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# THE DUAL LOCALE OF A SEMINORMED SPACE

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In the category of sheaves on a topological space $X$, a linear functional $f: A \to R_X$ into the Dedekind reals on $X$ from a subspace $A$ of a normed space $B$ cannot generally be extended to a linear functional $g: B \to R_X$ having identical norm. Moreover, there exist normed spaces on which only the zero functional may be defined, yet which are themselves non-zero. The canonical mapping $\hat{}: B \to B^{**}$ from a normed space to its dual is not, therefore, generally an embedding.

One way in which these difficulties may be resolved, at least partially, is to consider linear *functionals, rather than functionals; that is, to redefine the notion of functional to mean a bounded linear map $g: B \to {}^*R_X$ from the normed space into the space ${}^*R_X$ of MacNeille reals on $X$ [6, 8]. It may then be proved [5] that the Hahn-Banach Theorem holds for linear *functionals in the category of sheaves on $X$: that is, that any linear *functional $f: A \to {}^*R_X$ on a subspace $A$ can be extended to a linear *functional $g: B \to {}^*R_X$ with identical norm.

However, there remains the difficulty that there exist non-trivial spaces on which only the zero *functional can be defined. The canonical mapping $\hat{}: B \to {}^{**}B$ into the double *dual is an embedding exactly if the space is *normed [4, 6, 13]: that is, provided that the map $N: Q_X^+ \to \Omega^B$ defining the normed structure of $B$ [12] arises canonically from a map ${}^*\| \ \|: B \to {}^*R_X$. This condition requires that there must not exist elements $a \in B$ of which

$$inf \{ q \in Q_X^+ \mid a \in N(q) \}$$

is zero without satisfying the condition

$$\forall q \in Q_X^+ \quad a \in N(q)$$

which makes them zero, for it is these elements which lie in the kernel of every linear functional or *functional. It is this problem which is addressed here.

Instead of considering the dual of $B$ to be the space of linear functionals on $B$, the approach will be to define the *dual locale* $Fn\,B$, corresponding classically to the weak* topology on the unit ball of the space of linear functionals on $B$. In the category of sheaves on $X$, this locale will not generally be the topology of the unit ball of the dual space. The dual locale will instead be obtained directly by considering the theory $\mathrm{Fn}\,B$ of linear functionals on $B$ of norm not exceeding $1$. Another instance of a locale constructed from a propositional theory, in that case the locale of maximal ideals of a ring of continuous real functions, may be found in [3]. In the present case, the locale $Fn\,B$ will be shown to retain the information concerning $B$ which the dual space of $B$ may lose. In particular, it will be proved that each $a \in B$ may be identified isometrically with its evaluation functional on the dual locale $Fn\,B$, allowing any normed space $B$ in the category of sheaves on $X$ to be embedded in a double dual.

Evidently, an important consideration in any discussion of duals of spaces is whether a Hahn-Banach Theorem may be obtained. The theorem which will be proved here is that the canonical map $Fn\,B \to Fn\,A$ is a quotient map of locales for any subspace $A$ of a seminormed space $B$ in the category of sheaves on $X$. Classically, this is equivalent to the Hahn-Banach Theorem, giving that the unit ball of the dual of $A$ is a quotient of that of the dual of $B$. In the category of sheaves on $X$, it is the form of the Hahn-Banach Theorem which allows one to identify the space $B$ with the space of evaluation functionals on its dual locale. It will be proved elsewhere that the Hahn-Banach Theorem for linear *functionals [5] may be retrieved from that obtained here by considering the Gleason cover [9] of the category of sheaves on $X$.

For background on Banach spaces in categories of sheaves the reader is referred to [6, 12], and on propositional theories and locales to [2, 3].

## 1. THE LOCALE $FnB$.

Let $B$ denote a seminormed space in the category of sheaves on the topological space $X$. That is, $B$ is a module over the sheaf $Q_X$ of locally constant rational functions on $X$, together with a map $N: Q_X^+ \to \Omega^B$ from the sheaf of positive rationals to the sheaf of subsheaves of $B$, satisfying [7,11] in the category of sheaves on $X$ the conditions [6,12] defining a seminorm in the space $B$. The subsheaf $N(q)$ of $B$ assigned to a rational $q > 0$ is the open ball of radius $q$ at the zero of $B$.

Consider the propositional theory Fn$B$ in the category of sheaves on $X$ determined by introducing a proposition $a \in (r, s)$ for each $a \in B$ and each $r, s \in Q_X$, together with the following axioms:

| | | |
|---|---|---|
| (F1) | $true \vdash 0 \in (r, s)$ | whenever $r < 0 < s$ |
| (F2) | $0 \in (r, s) \vdash false$ | otherwise |
| (F3) | $a \in (r, s) \vdash -a \in (-s, -r)$ | |
| (F4) | $a \in (r, s) \vdash ta \in (tr, ts)$ | whenever $t > 0$ |
| (F5) | $a \in (r, s) \wedge a' \in (r', s') \vdash a + a' \in (r + r', s + s')$ | |
| (F6) | $a \in (r, s) \vdash a \in (r, s') \vee a \in (r', s)$ | whenever $r < r' < s' < s$ |
| (F7) | $true \vdash a \in (-1, 1)$ | whenever $a \in N(1)$ |
| (F8) | $a \in (r, s) \dashv\vdash \bigvee_{r < r' < s' < s} a \in (r', s')$. | |

The theory Fn$B$ is a geometric theory [15] in the category of sheaves on $X$, having propositions and axioms whose extents are open subsets of the space.

Now, denote by $Fn\,B$ the locale obtained by taking the propositions of the theory Fn$B$, in a language involving finite conjunctions and arbitrary disjunctions, partially ordered with respect to intuitionistically provable entailment within the theory. The locale $FnB$ will be called the *dual locale* of the seminormed space $B$. It will be shown to play the role of the weak* topology on the unit ball of the dual of $B$.

Like the space $B$, the locale $FnB$ is a sheaf on the topological space $X$. Algebraically, it is the locale obtained by taking the propositions $a \in (r, s)$ to be the generators of a complete Heyting algebra in the category of sheaves on $X$, with relations given by the axioms of the theory. An analogous construction of the locale of maximal ideals of the ring of continuous real functions on a locale [3] has been used to obtain the Stone-Čech compactification of a locale. A more explicit description of $Fn\,B$, identifying it as a sublocale of the ideal lattice of a distributive lattice, will be needed later.

The points of the locale $Fn\,B$ are exactly the models of the theory Fn $B$, which may in turn be identified with the linear functionals on $B$ having norm not exceeding $1$. This correspondence between models $F$ of the theory and linear functionals $f$ in the unit ball of $B^*$ is given by the relationship

(1.1) $\qquad F \models a \in (r, s)$ if and only if $r < f(a) < s$.

To each model $F$ of the theory may be assigned the linear functional $f$ of which the lower and upper cuts of $f(a)$ at each $a \in B$ are defined by taking those $r, s \in Q_X$ respectively for which $a \in (r, s)$ is satisfied in the model. Conversely, each linear functional $f$ in the unit ball of the dual space of $B$ determines a model of the theory by making $a \in (r, s)$ true to the extent that $r < f(a) < s$. To verify that these assignments give respectively a linear functional of norm not exceeding $1$, and a model of the theory, it need only be remarked that the axioms (F1)-(F5) correspond to the linearity of the functional, (F8) to the lower and upper cuts being open and (F6) to their giving a Dedekind real on $X$. The axiom (F7) is equivalent to the linear functional being bounded in norm by $1$. In verifying that the assignments are mutually inverse, one uses that:

(1.2) $a \in (r, s') \wedge a \in (r', s) \vdash a \in (r', s')$ whenever $r < r' < s' < s$,

which can be proved by an internally inductive application of (F5) and (F4).

Taking the space in the category of sheaves on $X$ given by the points of the locale $Fn\,B$, one obtains:

THEOREM. *The space Points($Fn\,B$) of the dual locale is isomorphic to the unit ball of the dual space $B^*$ of linear functionals on $B$ in the weak* topology.*

For, the relation between models of $\mathrm{Fn}B$ and linear functionals in the unit ball of $B^*$ has already been remarked to be bijective. Moreover, the canonical topology on the space of points of the locale admits a subbase of open sets of the form

$$\{ F \mid F \models a \in (r,s) \},$$

which corresponds exactly to that of open sets of the form

$$\{ f \mid r < f(a) < s \}$$

for the weak* topology on $B^*$.

The locale $FnB$ also admits a linear structure of the kind associated with the unit ball of a normed space: operations of zero, negation and mean can be defined on the locale, induced by those on the locale of reals [3,10] in the category of sheaves on $X$. The operation of mean (that is, of addition followed by halving) is taken, rather than addition itself, in order to remain within the unit ball. In turn, the mean gives the operation of halving, inducing that of multiplication by rationals between $-1$ and $1$. Taking points of the locale these operations give exactly those of the unit ball of the dual space.

The locale $FnB$ does not contain the elements corresponding to the open balls of the dual space of $B^*$. For the open balls of $B^*$ are not open in the weak* topology, but only in that given by the norm which they define. However, the open complements of the open balls of $B^*$ *are* open in the weak* topology. Moreover, the normed structure of the dual space $B^*$ is known [4,6] to be definable in terms of the subsheaves

$$A^*(q) = \{ f \in B^* \mid \| f \| > q \},$$

which are the open coballs of the space.

The normed, or rather conormed, structure of the dual locale $FnB$ is therefore defined by taking

$$A^*(q) = \bigvee_{a \in N(1)} a \in A(q) \tag{1.3}$$

for each $q \in Q_X^+$. The expression $a \in A(q)$ in this disjunction denotes the proposition

$$\bigvee_{p>q} a \in (-p,-q) \vee a \in (q,p)$$

satisfied by those linear functionals mapping $a \in B$ into the open subset

$$A(q) = \{ r \in \mathrm{R}^+_X \mid r < -q \vee q < r \}$$

of the reals. The condition $A^*(q)$ is therefore exactly that satisfied by linear functionals having norm greater than $q$. The element $A^*(q)$ of the locale $FnB$ will therefore be called the *open coball of radius* $q$ at the zero of the dual locale.

Joining these observations together, one has the following:

COROLLARY. *The isomorphism between Points(FnB) and the unit ball of the dual space $B^*$ in the weak* topology is an isometric isomorphism.*

It will be convenient later to denote by $A^*(x)$ the element of $FnB$ defined by

$$A^*(x) = \bigvee_{q>x} A^*(q) \tag{1.4}$$

for any non-negative element $x \in {}^*\mathrm{R}_X$ of the MacNeille reals on $X$.

Finally, the assignment to each space $B$ of the dual locale $FnB$ may be made functorial from the dual of the category of seminormed spaces to the category of conormed linear locales in the category of sheaves on $X$. To each linear map $\phi: B \to B'$ of norm not exceeding $1$ may be assigned the map $Fn\phi: FnB' \to FnB$ of locales, of which the inverse image maps the proposition $a \in (r,s)$ of the theory $\mathrm{Fn}B$ to the proposition $\phi(a) \in (r,s)$ of the theory $\mathrm{Fn}B'$. The map is linear, in the sense that it preserves the linear structure of $FnB'$, and has norm not exceeding $1$, in the sense that the inverse image of $A^*(q)$ in $FnB$ is contained in $A^*(q)'$ in $FnB'$. In particular, any linear subspace $A$ of a seminormed space $B$ determines a linear contraction $FnB \to FnA$ which will later be proved to be a quotient map of locales.

## 2. ALAOGLU'S THEOREM.

One of the reasons for considering the weak* topology on the dual of a space $B$ is that the unit ball of $B^*$ classically is compact in this topology. This fact is known as Alaoglu's Theorem. It will be proved now that it remains true of the dual locale $Fn\,B$ of a seminormed space $B$ in the category of sheaves on $X$. However, the unit ball of the dual space $B^*$ is not necessarily compact in the weak* topology in this context.

It may be recalled that a locale $L$ is said to be *compact* provided that $1 = \bigvee S$ implies that $1 \in S$ for any up-directed $S \subset L$. The locale $L$ is said to be *regular* provided that each $b \in L$ is the join of those $a \in L$ which are rather below it, where $a \in L$ is said to be *rather below* $b \in L$, written $a \lhd b$, provided that there exists $c \in L$ for which

$$a \wedge c = 0 \quad \text{and} \quad c \vee b = 1.$$

The locale $L$ is said to be *completely regular* provided that each $b \in L$ is the join of those $a \in L$ which are completely below it, where $a \in L$ is said to be *completely below* $b \in L$, written $a \lhd\!\lhd b$, provided that there exists an interpolation $d_{ik} \in L$, for $i = 0, 1, \ldots$ and $k = 0, 1, \ldots, 2^i$ dependent on $i$, such that:

i) $d_{00} = a$ and $d_{01} = b$,

ii) $d_{ik} \lhd d_{i\,k+1}$,

iii) $d_{ik} = d_{i+1\,2k}$

for all appropriate $i$, $k$.

Then Alaoglu's Theorem may be proved in the following form:

THEOREM. *The dual locale $Fn\,B$ of a seminormed space $B$ in the category of sheaves on $X$ is a compact, completely regular locale.*

The proof of this assertion is similar to that establishing that the locale of maximal ideals of a ring of continuous real functions is a compact regular locale [3].

It will be shown first that the locale is completely regular. For any element of the locale $Fn\,B$ is expressible in the form

$$a_1 \in (r_1, s_1) \wedge \ldots \wedge a_n \in (r_n, s_n).$$

Hence, it is enough to show that each conjunction

$$a_1 \in (r_1, s_1) \wedge \ldots \wedge a_n \in (r_n, s_n)$$

is the join of elements which are completely below it in the lattice $Fn\,B$. Since the completely below relation distributes over finite conjunctions, it is sufficient to prove this for each proposition $a \in (r, s)$. But $a \in (r, s)$ is equivalent to

$$\bigvee a \in (r', s') \quad ( r < r' < s' < s )$$

by (F8), so it is enough to establish that

$$a \in (r', s') \lll a \in (r, s) \quad \text{whenever} \quad r < r' < s' < s.$$

Let $r < r' < s' < s$ be given. Then, for $a \in B$, choose $t \in \mathrm{Q}_X^+$ such that $a \in N(t)$. Assume that $-t < r$ and $s < t$, and consider the proposition $a \in (-t, r) \vee a \in (s', t)$. Now, $true \vdash a \in (-t, t)$ (by (F4) and (F7)): hence

$$true \vdash ( a \in (-t, r') \vee a \in (s', t)) \vee a \in (r, s),$$

by (F6). And,

$$a \in (r', s') \wedge a \in (s', t) \vdash false \quad \text{and} \quad a \in (-t, r') \wedge a \in (r', s') \vdash false,$$

by (F2), (F3) and (F5); hence,

$$a \in (r', s') \wedge (a \in (-t, r') \vee a \in (s', t)) \vdash false.$$

Thus, one has

$$a \in (r', s') \lhd a \in (r, s) \quad \text{whenever} \quad r < r' < s' < s.$$

Then the propositions $a \in (r_{ik}, s_{ik})$ for $i = 0, 1, \ldots$ and $k = 0, 1, \ldots, 2^i$ dependent on $i$ define an interpolating family establishing that

$$a \in (r', s') \lll a \in (r, s),$$

where

$$r_{ik} = (k/2^i)r + (1-(k/2^i))r' \quad \text{and} \quad s_{ik} = (k/2^i)s' + (1-(k/2^i))s.$$

The locale $Fn\,B$ is therefore completely regular.

Now, consider the propositional theory $\mathrm{Fn}_f B$ obtained from the theory $\mathrm{Fn}\,B$ by replacing the axiom (F8), which is the only axiom involving an infinitary disjunction, by the axiom:

(F8') $\quad a \in (r', s') \vdash a \in (r, s)$ whenever $r < r' < s' < s$.

The axioms of the theory $\mathrm{Fn}_f B$ now involve only *finitary* disjunctions. The locale $Fn_f B$ of this theory is therefore compact, being the locale of a finitary geometric theory. Because the theory $\mathrm{Fn}\, B$ may be recovered from the theory $\mathrm{Fn}_f B$ by adding the axiom

$$(\mathrm{F8})'' \qquad a \in (r, s) \vdash \bigvee_{r<r'<s'<s} a \in (r', s'),$$

the locale $Fn\, B$ is evidently a sublocale of $Fn_f B$. In fact, $Fn\, B$ is a retract of $Fn_f B$, by the map of which the inverse image assigns to each proposition $a \in (r, s)$ the proposition

$$\bigvee_{r<r'<s'<s} a \in (r', s').$$

Provided that this assignment yields a map of locales, it is certainly a retraction of the inclusion, because $(\mathrm{F8})''$ is an axiom of $\mathrm{Fn}\, B$. That it yields a map of locales may be verified straightforwardly by checking that the axioms of $\mathrm{Fn}\, B$ are satisfied in this interpretation in the locale $Fn_f B$. The locale $Fn\, B$ is therefore compact, being a retract of the compact locale $Fn_f B$.

Together, these observations prove Alaoglu's Theorem. Before passing on, it may be noted that the argument giving the compactness of $Fn\, B$ also gives an explicit construction of the locale. The finitary theory $\mathrm{Fn}_f B$, considered in the language involving only finitary conjunctions and disjunctions, determines internally a distributive lattice, of which the locale $Fn_f B$ is the lattice of ideals. The fact that $Fn\, B$ is a retract of this ideal lattice will be needed later. Explicitly, one observes that it consists of those ideals which are joins of finite meets of ideals generated by families:

$$(a \in (r', s'))_{r<r'<s'<s} \quad \text{for} \quad a \in B \text{ and } r < s \text{ in } \mathrm{Q}_X.$$

One consequence of the theorem is the following:

COROLLARY. *If $X$ is discrete, then the locale $Fn\, B$ is exactly that of the weak* topology of the unit ball of the dual space $B^*$.*

For, the axiom of choice in the category of sheaves on the discrete space $X$ implies that the compact completely regular locale $Fn\, B$ is isomorphic to the topology of its space of points [2]. This has already been identified as that of the unit ball of the dual space $B^*$ in the weak* topology. Of course, the dual space $B^*$ is obtained in this context simply by taking the dual of each stalk of $B$.

Before leaving the compactness of $FnB$, one may observe the following fact about the normed structure of the dual locale. Given any rational $q$ with $0 < q < 1$, one may consider the closed sublocale

$$(FnB)(q) \rightarrowtail FnB$$

of $FnB$ determined by the open coball $A^*(q)$ of radius $q$. This is the quotient lattice of $FnB$ obtained by taking the image of the mapping

$$A^*(q) \vee -: FnB \to FnB.$$

The sublocale $(FnB)(q)$ will be called the *closed ball of radius* $q$ of the dual locale. It is a compact, completely regular locale.

One may also consider the sublocale $FnB(q) \rightarrowtail FnB$ of $FnB$ obtained by replacing the axiom (F7) of the theory Fn$B$ by the axiom:

$$(\mathrm{F7})_q \qquad true \vdash a \in (-q, q) \text{ whenever } a \in N(1).$$

The theory Fn$B(q)$ obtained in this way is evidently that of linear functionals on $B$ of norm not exceeding $q$. It is naturally equivalent to that of linear functionals of norm $\leq 1$ on the space $B(q)$ obtained from $B$ by dilating the open balls of $B$ by a factor of $1/q$. In particular, $FnB(q)$ is also a compact, completely regular locale.

The compactness of the sublocale $FnB(q)$ means that it is necessarily a closed sublocale of $FnB$, which will be proved to be naturally isomorphic to the closed ball of radius $q$. Firstly, it may be shown that the closed ball is actually a sublocale of $FnB(q)$. To prove this, it is enough to establish that the inverse image of the inclusion $(FnB)(q) \rightarrowtail FnB$ of the closed ball satisfies the axiom $(\mathrm{F7})_q$ which defines the sublocale $FnB(q)$. This is equivalent to showing that

$$true \vdash a \in (-q, q) \vee A^*(q)$$

is provable in the theory Fn$B$ whenever $a \in N(1)$ in $B$. But, if $a \in N(1)$, then $p < 1$ may be chosen with $a \in N(p)$. Then $a/p \in N(1)$ implies that $a \in A(pq) \vdash A^*(q)$, from which follows the required result since

$$true \vdash a \in (-q, q) \vee a \in A(pq)$$

is provable in Fn$B$ (by applying (F6) and (F7)) on remarking that $pq < q$. The closed ball of radius $q$ is therefore contained in the closed sublocale $FnB(q)$ of $FnB$. To show that this inclusion is an isomorphism it remains only to prove that the intersection of $A^*(q)$ with $FnB(q)$ is trivial. But $true \vdash a \in (-q, q)$ is provable in Fn$B(q)$ for any $a \in N(1)$ in $B$. Hence, $a \in A(q) \vdash false$ is provable in Fn$B(q)$. So

$$\bigvee_{a \in N(1)} a \in A(q) \vdash false$$

is provable in Fn$B(q)$, which is equivalent to the intersection of $A^*(q)$ with $FnB(q)$ being trivial. The closed ball of $FnB$ of radius $q$ is therefore naturally isomorphic to $FnB(q)$.

In particular, it follows that the closed ball of $FnB$ of radius $q$ has points given by the linear functionals on $B$ of norm not exceeding $q$.

## 3. THE HAHN-BANACH THEOREM.

It has already been remarked that any subspace $A$ of a seminormed space $B$ in the category of sheaves on $X$ determines a map of locales $FnB \to FnA$ of which the inverse image identifies each proposition $a \in (r, s)$ of the theory Fn$A$ with the same proposition in the theory Fn$B$. This map of locales will now be shown to be a quotient map, giving the Hahn-Banach Theorem on the category of sheaves on $X$ in the form considered here. It will prove more convenient throughout the proof to work with the $\wedge\bigvee$ map $FnA \to FnB$ which is the inverse image mapping of this map of locales. The Hahn-Banach Theorem is then the statement that this $\wedge\bigvee$ map is monic. This is equivalent to the corresponding map of theories being a conservative extension, which of course is just the import of the Hahn-Banach Theorem.

The Godement covering of the topos $Sh(X)$ of sheaves on $X$ is the geometric map $\gamma: \mathrm{B}(X) \to Sh(X)$ from the category of sheaves on the space $X_d$ obtained by taking the discrete topology on $X$. The map is that induced by the continuous mapping which is the identity on the underlying sets. The geometric map is a covering of toposes in the sense that the inverse image functor $\gamma^*$ reflects isomorphisms. The topos $B(X)$ has the advantage that (AC) is satisfied in it; in particular, $B(X)$ is a Boolean topos.

The inverse image of any seminormed space $B$ along a geometric map is again a seminormed space, of which the seminormed structure is given by the inverse images of the open balls of $B$. It is to obtain this that seminormed spaces over the rationals have been considered throughout: the observation is not valid for normed spaces over the reals.

The Hahn-Banach Theorem will be proved by examining the effect of taking the inverse image of an inclusion of spaces along the Godement covering of the category $Sh(X)$, applying the Hahn-Banach Theorem in the topos $B(X)$ satisfying (AC), then proving that this implies the result in the category of sheaves on $X$. Explicitly, it will be proved that any subspace $A$ of a seminormed space $B$ in $Sh(X)$ determines a commutative diagram

$$\begin{array}{ccc}
Fn\,B & \longrightarrow & \gamma_* Fn\, \gamma^* B \\
\uparrow & & \uparrow \\
Fn\,A & \longrightarrow & \gamma_* Fn\, \gamma^* A
\end{array}$$

of lattice homomorphisms in which the horizontal maps will be proved monic by examination of the constructions involved in obtaining the locales and the right hand map will be shown to be monic by the Hahn-Banach Theorem in $B(X)$. The theorem obtained is the following:

THEOREM. *For any subspace $A$ of a seminormed space $B$ in the category of sheaves on $X$, the canonical map $Fn\,B \to Fn\,A$ of dual locales is a quotient map.*

Before starting the proof recall that the inverse image functor $\gamma^*$ maps each sheaf on $X$ to its sheaf of stalks on $X_d$, while the direct image functor $\gamma_*$ maps each family $(S_x)_{x \in X}$ of sets indexed by $X$ to the sheaf of which the sections over an open subset $U \subset X$ are the elements of $\Pi_{x \in U} S_x$. The unit $1_{Sh(X)} \to \gamma_* \gamma^*$ of the adjunction is monic, which expresses that the geometric map is a covering while the counit $\gamma^*\gamma_* \to 1_{B(X)}$ is actually epic on any sheaf in $B(X)$ of which the stalks are non-empty.

Recall also that the direct image $\gamma_*L$ of a locale in $B(X)$ is a locale in $Sh(X)$, since both being a Heyting algebra and having a complete partial ordering are preserved under direct image functors. The completeness of $\gamma_*L$, for $L$ a locale in $B(X)$, is such that if $S \rightarrowtail L$ has join $s$, then $\gamma_*S \rightarrowtail \gamma_*L$ will have join given by $\gamma_*s \in \gamma_*L$ over any open subset over which $\gamma_*S$ has support: for the join of $\gamma_*S \rightarrowtail \gamma_*L$ is calculated by applying $\gamma^*$ and taking the join in $L$ of the image factorisation of the resulting map to $L$ [7]. However,

$$\begin{array}{ccc} \gamma^*\gamma_*S & \rightarrowtail & \gamma^*\gamma_*L \\ \downarrow & & \downarrow \\ S & \rightarrowtail & L \end{array}$$

is exactly this factorisation over any open subset contained in the support of $S$.

Now, given a seminormed space $B$ in the category of sheaves on $X$, define a canonical map $Fn\,B \to \gamma_*Fn\,\gamma^*B$ by assigning to each $a \in (r, s)$ in the theory $\mathrm{Fn}B$ the element $a^* \in (r^*, s^*)$ of $Fn\,\gamma^*B$ given by applying $\gamma^*$ to $a \in B$ and $r, s \in Q_X$. This assignment extends to give an $\wedge\vee$ map by observing that the axioms of the theory $\mathrm{Fn}\,B$ are satisfied in the locale $\gamma_*Fn\,\gamma^*B$. This is immediately true for any axiom involving only finitary disjunctions. Moreover, it is true for the axiom

$$a \in (r, s) \vdash \bigvee_{r<r'<s'<s} a \in (r', s')$$

by the above remark concerning the construction of joins in the direct images of locales.

Given the subspace $A$ of the space $B$, the diagram

$$\begin{array}{ccc} Fn\,B & \longrightarrow & \gamma_*Fn\,\gamma^*B \\ \uparrow & & \uparrow \\ Fn\,A & \longrightarrow & \gamma_*Fn\,\gamma^*A \end{array}$$

is commutative. It is enough to check this on an element $a \in (r, s)$ of the locale $Fn\,A$, for which it is clearly the case. The map on the right hand side of the square is the direct image of the canonical map

$$Fn\,\gamma^*A \to Fn\,\gamma^*B$$

determined by the subspace $\gamma^*A$ of the seminormed space $\gamma^*B$ in the topos $B(X)$. Since $B(X)$ satisfies (AC), this map is the inverse image mapping between the weak* topologies of the unit balls of the dual spaces concerned. Further, the map is monic, because the restriction map between the unit balls of the dual spaces is a quotient map, by the Hahn-Banach Theorem of the topos $B(X)$, which satisfies (AC).

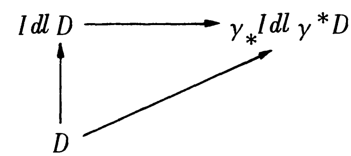
The theorem is therefore proved on establishing the following:

(3.1) *for any seminormed space* $A$ *the canonical map* $Fn\,A \to \gamma_* Fn \gamma^* A$ *is monic.*

For then the required map $Fn\,A \to Fn\,B$ will also be monic, hence the inverse image mapping of a quotient map of locales. The remaining effort lies in establishing (3.1) by examining the locales involved. It has already been remarked that $Fn\,A$ is a retract of the ideal lattice of the distributive lattice of the finitary theory $Fn_f A$. It is straightforward to compute that the inverse image of the distributive lattice is exactly that of the theory $Fn_f \gamma^* A$. The first step in the proof is therefore to show the following:

(3.2) *for any distributive lattice* $D$ *the canonical map* $Idl\,D \to \gamma_* Idl \gamma^* D$ *is monic.*

Suppose then that $D$ is a distributive lattice in $Sh(X)$. The map which is asserted to be monic is the unique $\wedge\vee$ map from the locale $Idl\,D$ of ideals of $D$ to the locale $\gamma_* Idl\, \gamma^* D$, for which the diagram




commutes, in which the map to $Idl\,D$ is the canonical embedding and the map to $\gamma_* Idl\, \gamma^* D$ is adjoint to the canonical embedding of $\gamma^* D$ in its completion. To see that this is monic, note that it also makes the diagram

$$\begin{array}{ccc} \Omega^D & \longrightarrow & \gamma_* \Omega^{\gamma^* D} \\ \uparrow & & \uparrow \\ Idl\ D & \longrightarrow & \gamma_* Idl\ \gamma^* D \end{array}$$

commute. For the top map can be seen to take ideals of $D$ to direct images of ideals of $\gamma^* D$. However, the top map is monic; hence, the required map is monic.

Applying this result to the distributive lattice of the finitary theory $\mathrm{Fn}_f A$, of which the inverse image is canonically isomorphic to the distritive lattice of the theory $\mathrm{Fn}_f \gamma^* A$, one obtains the following:

(3.3) *for any seminormed space A the canonical map* $Fn_f A \to \gamma_* Fn_f \gamma^* A$ *is monic.*

That yields that the required map is monic, provided that it can be shown that the diagram

$$\begin{array}{ccc} Fn_f A & \longrightarrow & \gamma_* Fn_f \gamma^* A \\ \uparrow & & \uparrow \\ Fn\, A & \longrightarrow & \gamma_* Fn\ \gamma^* A \end{array}$$

is commutative, in which the vertical maps are the inverse image mappings of the retractions from the locales of the finitary theories to the locales of $\mathrm{Fn}\, A$ and $\mathrm{Fn}\, \gamma^* A$ respectively.

Taking a proposition $a \in (r, s)$ of the theory $\mathrm{Fn} A$, it is mapped along the upper path, first into

$$\bigvee_{r < r' < s' < s} a \in (r', s'),$$

then into the join of the propositions $a^* \in (r'^*, s'^*)$ of $\mathrm{Fn}\, \gamma^* A$ for which $r < r' < s' < s$. Along the lower path it is taken first to the proposition $a^* \in (r^*, s^*)$ of $\mathrm{Fn} \gamma^* A$, then to the join of the propositions $a^* \in (p, q)$ of $\mathrm{Fn} \gamma^* A$ for which $r^* < p < q < s^*$. It must be proved that these joins in the locale $\gamma_* Fn\,\gamma^* A$ are equal. However, the propositions $a^* \epsilon (r'^*, s'^*)$ of $\mathrm{Fn}\,\gamma^* A$ are seen to be cofinal in $\mathrm{Fn}\,\gamma^* A$ among those of the form $a^* \epsilon (p, q)$ with $r^* < p < q < s^*$. The joins of these elements therefore coincide in $Fn\,\gamma^* A$. However, their joins in $\gamma_* Fn\,\gamma^* A$ are actually computed in $Fn\,\gamma^* A$, since the subsheaves of which the joins are taken are direct images of subsheaves of $Fn\,\gamma^* A$.

The commutativity of the diagram thus established gives the required result. For the vertical maps are monic, being the inverse image mappings of retractions. The top map has already been shown to be monic (3.3). Hence, the canonical map from $Fn\,A$ to $\gamma_* Fn\,\gamma^* A$ is indeed monic as asserted in (3.1), which completes the proof of the Hahn-Banach Theorem.

For any $x \epsilon {}^*\mathrm{R}_X$ for which $0 \leq x \leq 1$, denote by $FnB(x)$ the closed sublocale of $FnB$ determined by the proposition

$$A^*(x) = \bigvee_{q < x} A^*(q).$$

The locale $FnB(x)$ will be called the *closed ball of radius* $x$ about the zero of $FnB$. For any subspace $A$ of the space $B$, the canonical map from $FnB$ to $FnA$ is such that the inverse image of the open coball $A^*(x)$ of $A$ is contained in the corresponding open coball of $B$. Concerning the canonical map $FnB(x) \to FnA(x)$ thus obtained, one has the following:

COROLLARY. *For any subspace* $A$ *of a seminormed space* $B$ *in the category of sheaves on* $X$, *the canonical map* $FnB(x) \to FnA(x)$ *is a quotient map of locales for each* $x \epsilon {}^*\mathrm{R}_X$ *with* $0 \leq x \leq 1$.

For the case that $x$ is a positive rational $q$, the assertion may be proved by applying the Hahn-Banach Theorem above to the subspace $A(q)$ of the space $B(q)$, each obtained by changing the norm by a factor of $1/q$. The closed balls of radius $q$ have already been identified with the dual locales of these spaces. Hence, $FnB(q) \to FnA(q)$ is a quotient map. Note that it follows that $\phi \vdash A^*(q)$ is provable in $\mathrm{Fn}\,A$ whenever $\phi \vdash A^*(q)$ is provable in $\mathrm{Fn}B$, for any proposition $\phi$ of the theory $\mathrm{Fn}A$.

Now suppose that $x \epsilon {}^*\mathrm{R}_X$ is given, with $0 \leq x \leq 1$. The canonical map may be proved to be a quotient by showing that its inverse mapping reflects the zero of the locale, because the locales concemed are compact, completely regular. Given a proposition $\phi$ of the theory $\mathrm{Fn}\,A$, suppose that $\phi \vdash A^*(x)$ is provable in $\mathrm{Fn}B$. Choosing $\psi \lhd \phi$ in the locale $Fn\,A$ and obverving that the rather below relation is preserved by the inverse image mapping, one has $\psi \lhd A^*(x)$ in the locale $Fn\,B$. The compactness of $Fn\,B$ implies that

$$\psi \lhd A^*(q_1) \vee \dots \vee A^*(q_n) \text{ for finitely many } q_i < x,$$

from the definition of $A^*(x)$. Then

$$\psi \lhd A^*(q) \text{ in } Fn\,B \text{ for } q = min(q_1, \dots, q_n).$$

In particular, $\psi \vdash A^*(q)$ in $\mathrm{Fn}B$, hence $\psi \vdash A^*(q)$ in $\mathrm{Fn}\,A$, by the rational case. Then $\psi \vdash A^*(x)$ is provable in $\mathrm{Fn}A$ since $q < x$. However this implies that $\phi \vdash A^*(x)$ is provable in $\mathrm{Fn}\,A$, since the proposition $\phi$ is the disjunction of those $\psi$ with $\psi \lhd \phi$, by the regularity of the locale $FnA$. This proves that an element of $Fn\,A(x)$ is zero whenever its inverse image in $FnB(x)$ is zero. Since the locales are compact, completely regular, this yields that the canonical map is a quotient.

## 4. THE CANONICAL EMBEDDING.

By a *weak* functional* on the dual locale $FnB$ of a seminormed space $B$ in the category of sheaves on $X$ will be meant a map of locales $f: FnB \to \mathrm{R}_X$ from $Fn\,B$ to the locales of reals on $X$ [3,10], which preserves the linear structure of $FnB$. The sheaf $B^{w**}$ of weak* functionals on the dual locale admits a linear structure, obtained from that of the locale of reals, on which a norm may be defined by taking

$$N^{w**}(q) = \{ f \in B^{w**} \mid \exists\, q' < q \;\; f^*(-q', q') = 1_{Fn\,B} \}.$$

The space $B^{w**}$ will be called the *weak* double dual of* $B$.

For each $a \in B$, consider the weak* functional $\hat{a}: FnB \to \mathrm{R}_X$, of which the inverse image is defined by assigning to each open interval $(r, s)$ of the rationals on $X$ the proposition $a \in (r, s)$ of the theory $\mathrm{Fn}B$. That this defines a weak* functional on $FnB$ is established by verifying that each axiom of the theory of reals is taken into an entailment provable in the theory of the dual locale of $B$. This is straightforward and will be omitted. Assigning to each $a \in B$, the weak* functional $\hat{a} \in B^{w**}$ gives a linear map $\hat{}: B \to B^{w**}$, concerning which one has the following:

THEOREM. *The canonical map* $\hat{}: B \to B^{w**}$ *from a normed space* $B$ *in the category of sheaves on* $X$ *into the weak* double dual is an isometric embedding.*

Indeed, this is equivalent to $B$ being normed, rather than simply seminormed. One may conjecture that the embedding is actually an isomorphism, which is the case classically [14].

The theorem is proved by noting that for any $q \in \mathrm{Q}_X^+$ the open ball of $B^{w**}$ of radius $q$ contains the weak* functional determined by an element $a \in B$ exactly if there exists $q' < q$ such that $true \vdash a \in (-q', q')$ is provable in the theory $\mathrm{Fn}B$. It must be proved that this is equivalent to $a \in N(q)$. It is enough to prove this in the case $q = 1$. And, by the openness of $N(q)$ and the axiom (F8), it is enough to show that $a \in N(1)$ in $B$ is equivalent to $true \vdash a \in (-1, 1)$ being provable in $\mathrm{Fn}B$. In one direction, this is just the axiom (F7). Conversely, it is enough, by the Hahn-Banach Theorem, to prove that $a \in N(1)$ provided that the above entailment is provable in the theory $\mathrm{Fn}A$ corresponding to the subspace $A$ of $B$ generated by the given $a \in B$, since $Fn\,B \to Fn\,A$ is a quotient map.

To prove this, take the inverse image of the theory $\mathrm{Fn}A$ and of the space $A$ along any Boolean cover $\gamma : \mathrm{B} \to Sh(X)$ of the category of sheaves on $X$. For then $true \vdash a^* \in (-1, 1)$ is provable in the theory $\mathrm{Fn}\,\gamma^*A$ in the Boolean topos $\mathrm{B}$. But the norm of $\gamma^*A$ is determined classically, by a function $\| \ \| : \gamma^*A \to \mathrm{R}_{\mathrm{B}}$, for which one has

$$\|a^*\| < 1 \vee \|a^*\| \geq 1.$$

Now if $\|a^*\| \geq 1$, then the linear functional on $\gamma^*A$ defined by $f(q\,a^*) = q$ for any $q \in \mathrm{Q}_{\mathrm{B}}$ has norm not exceeding $1$. This contradicts the fact that $a^* \in (-1, 1)$ is provable in $\mathrm{Fn}\,\gamma^*A$, which states that any linear functional of norm $\leq 1$ must map $a^*$ into the open interval $(-1, 1)$. However, one may then conclude that $a \in N(1)$, since $\gamma^*$ is the inverse image functor of a geometric covering of $Sh(X)$. This completes the proof.

Applying the same reasoning to the canonical map $\hat{}: B \to \mathrm{R}(FnB)$ into the space of continuous real functions on the compact completely regular locale $FnB$ yields the following:

COROLLARY. *The canonical map* $\hat{}: B \to \mathrm{R}(FnB)$ *from a normed space* $B$ *in the category of sheaves on* $X$ *into the space of continuous real functions on the dual locale* $FnB$ *is an isometric embedding.*

This embedding is just the adjoint of the generic linear functional $B \to \mathrm{R}_{FnB}$ in the classifying topos of the theory $\mathrm{Fn}\,B$ of linear functionals on $B$.

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FOOTNOTE. The Hahn-Banach Theorem established here has subsequently been extended to any Grothendieck topos, of which details will appear elsewhere.

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