ is essentially epimorphic; thus we have a canonical essential equivalence in $\mathit{CAT}(\mathrm{E}^{\sim})$

$$c: \mathfrak{G} \to a\, \mathit{Cart}_{\mathrm{E}}(\mathrm{E}/\bullet, \mathit{TORS}_{\mathrm{E}}(\mathfrak{G}))$$

which thus admits a quasi-inverse iff $\mathfrak{G}$ is internally complete, and the theorem is established.

REMARK. Since $a\, \mathit{Cart}_{\mathrm{E}}(\mathrm{E}/\bullet, \mathit{TORS}_{\mathrm{E}}(\mathfrak{G}))$ is internally complete and c is an essential equivalence $a\mathit{Cart}_{\mathrm{E}}(\mathrm{E}/\bullet, \mathit{TORS}_{\mathrm{E}}(\mathfrak{G}))$ may be taken as the (internal) *completion of* $\mathfrak{G}$, for any functor from $\mathfrak{G}$ into a complete groupoid must factor uniquely through c. Thus in any Grothendieck topos any groupoid has a completion. If we only have an elementary topos (without generators), the fibered category $TORS(\mathfrak{G})$ furnishes an *external* completion for $\mathfrak{G}$ which may be too large to be internalized.

Any sheaf of groups G is precomplete since any sheaf of categories is precomplete. Since a sheaf of groups has only one X-object it is complete iff every torsor under G over X is isomorphic to the split torsor G_d pulled back to X, i.e. iff the set of isomorphism classes of $TORS(X;G)$ has a single element. But this set is, by definition, $H^1(X;G)$; thus *a group object G in $\mathrm{E}^{\sim}$ is complete iff $H^1(X;G)$ is trivial.*

GROUP EXTENSIONS AND BOUQUETS OF E-SETS.

If $F: \mathrm{F} \to \mathrm{E}$ is a homomorphism of groups, then, viewed as groupoids with a single object, it is easy to see that since every arrow of F (being an isomorphism) is necessarily cartesian, *$F: \mathrm{F} \to \mathrm{E}$ is a fibration iff F is surjective*, that is defines an extension of E by the kernel of F, which is just the fiber F_e of F above the single object e of E. A cleavage of this fibration is then nothing more than a set-theoretic section of F, which is a splitting iff it is a homomorphism of groups, i.e. if $F: \mathrm{F} \to \mathrm{E}$ is a split extension of E by its kernel. The corresponding pseudo-functor which any cleavage defines is easily seen to be entirely equivalent to the classical *Schreier factor system* defined by the extension and the classical theorem that every group extension is determined up to isomorphism by such a factor system is seen as a corollary of the Grothendieck theorem that every fibration is determined up to isomorphism by an associated pseudo-functor. Since, in general, not every group extension is split, this furnishes a convincing example that not every fibration is split and thus one cannot, in general, replace pseudo-functors with functors.

Never-the-less the Grothendieck-Giraud theory is applicable and thus every fibration, including that determined by a group extension F is E-*equivalent* to a split fibration SF determined by the externalization of its presheaf of cartesian functors, $Cart_{\mathrm{E}}(\mathrm{E}/-, \mathrm{F})$,

$$\begin{tikzcd}
\mathrm{F} \arrow[rr, "sub"'] \arrow[dr, "F"'] & & \mathrm{SF} \arrow[ll, bend right, "ev"'] \arrow[dl] \\
& \mathrm{E} &
\end{tikzcd}$$

But here SF is not a group, but rather a split fibration, *fibered in groupoids*. The E-functor *sub:* $\mathrm{F} \to \mathrm{SF}$, which is quasi-inverse to that defined by evaluation and only depends on the choice of a cleavage, is fully faithful and thus allows the *complete recovery* (up to isomorphism) *of the group extension as the subgroup of automorphisms of any one of the objects of SF.* This means that all information about group extensions is contained in the category $\mathrm{E}^{\wedge}$ of presheaves on the group E, more commonly (equivalently) known as the category of (right) E-sets.

More precisely, if $\mathrm{F} \to \mathrm{E}$ is a group extension, then $Cart_{\mathrm{E}}(\mathrm{E}/- ,\mathrm{F})$ is a bouquet of E-sets, i.e., a non empty, connected E-groupoid and we have:

THEOREM 6'. *(a) The class of isomorphism classes of extensions of* E *is in 1-1 correspondance with the class of connected components* (under essential equivalence) *of the category of* E*-bouquets.*

(b) Moreover, if we make the essential identification of «abstract kernels» (i.e., homomorphisms of E into the group

$$Out(N) = Aut(N)/Int(N) \;)$$

with the liens of the category of E*-sets, « $\theta \mapsto lien(\theta)$ », then for any extension* $\mathrm{F} \to \mathrm{E}$ *which induces* $\theta : \mathrm{E} \to Out(N)$,

$$lien(Cart_{\mathrm{E}}(\mathrm{E}/-,\mathrm{F})) \xrightarrow{\sim} lien(\theta),$$

and we have a bijection of sets

$$c: Ext(\mathrm{E};N,\theta) \xrightarrow{\sim} \mathrm{H}^2(\mathrm{E}^{\wedge};lien(\theta)).$$

We content ourselves with showing how one directly passes from E-bouquets to extensions of E. Thus let $\mathfrak{G}: A \rightrightarrows O$ be a bouquet of E sets; then, by definition of the restriction of the externalization of $\mathfrak{G}$ to

E, we have for any $a \in O$ that an arrow $f : a \to a$ of projection $x \in$ E is an arrow $f \in A$ of the form $f : a \to a^x$, where $a^x \in O$ is the result of the action of $x \in$ E on a. If $f : a \to a^x$ and $g : a \to a^y$, then $g.f : a \to a^{yx}$ is an arrow of projection yx which is given by the composition $g^x f$:

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[dr, "g^x f"'] & a^x \arrow[d, "g^x"] \\ & a^{yx} \end{tikzcd}$$

in $\mathfrak{G}$. This multiplication is always associative with $id_a : a \to a^e \ (= a)$ as unity element. The inverse of $f : a \to a^x$ is given by the arrow

$$(f^{-1})^{x^{-1}} : a \to a^{x^{-1}}$$

in $\mathfrak{G}$. Since $\mathfrak{G}$ is non empty, there exists an $a \in O$ and since $\mathfrak{G}$ is connected, for any $x \in$ E an arrow $f : a \to a^x$ always exists, i.e.,

$$\mathrm{F} = Aut_{\mathrm{Ex}}(\mathfrak{G})(a) \to \mathrm{E}$$

is a surjective homomorphism of groups. Note that the kernel of this map is just the fiber $N = Aut(a) \subset \mathfrak{G}$. Again since $\mathfrak{G}$ is connected, any two such extensions are E-isomorphic.

In spite of its appearance, the so defined extension is not split in general: Choose, for each $x \in$ E, an arrow $s(x) : a \to a^x$ in $\mathfrak{G}$, then, for each $f : a \to a^x$, define $n : a \to a$ so that the square

$$\begin{tikzcd} a \arrow[r, "id"] \arrow[d, "n"'] & a \arrow[d, "f"] \\ a \arrow[r, "s(x)"'] & a^x \end{tikzcd}$$

is commutative in $\mathfrak{G}$. The mapping $f \mapsto (n, x)$ then defines a bijection of F onto $N \times$ E. Then the mapping

$$(x \mapsto (m : a \to a \mapsto m^{(x)} = s(x)^{-1} m^x s(x))) : N \to N$$

defines a system of automorphisms for which the mapping

$$(y, x) \mapsto s(y, x) = s(yx)^{-1} s(y)^x s(x) : a \to a$$

forms the corresponding factor set. The multiplication on $\mathrm{E} \times N$ is then given by

$$(m, y)(n, x) = (s(y, x), n^{(y)} m, yx).$$

If $\mathfrak{G}$ has an invariant object ($a^x = a$ for all x) then the corresponding extension is split.

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Department of Mathematics
S.U.N.Y. CTR at Buffalo
106 Diefendorf Hall
BUFFALO, N.Y. 14214
U.S.A.