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OPEN COVERS AND INFINITARY OPERATIONS IN C^{∞} -RINGS¹) by Eduardo J. DUBUC

INFINITE ADDITIONS.

In the C[∞]-ring $C^{\infty}(\mathbb{R}^n)$ of smooth real valued functions, one can define elements (functions) by adding certain infinite families. A typical application of this method is the following: Suppose $M \to \mathbb{R}^n$ is a closed smooth submanifold, and let $h: M \to \mathbb{R}$ be a smooth function defined on M. By definition this means that there are open sets $U_a \subset \mathbb{R}^n$, $a \in \Gamma$, such that they cover M, and smooth functions $h_a: U_a \to \mathbb{R}$ such that

$$\forall p \in U_a \cap M, \quad h_a(p) = h(p).$$

Let U_0 be $\mathbb{R}^n - M$, and take any function (e.g. the zero function) $h_0: U_0 \to \mathbb{R}$. We then have an open covering U_a , $a \in \Gamma + \{0\}$, of the whole space \mathbb{R}^n , and functions $h_a: U_a \to \mathbb{R}$ such that

$$\forall p \in U_a \cap M, \quad h_a(p) = h(p).$$

(since $U_0 \cap M = \emptyset$). This family does not agree in the intersections $U_a \cap U_\beta$; thus it does not define a global function $f: \mathbb{R}^n \to \mathbb{R}$ which extends h. However, we can construct an extension of h. Let W_i be a locally finite refinement of U_a , for each i take a such that $W_i \subset U_a$, and let $g_i: W_i \to \mathbb{R}$ be equal to $h_a \mid W_i$. Let ϕ_i be an associated partition of unity. The functions $f_i = \phi_i g_i$ are defined globally, $f_i: \mathbb{R}^n \to \mathbb{R}$, and have support contained in W_i . (This is so since $support(\phi_i) \subset W_i$ by definition.) It follows that the equality $f = \sum_i f_i$ defines a function $f: \mathbb{R}^n \to \mathbb{R}$ such that

$$\forall x \in \mathbb{R}^n, f(x) = \sum_i f_i(x)$$

(which is a *finite sum* on an open neighborhood of x). If we compute f at 1) Partially supported by the Danish Natural Science Research Council. a point $p \in M$, we have:

$$\begin{split} f(p) &= \sum_{i} f_{i}(p) = \sum_{i \mid p \in W_{i}} f_{i}(p) = \sum_{i \mid p \in W_{i}} \phi_{i}(p) g_{i}(p) \\ &= \sum_{i \mid p \in W_{i}} \phi_{i}(p) h(p) = h(p) \sum_{i} \phi_{i}(p) = h(p). \end{split}$$

Thus f is an extension of h.

Two questions arise:

1. Which is the algebraic meaning of these infinitary additions?

11. To which extent can they be defined and performed in arbitrary C^{∞} -rings of finite type $A = C^{\infty}(\mathbb{R}^n)/1$?

In particular, can we give a meaning to expressions of the form $a = \sum_{\alpha} a_{\alpha}$, $a_{\alpha} \in A$? In a way such that if $a_{\alpha} = [f_{\alpha}]$, $f_{\alpha} \in C^{\infty}(\mathbb{R}^{n})$, and the f_{α} can be added in $C^{\infty}(\mathbb{R}^{n})$ defining a function $f = \sum_{\alpha} f_{\alpha}$ (as before), then a = [f] (where brackets indicate equivalence class modulo 1). That this is not always possible is seen as follows: Let

 $l = (h \mid h \text{ is of compact support})$

and let ϕ_a be a partition of unity such that $\phi_a \epsilon I$ for all a. Let

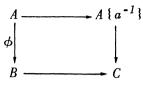
 $a_a = [\phi_a]$ and $a = \sum_a a_a$.

Then a = 1 since $l = \sum_{\alpha} \phi_{\alpha}$ in $C^{\infty}(\mathbb{R}^{n})$. But also $a_{\alpha} = [0]$ and then a = 0 since $0 = \sum_{\alpha} 0$ in $C^{\infty}(\mathbb{R}^{n})$. Thus l = 0 in A, which is impossible since $l \notin l$. We see that a condition is needed in the ideal that presents A. Namely, that if $l_{\alpha} \in l$ and the l_{α} can be added in $C^{\infty}(\mathbb{R}^{n})$ defining a function $l = \sum_{\alpha} l_{\alpha}$ (as before), then $l \in l$. This leads to the notion of *ideal of local character* (cf. Definition 6, iv).

ANSWER TO QUESTION II.

The first step is to define the open cover topology in the category $\mathfrak{A}_{f,t}^{op}$, dual of the category of \mathbb{C}^{∞} -rings of finite type. Given a \mathbb{C}^{∞} -ring A and an element $a \in A$, we denote $A \to A \{a^{-1}\}$ the solution in \mathfrak{A} to the universal problem of making a invertible (cf. [1, 2]). The following is straightforward:

1. PROPOSITION. Given any morphism $\phi: A \rightarrow B$ and element $a \in A$, the diagram



is a pushout diagram iff $C = B\{\phi(a)^{-1}\}$.

We recall the following

2. PROPOSITION. Let $U \subset \mathbb{R}^n$ be an open set, and let f be such that $U = \{x \mid f(x) \neq 0\}$. Then $C^{\infty}(\mathbb{R}^n) \rightarrow C^{\infty}(U) = C^{\infty}(\mathbb{R}^n) \{f^{-1}\}$.

PROOF. The basic idea is to consider the map

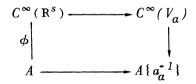
$$U \rightarrow \mathbb{R}^{n+1}: p \mapsto (p, f(p, f))$$

which makes U the closed sub-manifold of \mathbb{R}^{n+1} defined by the equation $1 \cdot x_{n+1}f(p) = 0$. Then use the fact that this equation is independent to deduce that the equalizer is preserved when taking the C[∞]-rings of smooth functions (cf. [1, 2]). A different proof of the preservation of this equalizer is given in [4].

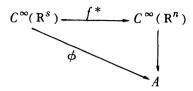
3. DEFINITION. The open cover topology in $\mathfrak{A}_{f,t}^{op}$ is the topology generated by the empty family covering $\{0\}$, and families of the form

 $C^{\infty}(\mathbb{R}^n) \to C^{\infty}(U_{\alpha})$, for all *n* and all open coverings U_{α} of \mathbb{R}^n .

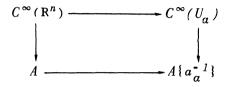
It follows from Propositions 1 and 2 that basic (generating) covers of an arbitrary C^{∞} -ring of finite type A are families $A \rightarrow A\{a_{a}^{-1}\}$ which can be completed into pushout diagrams



where $V_a = \{x \mid g_a(x) \neq 0\}$ is an open cover of \mathbb{R}^s and $\phi(g_a) = a_a$. The morphism ϕ can be chosen to be a quotient map. Let $A = C^{\infty}(\mathbb{R}^n)/I$; then, since $C^{\infty}(\mathbb{R}^s)$ is free, there is a smooth function $f: \mathbb{R}^n \to \mathbb{R}^s$ making the following triangle commutative :



One checks then that the following diagrams are pushouts:



where

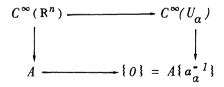
$$U_{a} = f^{-1}(V_{a}) = \{ x \mid f_{a}(x) \neq 0 \}, f_{a} = g_{a}f,$$

is an open cover of \mathbb{R}^n and $a_a = [f_a]$.

Open covers $C^{\infty}(\mathbb{R}^n) \to C^{\infty}(U_{\alpha})$ are effective epimorphic families. This just means that given a compatible family $f_{\alpha} \in C^{\infty}(U_{\alpha})$ (meaning that they agree in the intersections), there exists a unique $f \in C^{\infty}(\mathbb{R}^n)$ such that $f|_{\alpha} = f$. (Remark that $C^{\infty}(U_{\alpha} \cap U_{\beta})$ is the pushout of $C^{\infty}(U_{\alpha})$ with $C^{\infty}(U_{\beta})$ over $C^{\infty}(\mathbb{R}^n)$.) However, they are not universal. Open covers of an arbitrary A will not, in general, be effective epimorphic families. For example, let as before I be the ideal

 $l = (h \mid h \text{ is of compact support}),$

and let ϕ_a be a partition of unity such that $\phi_a \epsilon l$ for all a. Then the diagrams



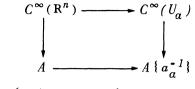
are pushout diagrams for all a, where U_a is the open covering

$$U_a = \{x \mid \phi_a(x) \neq 0\}, A = C^{\infty}(\mathbb{R}^n)/I, \text{ and } a_a = [\phi_a] = 0.$$

On the way we see that the empty family covers A since it covers $\{0\}$

(use composition of coverings).

Consider now an open covering of an arbitrary C^{∞} -ring $A = C^{\infty}(\mathbb{R}^n)/I$



$$U_a = \{x \mid f_a(x) \neq 0\} \text{ and } a_a = [f_a].$$

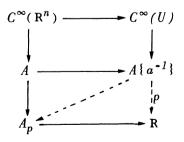
For $b \in A$, we denote $b_{|\alpha|}$ its image in $A\{a_{\alpha}^{-1}\}$. Suppose b, $b' \in A$ are such that $b_{|\alpha|} = b'_{|\alpha|}$ for all α . Let b = [f] and b' = [f']. Then

$$[f|U_{\alpha}] = b|_{\alpha}$$
 and $[f'|U_{\alpha}] = b'|_{\alpha}$.

Thus $[f|U_{\alpha}] = [f'|U_{\alpha}]$, which means that $(f - f')|U_{\alpha} \in I|U_{\alpha}$. If the covering is going to be effective epimorphic, it should follow that $(f - f') \in I$. This leads to the notion of *ideal of local character* (cf. Definition 6, ii).

We recall the following elementary facts in order to fix the notation.

4. PROPOSITION. Let A be a C^{∞} -ring and $p: A \rightarrow \mathbb{R}$ a morphism into \mathbb{R} , which we will also call a point of A. We denote $A \rightarrow A_p$ the solution in the category of C^{∞} -rings to the universal problem of making invertible all the elements $a \in A$ such that $p(a) \neq 0$. There is a factorization of p, $A \rightarrow A_p \rightarrow \mathbb{R}$, and A_p is a C^{∞} -local ring. If $a \in A$, we denote $a_{\mid p} \in A_p$ its image in A_p . Suppose $A = C^{\infty}(\mathbb{R}^n)/I$, let $U \subset \mathbb{R}^n$ open, f such that $U = \{x \mid f(x) \neq 0\}$, and a = [f]. Consider the following diagram (where the upper square is a pushout):



Given a point p of A, $p: A \rightarrow \mathbb{R}$, since $C^{\infty}(\mathbb{R}^n)$ is free, p can be identified with a point of $Zeros(I) \subset \mathbb{R}^n$ which we will also denote p. When

there exist factorizations (necessarily unique) as shown in the dotted arrows, p can be identified with a point of $A\{a^{-1}\}$, which we will also denote by p. Then

p is a point of $A\{a^{-1}\} \iff p(a) \neq 0 \iff p \in U$.

It follows that if $A \to A\{a_a^{-1}\}$ is an open cover and p is a point of A then there exists a such that p is a point of $A\{a_a^{-1}\}$ (since there exist a such that $p \in U_a$).

PROOF. This is all rather straightforward. For a proof, cf. [2, Exposé 11].

Suppose now b, b' ϵ A are such that $b|_p = b'|_p$ for all points p of A. Let b = [f] and b' = [f']. Then

$$b_{|p} = [f_{|p}] \text{ and } b'_{|p} = [f'_{|p}].$$

Thus $[f|_p] = [f'|_p]$, which means that $(f - f')|_p \epsilon l|_p$. If we want to deduce that b = b', it should follows that $(f - f') \epsilon l$. This leads to the notion of *ideal of local character* (cf. Definition 6, i).

5. DEFINITION. A family $l_i \in C^{\infty}(\mathbb{R}^n)$ (indexed by an arbitrary set) is locally finite if there is an open covering U_{α} such that, for every α , $l_i | U_{\alpha} = 0$ except for a finite number of *i*. Equivalently, if each point *p* of \mathbb{R}^n has an open neighborhood *U* such that $l_i | U = 0$ except for a finite number of *i*.

Given a locally finite family l_i , the finite sums $\sum_i l_i | U_a \in C^{\infty}(U_a)$ form a compatible family. Thus there exists a unique

 $l \in C^{\infty}(\mathbb{R}^n)$ such that $l \mid U_{\alpha} = \sum_i l_i \mid U_{\alpha}$.

We denote this l by the formula $l = \sum_{i} l_i$.

6. DEFINITION. An ideal $l \in C^{\infty}(\mathbb{R}^n)$ is of *local character* if it satisfies any one of the following equivalent conditions:

- i) $(f_{|p} \epsilon l_{|p} \text{ for all } p \epsilon Zeros(l)) \Rightarrow f \epsilon l.$
- ii) $(f | U_a \epsilon I | U_a$ for some open cover U_a) => $f \epsilon I$.
- iii) $\phi_a f \epsilon l$ for some partition of unity $\phi_a \Rightarrow f \epsilon l$.
- iv) $l_i \epsilon l$ for all $i \Rightarrow (\sum_i l_i) \epsilon l$ for every locally finite family l_i .

PROOF OF THE EQUIVALENCE. The antecedents of the first three implications are themselves equivalent, cf. [1], Lemma 10, and for detailed proof, [2] Théorème 1.5; thus the equivalence of the first three conditions. That $iv \Rightarrow iii$ is immediate since $\phi_a f$ is a locally finite family and $f = \Sigma \phi_a f$. Finally, it is evident that $ii \Rightarrow iv$. Notice that the implications in the first three conditions always hold in the other sense.

It is clear that any ideal l has a «closure» of local character; namely

$$I = \{ f \mid f|_p \in I|_p \quad \forall p \in Zeros(I) \}.$$

 \tilde{l} can also be seen as the closure of l under additions of locally finite families. Let \mathcal{B} be the category of C^{∞}-rings presented by an ideal of local character. Let

$$A = C^{\infty}(\mathbb{R}^n)/I$$
 and $rA = C^{\infty}(\mathbb{R}^n)/I$.

Then we have a canonical (quotient) map $A \to rA$ and the passage $A \to rA$ is clearly a left adjoint for the inclusion $\mathcal{B} \to \mathcal{A}_{f,t}$. Thus \mathcal{B} is closed under all inverse limits and has all colimits. These colimits will not coincide in general with the respective construction in $\mathcal{A}_{f,t}$. We remark that, since Zeros(l) = Zeros(l), A and rA have the same points.

7. EXAMPLES. 1° Let $A = C_0^{\infty}(\mathbb{R})$, $B = C^{\infty}(\mathbb{R})$. Then $A, B \in \mathcal{B}$. By construction, $A \otimes_{\infty} B = C^{\infty}(\mathbb{R}^2)/l$, where

$$I = \{ f \mid \exists \eta > 0, f(x, y) = 0 \forall x \mid |x| < \eta \}.$$

This ideal is not of local character:

 $\hat{I} = \{ f \mid f = 0 \text{ on an arbitrary neighborhood of the y-axis } \}.$

The coproduct of A with B in \mathcal{B} does not coincide then with the one performed in $\mathfrak{A}_{f,t}$.

20 Let
$$A = C_0^{\infty}(\mathbf{R})$$
 and $a = x|_0$. Then
 $A\{a^{-1}\} = C_0^{\infty}(\mathbf{R})\{x|_0^{-1}\} = C^{\infty}(\mathbf{R}^*)/J$

where $R^* = R - \{0\}$ and

$$\mathbf{J} = \{ f \mid \exists U \mid 0 \in U, f \mid U = 0 \} \subset C^{\infty}(\mathbf{R}^*).$$

We see this because the following is a pushout diagram:

$$C^{\infty}(\mathbf{R}) \longrightarrow C^{\infty}(\mathbf{R}^{*}) = C^{\infty}(\mathbf{R}) \{ x^{-I} \}$$

$$\downarrow$$

$$C^{\infty}_{0}(\mathbf{R}) \longrightarrow C^{\infty}(\mathbf{R}^{*})/J = C^{\infty}_{0}(\mathbf{R}) \{ x_{|0}^{-I} \}$$

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(cf. Proposition 1). Now, the ring $A\{a^{-1}\}$ has the presentation

$$A\{a^{-1}\} = C^{\infty}(\mathbb{R}^2)/(1, 1-xy),$$

where *l* is the ideal in 1 above. Thus the localization $A\{a^{-1}\}$ in \mathcal{B} does not coincide with the one performed in $\mathfrak{A}_{f,t}$. Since Zeros $(1, 1 - xy) = \emptyset$, we have that $1 \in (1, 1 - xy)^{*}$. Thus $A\{a^{-1}\} = \{0\}$ in \mathcal{B} . This means that if $\phi: C_{0}^{\infty}(\mathbb{R}) \rightarrow B$ is any morphism, *B* is in \mathcal{B} , and $\phi(x_{|0})$ invertible, then $B = \{0\}$. This is not so if *B* is not in \mathcal{B} .

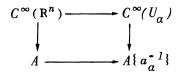
In what follows we will utilize the same notation as in $\mathfrak{A}_{f,t}$ for the constructions performed in \mathfrak{B} . Since they are defined by the same universal properties, we have:

8. PROPOSITION. Propositions 1 and 4 remain valid for the category \mathfrak{B} . In addition, since all finitely generated ideals are of local character (cf. [1, 2]), if A is of finite presentation, then for any $a \in A$, $A\{a^{-1}\}$ constructed in $\mathfrak{A}_{f,t}$ is already in \mathfrak{B} . Thus Proposition 2 remains valid also for the category \mathfrak{B} . Furthermore, if A is in \mathfrak{B} , then for any elements b, b' ϵA ,

$$b = b'$$
 iff $b|_p = b'|_p$ for all points p of A.

9. PROPOSITION. The open coverings are universal effective epimorphic families in the category \mathbb{B}^{op} .

PROOF. The proof is essentially a repetition of the argument given at the beginning of this article. Let $A = C^{\infty}(\mathbb{R}^n)/l$, l of local character, U_{α} and open cover of \mathbb{R}^n , f_{α} such that $U_{\alpha} = \{x \mid f_{\alpha}(x) \neq 0\}$ and $a_{\alpha} \in A$, $a_{\alpha} = [f_{\alpha}]$. We consider the pushout diagram in \mathcal{B} :



Let $b_a \in A\{a_a^{-1}\}$ be a compatible family. This implies that for all points of $A\{a_a^{-1}, a_\beta^{-1}\}$, $b_{a|p} = b_{\beta|p}$. Thus, for all p in A there is a well defined $b(p) \in A_p$, $b(p) = b_{a|p}$ for any a such that p is in A_a . We shall construct an element

$$b \in A$$
 such that $b|_p = b(p)$ for all p in A.

Let $h_a \in C^{\infty}(U_a)$ such that $b_a = [h_a]$, let W_i be a locally finite refinement of U_a , for each *i* take *a* such that $W_i \subset U_a$, and let $g_i \in C^{\infty}(W_i)$ be equal to $h_a | W_i$. Thus, for all *p* in W_i , $[g_i|_p] = b(p)$. Let ϕ_i be an associated partition of unity. The functions $l_i = \phi_i g_i$ are defined globally, $l_i \in C^{\infty}(\mathbb{R}^n)$, and have support contained in W_i . Thus, given any *p* in *A*, if $p \notin W_i$, $l_i|_p = 0$, and if $p \in W_i$,

$$[l_{i|p}] = [\phi_{i|p}] b(p),$$

Since the family l_i is locally finite (Definition 5), we have a function $l = \sum l_i$. Let b = [l]. Then, for any p in A,

$$b_{\mid P} = \left[\sum_{i} l_{i \mid P}\right] = \sum_{i \mid p \in W_{i}} \left[l_{i \mid P}\right] = b(p) \Sigma\left[\phi_{i \mid P}\right] = b(p).$$

Given any a, $b_{|a} = b_a$ since for all p in $A\{a_a^{-1}\}$, $b_{|p} = b_{a|p}$ and $A\{a_a^{-1}\}$ is in \mathcal{B} . In the same way one checks the uniqueness of such a, b, since all points of A are in some $A\{a_a^{-1}\}$. This finishes the proof.

10. DEFINITION - PROPOSITION. Given any C^{∞} -ring A presented by an ideal of local character, a family $b_i \in A$ (indexed by an arbitrary set) is *locally finite* if there is an open covering $A \rightarrow A\{a_a^{-1}\}$ such that for every $a, b_i|_a = 0$ except for a finite number of i. Given such a family, the finite sums $\sum_i b_i|_a \in A\{a_a^{-1}\}$ form a compatible family. It follows then from the previous proposition that there exists a unique

 $b \in A$ such that $b \mid a = \sum_{i} b_i \mid a$.

We denote this b by the formula $b = \sum_{i} b_{i}$. Given any morphism $\phi: A \to B$ in \mathcal{B} , the family $\phi(b_{i}) \in B$ is also locally finite, and $\phi \sum_{i} b_{i} = \sum_{i} \phi(b_{i})$.

This finishes the answer to Question II. Infinite additions of locally finite families make sense and can be performed in certain C^{∞} -rings. The

E. DUBUC 10

 C^{∞} -rings which have this extra structure are precisely those presented by an ideal of local character. Before passing to Question I, we make a final remark on the open covering topology.

11. PROPOSITION. Given any C^{∞} -ring $A \in \mathfrak{A}_{f,t}$ and a family $a_a \in A$, $A \rightarrow A\{a_a^{-1}\}$ covers in the open covering topology iff for all points p in A there exists a such that p is in $A\{a_a^{-1}\}$. Thus any C^{∞} -ring without points is covered by the empty family.

PROOF. One of the implications has already been seen (Proposition 4). Suppose then that for all p in A there is a such that p is in $A \{a_a^{-1}\}$. Let $A = C^{\infty}(\mathbb{R}^n)/I$, $f_a \in C^{\infty}(\mathbb{R}^n)$ such that

$$a_{a} = [f_{a}] \text{ and } U_{a} = \{ x \mid f_{a}(x) \neq 0 \}.$$

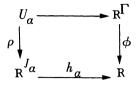
The hypothesis means that for all $p \in Zeros(I) \subset \mathbb{R}^n$ there is α such that $p \in U_{\alpha}$. If $p \notin Zeros(I)$, there is

$$l \in I$$
 such that $p \in U_l = \{ c \mid l(x) \neq 0 \}$.

Thus the U_{α} together with the U_{l} form an open cover of \mathbb{R}^{n} . Let $b_{l} \in A$, $b_{l} = [l]$. Then $A \rightarrow A\{a_{\alpha}^{-1}\}$ together with $A \rightarrow A\{b_{l}^{-1}\}$ form an open cover of A. But $A\{b_{l}^{-1}\} = \{0\}$; thus it is covered by the empty family. By composition of coverings, it follows that $A \rightarrow A\{a_{\alpha}^{-1}\}$ is an open covering of A.

ANSWER TO QUESTION I.

We consider the free C^{∞} -ring in λ generators, $\lambda \in \Gamma$, cf. [2]. This ring, which we denote $C^{\infty}(\mathbb{R}^{\Gamma})$, is the ring of functions $\mathbb{R}^{\Gamma} \to \mathbb{R}$ which depend only on a finite number of variables, and which are smooth on these variables. Clearly, this is a ring of continuous functions (for the product topology in \mathbb{R}^{Γ}). If we take a locally finite family of functions $\mathbb{R}^{\Gamma} \to \mathbb{R}$ in $C^{\infty}(\mathbb{R}^{\Gamma})$, and add it up, we get a continuous function which will not be in general in $C^{\infty}(\mathbb{R}^{\Gamma})$. However, every point will have a neighborhood in which this function will coincide with the restriction of a function in $C^{\infty}(\mathbb{R}^{\Gamma})$. We consider all functions ϕ which locally depend on a finite number of variables. More precisely, functions for which there exists an open cover $U_{\alpha} \subset \mathbb{R}^{\Gamma}$, finite sets J_{α} , smooth functions h_{α} and factorizations as indicated in the following diagram (where ρ is the projection):



for all α . Thus we add to $C^{\infty}(\mathbb{R}^{\Gamma})$ all the functions needed to render the open covers of \mathbb{R}^{Γ} effective epimorphic families. We will, for the lack of a better notation, denote this ring by ${}_{\infty}C^{\infty}(\mathbb{R}^{\Gamma})$. Thus a function ϕ is in ${}_{\infty}C^{\infty}(\mathbb{R}^{\Gamma})$ iff for each point $p \in \mathbb{R}^{\Gamma}$, there is an open neighborhood U of p, a finite set of indices $\{\lambda_1, \lambda_2, ..., \lambda_n\}$ and a smooth function of n variables h such that for all $(x_{\lambda}) \in U$, $\lambda \in \Gamma$,

$$\phi(x_{\lambda}) = h(x_{\lambda_1}, \dots, x_{\lambda_n}).$$

We leave to the reader the proof of the following (where Γ and Λ are any sets, including finite):

12. PROPOSITION. Given any $\phi \in {}_{\infty}C^{\infty}(\mathbb{R}^{\Gamma})$ and a Γ -tuple

$$\phi_{\lambda} \epsilon_{\infty} C^{\infty}(\mathbb{R}^{\Lambda}), \ (\phi_{\lambda}): \mathbb{R}^{\Lambda} \to \mathbb{R}^{\Gamma},$$

the composite $\phi(\phi_{\lambda}) \epsilon_{\infty} C^{\infty}(\mathbb{R}^{\Lambda})$.

It follows then that there is an infinitary algebraic theory in the sense of Linton (cf. [3]) which has as Γ -ary operations the rings ${}_{\infty}C^{\infty}(\mathbb{R}^{\Gamma})$. We shall call this theory the *infinitary theory of* C^{∞} -rings. Its finitary part is the theory of C^{∞} -rings, since ${}_{\infty}C^{\infty}(\mathbb{R}^{n}) = C^{\infty}(\mathbb{R}^{n})$. Thus, this ring is still free on n generators for the infinitary theory. (The action of a Γ -ary operation on $C^{\infty}(\mathbb{R}^{n})$ is given in Proposition 12, with $\Lambda = n$.) All ${}_{\infty}C^{\infty}$ -rings of finite type are then quotients of $C^{\infty}(\mathbb{R}^{n})$ presented by ${}_{\infty}C^{\infty}$ -ideals, that is, ideals I such that the congruence

is an ${}_{\infty}C^{\infty}$ -congruence. Recall that this congruence is always a C^{∞}-congruence (cf. [1, 2]). 13. PROPOSITION. Let $A = C^{\infty}(\mathbb{R}^n)/l$ be a C^{∞} -ring presented by an ideal l of local character. Then, A is an ${}_{\infty}C^{\infty}$ -ring. Or, equivalently, l is an ${}_{\infty}C^{\infty}$ -ideal.

PROOF. Let $\phi \epsilon_{\infty} C^{\infty}(\mathbf{R}^{\Gamma})$, and let

 $g_{\lambda}, f_{\lambda} \in C^{\infty}(\mathbb{R}^{n}), \lambda \in \Gamma$, such that $g_{\lambda} - f_{\lambda} \in I$.

Let p be any point of \mathbb{R}^n and let $U \subset \mathbb{R}^{\Gamma}$ be an open neighborhood of the two points $(g_{\lambda}(p))$, $(f_{\lambda}(p)) \in \mathbb{R}^{\Gamma}$ where ϕ depends on a finite number of variables:

$$\phi(x_{\lambda}) = h(x_{\lambda_{1}}, \dots, x_{\lambda_{k}})$$
 over U ,

for some $h \in C^{\infty}(\mathbb{R}^k)$. Let $\mathbb{V} \subset \mathbb{R}^n$ be an open neighborgood of p such that

$$(g_{\lambda})(W) \subset U$$
 and $(f_{\lambda})(W) \subset U$.

Then

$$(\phi(g_{\lambda}) \cdot \phi(f_{\lambda})) | \mathbb{W} = (h(g_{\lambda_{1}}, \dots, g_{\lambda_{k}}) \cdot h(f_{\lambda_{1}}, \dots, f_{\lambda_{k}})) | \mathbb{W} \in I | \mathbb{W},$$

because

$$h(g_{\lambda_1}, \dots, g_{\lambda_k}) - h(f_{\lambda_1}, \dots, f_{\lambda_k}) \in I,$$

since all ideals are C^{∞} -ideals. Thus the term $(\phi(g_{\lambda}) - \phi(f_{\lambda}))|_{p}$ is in $I|_{p}$. Since I is of local character, this implies $\phi(g_{\lambda}) - \phi(f_{\lambda}) \epsilon I$.

The converse of this proposition says, in a way, that there are enough operations in the infinitary theory of C^{∞}-rings to force any ${}_{\infty}C^{\infty}$ -ideal to be of local character. This is actually the case, and we prove it by showing that given any locally finite family $l_{\lambda} \in C^{\infty}(\mathbb{R}^{n})$, there is an infinitary operation that adds it up.

14. PROPOSITION. Let $l_{\lambda} \in C^{\infty}(\mathbb{R}^n)$, $\lambda \in \Gamma$, be any locally finite family. Then, the following function:

$$L: \mathbb{R}^{n} \times \mathbb{R}^{1} \to \mathbb{R}, \ L(x_{1}, \dots, x_{n}, x_{\lambda}) = \sum_{\lambda} l_{\lambda}(x_{1}, \dots, x_{n}) x_{\lambda}$$

is in ${}_{\infty}C^{\infty}(\mathbb{R}^{n+\Gamma}).$

PROOF. Given any point $p \in \mathbb{R}^{n+\Gamma}$, $p = (a_1, \dots, a_n, a_{\lambda})$, take an open neighborhood $U \subset \mathbb{R}^n$ of (a_1, \dots, a_n) , where all but a finite number of l_a

are zero. Then the open set $U \times \mathbb{R}^{\Gamma}$ is a neighborhood of p where L depends on only a finite number of variables.

Let h_1, h_2, \ldots, h_n and f_{λ} be any $(n + \Gamma)$ -tuplet of smooth functions in k variables. The family (indexed by Γ) $l_{\lambda}(h_1, \ldots, h_n)f_{\lambda}$ is a locally finite family in $C^{\infty}(\mathbb{R}^k)$, and it follows from Proposition 12 (with $k = \Gamma$) that the action of L for the ${}_{\infty}C^{\infty}$ -ring structure of $C^{\infty}(\mathbb{R}^k)$ is given by the formula

$$L(h_1, \dots, h_n, f_{\lambda}) = \sum_{\lambda} l_{\lambda}(h_1, \dots, h_n) f_{\lambda}$$

We have

15. PROPOSITION. Let l_{λ} and L be as in Proposition 14.

i) Given any n-tuple $h_1, \ldots, h_n \in C^{\infty}(\mathbb{R}^k)$, $L(h_1, \ldots, h_n, 0) = 0$.

ii) Given any ${}_{\infty}C^{\infty}$ -ideal $l \in C^{\infty}(\mathbb{R}^{k})$ and any $n + \Gamma$ -tuple $h_{1},..., h_{n}, f_{\lambda} \in C^{\infty}(\mathbb{R}^{n})$, if $f_{\lambda} \in l$ for all λ , then $\sum_{\lambda} l_{\lambda}(h_{1},...,h_{n})f_{\lambda} \in l$.

iii) Given any ${}_{\infty}C^{\infty}$ -ideal $l \in C^{\infty}(\mathbb{R}^n)$ and any Γ -tuple $f_{\lambda} \in C^{\infty}(\mathbb{R}^n)$ if $f_{\lambda} \in I$ for all λ , then $\sum_{\lambda} l_{\lambda} f_{\lambda} \in I$.

iv) Given any ${}_{\infty}C^{\infty}$ -ideal $I \subset C^{\infty}(\mathbb{R}^n)$ and any locally finite family $f_{\lambda} \in C^{\infty}(\mathbb{R}^n)$, if $f_{\lambda} \in I$ for all λ , then $\sum_{\lambda} f_{\lambda} \in I$.

PROOF. i is clear, and ii follows clearly from i. We get iii by putting

k = n and $h_i = \pi_i$ the projections.

Finally, observe that given any open set $U \,\subset R^n$ and any smooth function f with $Supp(f) \subset U$, then there exists a function l with $supp(l) \subset U$ and such that f = fl. To prove iv, we apply this observation to each of the functions f_{λ} . The family l_{λ} so obtained is also locally finite; thus it has an associated operation L. iv follows then from iii.

16. COROLLARY. Let A be a ${}_{\infty}C^{\infty}$ -ring of finite type. Then A is a C^{∞} -ring presented by an ideal of local character.

PROOF. Immediate from iv in the previous proposition and Definition 6, iv.

Thus, the ideals of local character are exactly the congruences for

E. DUBUC 14

the infinitary theory of C^{∞} -rings. Remark that we have also proved that given any locally finite family f_{λ} in $C^{\infty}(\mathbb{R}^n)$, $\lambda \in \Gamma$, there is a $(n + \Gamma)$ -ary operation L such that

$$L(\pi_1,\ldots,\pi_n,f_\lambda) = \sum_\lambda f_\lambda,$$

where π_i are the projections. Suppose now l is an ideal of local character, let $A = C^{\infty}(\mathbb{R}^n)/l$, let $e_i = [\pi_i]$ be the generators of A, and let

$$b_{\lambda} = f_{\lambda}(e_{I}, \dots, e_{n}) = [f_{\lambda}].$$

Then, by Definition-Proposition 10, b_{λ} is a locally finite family, and $\left[\sum_{\lambda} f_{\lambda}\right] = \sum_{\lambda} b_{\lambda}$. But since *l* is a ${}_{\infty}C^{\infty}$ -ideal, we also have

$$[L(\pi_1,\ldots,\pi_n,f_{\lambda})] = L(e_1,\ldots,e_n,b_{\lambda}).$$

Thus, given any locally finite family $b_{\lambda} = [f_{\lambda}]$ in a C^{∞}-ring A presented by an ideal of local character, $A = C^{\infty}(\mathbb{R}^n)/I$, if f_{λ} is locally finite in $C^{\infty}(\mathbb{R}^n)$, there is an infinitary operation L such that

$$L(e_1,\ldots,e_n,b_\lambda) = \sum_{\lambda} b_{\lambda},$$

where e_1, \ldots, e_n are generators of A. With this we finish the answer to Question I.

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