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A HOMOLOGICAL ALGEBRA FOR VALUATED GROUPS

by Temple H. FAY

Recently there have been some significant developments in classifying mixed abelian groups obtained by viewing such groups equipped with the added data of a valuation. A valuation is a natural generalization of the notion of height. The category of valuated groups provides the proper context for viewing such classification theorems as the Ulm-Zippin theory of countable p-groups, the Baer-Kaplansky theory of completely decomposable torsion-free groups and more recent classifications involving even more sophisticated use of heights. Moreover, valuated groups are also a rich source of pleasing structures to study in their own right.

In this paper we lay the foundations for a homological algebra in the category of valuated groups. This category is preabelian and not abelian, hence the usual methods of examining short exact sequences, bilinear maps, projective presentations and the like fail or are not applicable.

For valuated groups A and B, valuations are introduced on $A \otimes B$ and $Hom(A, B)$ in such a manner that the adjoint situation between Hom and tensor product for the category of abelian groups is lifted to the category of valuated (abelian) groups. By considering free presentations and what will be called proper (short exact) sequences, we obtain in a classical way internal (i.e. valuated) biadditive Ext and Tor functors.

Once one has obtained an internal hom-functor, an internal Ext-functor can be obtained as a cokernel of a canonical hom-sequence. This gives rise to a relative homological theory when compared to the non-internal theory for Ext developed by Richman and Walker [9] for preabelian categories. We identify this relative theory by considering equivalence classes of «proper» exact sequences and Baer addition. In the localized case, for p-local groups, the Richman-Walker *Ext* and the one developed herein turn to be the same.

Valuations on $Tor(A,B)$ are obtained by considering Tor classically as a kernel of a tensor sequence formed by a canonical free resolution of A (or of B). This procedure is not symetric as the valuation obtained depends on whether the free resolution is taken of A or of B. The valuations obtained are identified by considering the generators and relations for Tor.

The snake lemma holds for proper stable exact sequences. Converquently, connecting (valuated) homomorphisms for *Hom-Ext* sequences are obtained. However, while the sequences obtained are exact as sequences of abelian groups, they are not exact in the category of valuated groups.

This work was completed while the author held a visiting position at New Mexico State University and he is pleased to be able to express his appreciation to his colleagues in the abelian group seminar for the opportunity to present much of this work and in particular to Fred Richman for many helpful and enlightening conversations.

1. PRELIMINARIES.

Let Γ denote the set of ordinals together with the symbol ∞. As usual, we make the convention that $\lambda < \infty$ for all $\lambda \in \Gamma$, so in particular, $\infty > \infty$. Let A be an abelian group and p a prime. A *p-valuation* ν_p on A is a function assigning to each element of A an element of Γ such that for all $a, a' \in A$,

$$\nu_p(a+a') \geq min(\nu_p(a), \nu_p(a')) \quad \text{and} \quad \nu_p(p.a) > \nu_p(a).$$

A valuated group is a group A together with a p-valuation on A for each prime p. When there is little likelihood of confusion, we shall use the generic symbol ν to denote a valuation.

A morphism of valuated groups $f: A \to B$ is a homomorphism which does not decrease the valuation, that is,

$$\nu(f(a)) \geq \nu(a) \quad \text{for all} \quad a \in A.$$

The category of all valuated groups and valuated group morphisms is denoted by $Val\,Ab$. There is an obvious forgetful functor U from $Val\,Ab$ to the category of abelian groups Ab. This functor has some nice properties (in fact it is an absolutely topological functor in the sense of H.Herrlich, «Topological functors», *Gen. Top. Appl.* 4 (1974), 125-142). For example, U has both a left and right adjoint. The left adjoint assigns the valuation given by height (denoted by the generic symbol h) to each abelian group. Height is the least valuation on a group and groups having valuation height behave much like discrete topological spaces in that, if B is any valuated group and A has valuation height, any group homomorphism $f: U\,A \to U\,B$ is a valuated group morphism from A to B. The right adjoint to U assigns to each abelian group the valuation which has constant values ∞. This is the largest valuation on a group and groups having this valuation behave like indiscrete topological spaces.

If A is an abelian group and $\{ A_i \mid i \in I \}$ is a family of valuated groups indexed by a class I (which may be proper, a set, or empty), and if $\{ f_i: A \to U\,A_i \mid i \in I \}$ is any family of homomorphisms (such a family is called a source), there is a largest valuation on A for which each f_i is a valuated group morphism. This valuation is obtained by setting

$$\nu(a) = \min(\nu(f_i(a)) \mid i \in I)$$

and has the important property that if B is any valuated group and if $g: U\,B \to U\,A$ is a homomorphism, then g is a valuated group morphism iff $f_i g$ is a valuated group morphism for each $i \in I$. To construct limits in $Val\,Ab$, construct the limit in Ab and endow it with this valuation determined by the limit source. For example, the product of a family of valuated groups $\{ A_i \mid i \in I \}$ is the group $\prod_{i \in I} U\,A_i$ endowed with the valuation determined by minimum on coordinates.

Dually, given a family of group homomorphisms $\{ f_i: U\,A_i \to A \mid i \in I \}$ (called a sink), there is a least valuation on A for which each f_i becomes a valuated group morphism. This valuation is the least valuation satisfying

$$\nu(a) \geq \sup\{ \nu(f_i(a')) \mid f_i(a') = a,\ i \in I \}.$$

It has the property that a homomorphism of the form $g: UA \to UB$ is a valuated group morphism iff gf_i is a valuated group morphism for each $i \in I$. Colimits are formed in *Val Ab* by forming the colimit in *Ab* and endowing it with the least valuation determined by the colimit sink. It is immediate that *Val Ab* has finite biproducts and that the direct sum of a family of valuated groups is the group theoretic direct sum having valuation minimum on coordinates.

An *embedding* in *Val Ab* is an injective valuated group morphism having the largest valuation on its domain for which the homomorphism is a valuated group morphism; that is, iff $f: A \to B$ is injective and

$$\nu(a) = \nu(f(a)) \quad \text{for each} \quad a \in A.$$

Dually we define a *quotient*. It is clear that embeddings are the kernels and quotients are the cokernels in *Val Ab*.

If $f: A \to B$ is a valuated group morphism, then the image of f is $ker\, coker(f)$ and its coimage is $coker\, ker(f)$. Because *Val Ab* fails to be balanced, the image of f and coimage of f need not coincide. By a *sequence* in *Val Ab* we shall mean

$$\cdots \longrightarrow A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \longrightarrow \cdots$$

where

$$\cdots \longrightarrow UA_{i-1} \xrightarrow{Uf_{i-1}} UA_i \xrightarrow{Uf_i} UA_{i+1} \longrightarrow \cdots$$

is an exact sequence in Ab. If for each $i \in I$, $coim(f_{i-1}) = ker(f_i)$, we say the sequence is *exact in Val Ab*. Thus for a sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

to be short exact we require

$$f = ker(g) \quad \text{and} \quad g = coker(f).$$

Short exact sequences need not remain exact under the pushout and pullback operations that make *Ext* a group valued functor. In any preabelian category there is a largest class of exact sequences that do remain exact under these operations and these sequences do define an abelian group valued functor *Ext* [9]. These sequences are called *stable* exact sequences. It follows that

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is stable exact in *Val Ab* iff f is an embedding and for every $b \in B$,

$$\{ n.b + a \mid a \in A,\ n \text{ relatively prime to } p \}$$

contains an element of maximal value [8].

The valuated groups which are modules over the integers localized at a prime p form a pre-abelian category $Val\,Ab_p$. Since $\nu_p(x) = \infty$ for q a prime different from p, we need only consider the p-valuation for such groups. There is an obvious forgetful functor $U_p : Val\,Ab_p \to Mod(Z_p)$ where $Mod(Z_p)$ denotes the category of Z_p-modules. We reserve Z_p to denote the integers localized at p, and use $Z(p)$ to be the cyclic group $Z/_pZ$. The functor U_p has essentially the same properties as the functor U and $Val\,Ab_p$ forms a reflective subcategory of $Val\,Ab$. If A is a valuated group, A_p denotes the abelian group $U A \otimes_Z Z_p$ endowed with the valuation $\nu(a \otimes c/n) = \nu(c.a)$ [8], and the canonical map $\Phi p : A \to A_p$ (front adjunction) preserves p-valuation.

2. TENSOR PRODUCTS.

Let A and B be valuated groups. In this section a functorial valuation shall be defined on the group $U A \otimes_Z U B$ in such a manner that the adjoint situation between $(\cdot)\otimes_Z(\cdot)$ and $Hom_Z(\cdot,\cdot)$ over Ab is lifted to an adjoint situation over $Val\,Ab$. A similar lifting for p-localized groups can be obtained.

If C is a valuated group, we call a bilinear map $f: U A \times U B \to U C$ *admissible* provided

$$\nu(f(a,b)) \geq max(\nu(a),\nu(b)) \quad \text{for all } (a,b) \in U A \times U B.$$

Each admissible bilinear map determines an *induced homomorphism*

$$\hat{f}: U A \otimes_Z U B \to U C \quad \text{satisfying } \hat{f}\eta = f$$

where η is «insertion of generators». Consider the source $(U A \otimes_Z U B, \hat{f}_i)_I$

of all induced homomorphisms. By endowing $UA \otimes_Z UB$ with the least valuation making each $\hat{f}_i$ a valuated group morphism, we obtain a valuated group $A \otimes B$ with generators $\{ a \otimes b \mid a \in A, \ b \in B \}$ (subject to the usual relations, and having $\nu(a \otimes b) \geq max(\nu(a), \nu(b))$. In fact, the valuation on $A \otimes B$ is the smallest group valuation on $UA \otimes_Z UB$ satisfying the previous inequality on generators.

Next we give a valuation on $Hom(A, B)$, the set of valuated morphisms from A to B. If $f: A \to B$ is a morphism of valuated groups, define

$$\nu_p(f) = inf\{ \nu_p(f(a)) \mid a \in A \}$$

for each prime p. This defines a functorial valuation on $Hom(A, B)$.

2.1. THEOREM. *For any valuated group A, the functor $A \otimes (-)$ is the left adjoint to the functor $Hom(A, -)$.*

PROOF. Let A, B and C be valuated groups. Define

$$\psi : Hom(A \otimes B, C) \to Hom(B, Hom(A, C))$$

by

$$\psi(f) = \hat{f} \text{ where } \hat{f}(b)(a) = f(a \otimes b).$$

It is straightforward to verify that ψ is a group isomorphism preserving valuation.

These definitions immediately apply for $Val\,Ab_p$ and a lifting of the adjoint situation between $A \otimes_Z (-)$ and $Hom(A, -)$ can be obtained as well. It follows that $Val\,Ab$ and $Val\,Ab_p$ are symmetric monoidally closed categories.

2.2. PROPOSITION. *Let A and B be valuated groups with valuation given by height. Then:*

(1) $A \otimes B$ has valuation given by height;

(2) $Hom(A, B)$ need not carry height.

PROOF. That $A \otimes B$ carries height follows from the inequality

$$h(a \otimes b) \geq h(a) + h(b) \geq max(h(a), h(b)).$$

To see $Hom(A, B)$ need not have height, let $A = Z(2)$ and $B = Z(8)$. Let $f \in Hom(A, B)$ with $f(1) = 4$. Then $f(A) \leq 2.B$ so $\nu_2(f) \geq 1$. But

$h_2(f) = 0$ since $2.Hom(A,B) = 0$.

The tensor product defined above yields the p-localized groups as defined by Richman and Walker [8].

2.3. THEOREM. *If $E_p : Val\,Ab_p \to Val\,Ab$ is the inclusion functor, then for each valuated group A, $E_p A_p = A \otimes Z_p$ where the tensor product is performed in $Val\,Ab$.*

PROOF. The underlying groups are the same, hence we need only consider the valuations. For a prime $q \neq p$, the valuations in both groups are constantly valued ∞. Thus we need only check the valuations for the prime p. In the following diagram

$$\begin{tikzcd} (A \times Z_p, max) \arrow[r, "\eta"] \arrow[dr, "\eta"'] & A \otimes Z_p \arrow[d, dashed, "1"] \\ & E_p A_p \end{tikzcd}$$

observe that «insertion of generators» η from $A \times Z_p$ to $E_p A_p$ is an induced morphism. Let $a \in A$ and $c/m \in Z_p$. Then

$$\nu(a \otimes c/m) = \nu(c.a) \geq \nu(a);$$

if $h(c) = \alpha$, where h is the height valuation on Z_p, then

$$\nu(c.a) \geq \nu(a) + \alpha \geq \alpha.$$

Thus

$$\nu(\eta(a, c/m)) \geq max(\nu(a), h(c)) = max(\nu(a), h(c/m)).$$

Next recall that there is a canonical p-valuation preserving map

$$\ast_p : A \to E_p A_p \quad \text{defined by} \quad \ast_p(a) = a \otimes 1 \text{ for each } a \in A.$$

This map is the reflection (front adjunction) and hence the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "\ast_p"] \arrow[dr, "\ast_p"'] & E_p A_p \arrow[d, dashed, "1"] \\ & A \otimes Z_p \end{tikzcd}$$

Thus the valuations on $E_p A_p$ and $A \otimes Z_p$ coincide.

2.4. COROLLARY. *If L_p denotes the p-localization functor, then L_p preserves tensor products.*

REMARK. While the tensor product functor and hom functor have many nice properties, neither turns out to be an exact functor. We postpone the demonstration that $Hom(-,Z)$ fails to be exact until the next section. To see tensor product fails to be an exact functor we consider the following example: Let $Z^{(n)}$ denote Z having valuation determined by $\nu(1) = n$, and $Z^{(n)}_{(p)}$ defined similarly. Then the following sequence is exact (it is in fact a free presentation):

$$0 \longrightarrow Z^{(2)} \xrightarrow{m} Z^{(1)} \longrightarrow Z^{(1)}_{(p)} \longrightarrow 0.$$

The morphism m is multiplication by p. Tensoring with $Z^{(3)}$ we obtain

$$\begin{tikzcd}
0 \arrow[r] & Z^{(2)} \otimes Z^{(3)} \arrow[r] \arrow[d, equal, "\wr"] & Z^{(1)} \otimes Z^{(3)} \arrow[r] \arrow[d, equal, "\wr"] & Z^{(1)}_{(p)} \otimes Z^{(3)} \arrow[r] \arrow[d, equal, "\wr"] & 0 \\
0 \arrow[r] & Z^{(3)} \arrow[r] & Z^{(3)} \arrow[r, "\bar{m}"] & Z^{(3)}_{(p)} \arrow[r] & 0
\end{tikzcd}$$

If this sequence were exact, $\bar{m}$ (multiplication by p) would be an embedding, but $\nu(1) = 3$ and $\nu(\bar{m}(1)) = \nu(p) \geq 4$.

In the next section, we shall exploit the notion of a proper cokernel. A cokernel $\beta: B \to A$ of valuated groups is called *proper* provided that for each $a \in A$, there is an element $x_a \in B$ with the property that

$$\beta(x_a) = a \quad \text{and} \quad \nu(x_a) = \nu(a).$$

2.5. THEOREM. *If $\beta: B \to A$ is a proper cokernel and C is any valuated group, then $\beta \otimes 1: B \otimes C \to A \otimes C$ is a proper cokernel.*

PROOF. The functor $(-) \otimes C$ preserves all colimits, hence $\beta \otimes 1$ is a cokernel. For each $a \in A$, there is an element $x_a \in B$ with $\beta(x_a) = a$ and $\nu(x_a) = \nu(a)$. Let $\Sigma a_i \otimes c_i \in A \otimes C$. Then

$$\nu(\Sigma a_i \otimes c_i) = \nu(\Sigma \beta(x_{a_i}) \otimes c_i) \geq \nu(\Sigma x_{a_i} \otimes c_i).$$

Define $\bar{\nu}$ on $A \otimes C$ by

$$\bar{\nu}(\Sigma a_i \otimes c_i) = \nu(\Sigma x_{a_i} \otimes c_i) ;$$

then $\bar{\nu}$ is a valuation on $A\otimes C$ and

$$\bar{\nu}(a\otimes c) = \nu(x_a\otimes c) \geq max(\nu(x_a),\nu(c)) = max(\nu(a),\nu(c)).$$

Thus the valuation ν on $A\otimes C$ satisfies the inequality $\nu \leq \bar{\nu}$ and so, in particular,

$$\nu(\Sigma\, a_i\otimes c_i) \leq \bar{\nu}(\Sigma\, a_i\otimes c_i) = \nu(\Sigma\, x_{a_i}\otimes c_i).$$

Hence

$$\nu(\Sigma\, a_i\otimes c_i) = \nu(\beta\otimes 1\,(\Sigma\, x_{a_i}\otimes c_i))$$

and $\beta\otimes 1$ is a proper cokernel.

3. FREE PRESENTATIONS, PROPER SEQUENCES AND EXT.

Let (X,μ) be a valuated set and FX denote the free valuated group over (X,μ). Then $FX = \bigoplus_{x\in X} \mathrm{Z}_x$ where Z_x is the group Z with valuation determined by $\nu(1) = \mu(x)$. A similar result holds for the free p-localized valuated group. That is,

$$F_p X = \bigoplus_{x\in X} \mathrm{Z}_x\otimes \mathrm{Z}_p = \bigoplus_{x\in X} \mathrm{Z}_{px}$$

where $\mathrm{Z}_{px} = \mathrm{Z}_p$ with valuation determined by $\nu(1) = \mu(x)$.

If A is a valuated group, let $F = \bigoplus_{a\in A} \mathrm{Z}_a$ as above and $\epsilon: F\to A$ be that unique valuated group morphism making the following diagram commute :

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[dr, "1"'] & \bigoplus_{a\in A} \mathrm{Z}_a = F \arrow[d, dashed, "\epsilon"] \\ & A \end{tikzcd}$$

The function η is insertion of generators and for $a\in A$, we denote the image of a under η by e_a. Let R be the kernel of ϵ with the induced valuation from F. Then it is easy to see that the following sequence is stable exact (i is simply inclusion):

$$0 \longrightarrow R \xrightarrow{\;i\;} F \xrightarrow{\;\epsilon\;} A \longrightarrow 0 \qquad\qquad ⊛$$

One way to observe this is to localize and note that the sequence

$$0 \longrightarrow R_p \xrightarrow{i_p} F_p \xrightarrow{\epsilon_p} A_p \longrightarrow 0$$

is stable exact for each prime p. Moreover, this localized sequence is a projective presentation of A_p in $Val\,Ab_p$ for each prime p. This is in contrast to the situation for the sequence $\circledast$; that is $\circledast$ need not be a projective presentation of A in $Val\,Ab$.

3.1. PROPOSITION. *Let* $\circledast$ *be as above and* B *be any valuated group. Then*

$$i_\# : Hom(B,R) \to Hom(B,F) \quad \text{and} \quad \epsilon^\# : Hom(A,B) \to Hom(F,B)$$

are embeddings.

In analogy with the classical procedure for abelian groups, we define $Ext(A,B)$, for a pair of valuated groups, to be the cokernel making the following sequence exact:

$$0 \to Hom(A,B) \xrightarrow{\epsilon^\#} Hom(F,B) \xrightarrow{i^\#} Hom(R,B) \xrightarrow{*} Ext(A,B) \longrightarrow 0.$$

Since $Val\,Ab$ is a preabelian category which is not abelian, the classical definition of Ext in terms of short exact sequences and Baer addition need not yield a group. Richman and Walker [9] (see also [2]) have defined Ext in preabelian categories by considering the stable short exact sequences and Baer addition, and subsequently, in [8], they consider Ext in $Val\,Ab_p$. Their definition of Ext in $Val\,Ab_p$ is not that of a valuated group. Our next theorem shows that the valuated group $Ext(A,B)$ as defined above has as its underlying group a subgroup of the group defined by Richman and Walker. In the p-local case, the two groups coincide.

A cokernel $\beta : E \to A$ of valuated groups is called *proper* provided when

$$0 \longrightarrow R \longrightarrow F \xrightarrow{\epsilon} A \longrightarrow 0$$

is a free presentation of A, there is a valuated group morphism $\pi : F \to E$ satisfying $\beta\pi = \epsilon$. That is, for each $a \in A$, there is an element $\pi(e_a) = x_a$ in E so that $\nu(x_a) = \nu(a)$ ($\nu_p(x_a) = \nu_p(a)$ for all primes p). We call a short exact sequence

$$0 \longrightarrow B \xrightarrow{\alpha} E \xrightarrow{\beta} A \longrightarrow 0$$

a proper sequence provided β is a proper cokernel. Such sequences are stable exact. Moreover, it is easy to verify that the class of all proper sequences is a «proper» class of stable exact sequences, thus giving rise to a relative Ext. A class of stable exact sequences is a proper class provided the class is closed under formations of pushouts, pullbacks, and compositions (see [9]).

For valuated groups A and B, denote the set of stable exact sequences $B \to . \to A$ by $E(A,B)$ where two sequences are considered equal if there is a map $(1_B, \phi, 1_A)$ between them with ϕ an isomorphism. Endow this set with Baer addition and $E(A,B)$ becomes an abelian group ([9], see also [2]).

3.2. THEOREM. *Given* $Ext(A,B)$ *and* $E(A,B)$ *as defined above, then* $U(Ext(A,B))$ *is that subgroup of* $E(A,B)$ *consisting of all proper sequences. Moreover, for p-localized groups* A *and* B,

$$U_p(Ext(A,B)) = E(A,B)$$

where all sequences are taken in $Val\,Ab_p$.

3.3. LEMMA. If F is a free valuated group over the valuated set (X,μ) and

$$0 \longrightarrow B \xrightarrow{\alpha} E \xrightarrow{\beta} A \longrightarrow 0$$

is a proper sequence, then the associated hom sequence

$$0 \longrightarrow Hom(F,B) \xrightarrow{\alpha_\#} Hom(F,E) \xrightarrow{\beta_\#} Hom(F,A) \longrightarrow 0$$

is a proper sequence.

3.4. THEOREM. *Let*

$$0 \longrightarrow B \xrightarrow{\alpha} B' \xrightarrow{\beta} B'' \longrightarrow 0$$

be a proper sequence. Then there is a connecting valuated group morphism making the following a sequence in $Val\,Ab$:

$$\begin{aligned} 0 \longrightarrow Hom(A,B) \longrightarrow Hom(A,B') \longrightarrow Hom(A,B'') \\ \xrightarrow{\omega} Ext(A,B) \longrightarrow Ext(A,B') \longrightarrow Ext(A,B'') \end{aligned}$$

3.5. THEOREM. *A valuated group A is free iff $Ext(A,B)=0$ for all valuated groups B.*

PROOF. By the lemma, A free implies $Ext(A,B)=0$ for all valuated groups B. The converse follows from the fact that the class of free valuated groups is closed under summands (see Arnold, Hunter and Richman [0], and Stanton [10].

For a valuated group C and $\lambda \in \Gamma$, define

$$C(\lambda) = \{ c \in C \mid \nu(c) \geq \lambda \}.$$

3.6. THEOREM. *Let $Z_{(p)}^{(\lambda)}$ denote the group $Z(p)$ with valuation determined by $\nu(1)=\lambda$. Then if B is any valuated group,*

$$Ext(Z_{(p)}^{(\lambda)}, B) \approx B(\lambda+1)/pB(\lambda).$$

3.7. COROLLARY. *$Hom(\ ,-Z)$ is not an exact functor.*

3.8. COROLLARY. *The connecting homomorphism sequence for Ext and Hom need not be exact.*

PROOF. For 3.7 apply $Hom(-,Z)$ and for 3.8 apply $Hom(Z_{(p)}^{(1)},-)$ to the free presentation

$$0 \longrightarrow Z^{(2)} \longrightarrow Z^{(1)} \longrightarrow Z_{(p)}^{(1)} \longrightarrow 0.$$

3.9. THEOREM. *Let B be a valuated group and*

$$0 \longrightarrow A \longrightarrow A' \longrightarrow A'' \longrightarrow 0$$

be a proper sequence. Then there exists a connecting valuated group morphism making the following a sequence in Val Ab

$$0 \longrightarrow Hom(A'',B) \longrightarrow Hom(A',B) \longrightarrow Hom(A,B) \longrightarrow$$
$$\longrightarrow Ext(A'',B) \longrightarrow Ext(A',B) \longrightarrow Ext(A,B).$$

4. TORSION PRODUCTS.

In this section we consider natural ways to produce valuations on the torsion product of two valuated groups. Let

$$0 \longrightarrow R \xrightarrow{i} F \xrightarrow{\epsilon} A \longrightarrow 0$$

be a free presentation of the valuated group A and let B be any valuated group. We obtain a functorial definition for the torsion product of A and B by declaring $\bar{T}or(A,B) = ker(i \otimes 1_B)$, thus obtaining the following exact sequence:

$$0 \to \bar{T}or(A,B) \xrightarrow{k} R \otimes B \xrightarrow{i \otimes 1_B} F \otimes B \xrightarrow{\epsilon \otimes 1_B} A \otimes B \to 0.$$

This definition is independent of the free presentation of A since all such presentations are proper sequences.

Another view of this valuation on $\bar{T}or(A,B)$ is obtained by considering generators and relations. Let

$$X = \{(a,m,b) \mid a \in A,\ b \in B,\ m \in Z,\ ma = 0 = mb\}.$$

For $a \in A$, pick

$$x_a \in F \text{ with } \nu(x_a) = \nu(a) \text{ and } \epsilon(x_a) = a.$$

Pick $r_a \in R$ with $i(r_a) = m x_a$. This can be done since $\epsilon(m x_a) = ma = 0$. Define

$$k(a,m,b) = r_a \otimes b \in R \otimes B.$$

These elements $k(a,m,b)$ generate $\bar{T}or(A,B)$. Observe

$$\nu(r_a \otimes b) \geq max(\nu(r_a), \nu(b)) = max(\nu(m x_a), \nu(b)) \geq$$
$$max(\nu(x_a) + h(m), \nu(b)) = max(\nu(a) + h(m), \nu(b)).$$

Consequently, if one defines $\nu(a,m,b) = max(\nu(a) + h(m), \nu(b))$, the valuation on $\bar{T}or(A,B)$ is the least group valuation making the map $k: X \to \bar{T}or(A,B)$ a valuated function.

Repeat this same argument by taking a free presentation of B, to obtain another valuation on the torsion product of A and B. We denote this group by $\underline{Tor}(A,B)$. This valuation is the least on generators satisfying the inequality

$$\nu(a,m,b) \geq max(\nu(a), \nu(b) + h(m)).$$

4.1. THEOREM. *Using the above notations,* $\underline{Tor}(A,B) \approx \bar{T}or(A,B)$. $\underline{Tor}(A,B)$ *and* $\bar{T}or(A,B)$ *need not be isomorphic as valuated groups.*

Richman has given a valuation group $Tor(A,B)$ which has the property that if A and B have height, so has $Tor(A,B)$. This valuation is different (smaller) than either given above.

Finally, we note that the connecting homomorphism theorem fails for our valuations on $Tor(A,B)$. That is, given a proper sequence

$$0 \to B \longrightarrow E \longrightarrow A \longrightarrow 0$$

and valuated group C, the connecting homomorphism $\omega : Tor(A,C) \to B \otimes C$ need not be a morphism of valuated groups. Consider the proper sequence

$$0 \longrightarrow Z^{(1)} \longrightarrow Z^{(0)} \longrightarrow Z^{(0)}_{(p)} \longrightarrow 0$$

and the valuated group $Z^{(2)}_{(p)}$. Then

$$Z^{(1)} \otimes Z^{(1)}_{(p)} \approx Z^{(1)}_{(p)} \quad \text{and} \quad Tor(Z^{(0)}_{(p)}, Z^{(0)}_{(p)}) \approx Z^{(2)}_{(p)}.$$

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