CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

R. F. C. WALTERS

Sheaves and Cauchy-complete categories

Cahiers de topologie et géométrie différentielle catégoriques, tome 22, nº 3 (1981), p. 283-286

<http://www.numdam.org/item?id=CTGDC_1981__22_3_283_0>

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE
Vol. XXII-3 (1981)

3e COLLOQUE SUR LES CATEGORIES
DEDIE A CHARLES EHRESMANN
Amiens, Juillet 1980

SHEAVES AND CAUCHY-COMPLETE CATEGORIES

by R. F. C. WALTERS

I want to consider the point of view (see [2, 4]) that sheaves are sets with a generalized equality, in the context of enriched category theory (see [3]), where such structures as metric spaces and additive categories are regarded as categories with a generalized hom-functor. In this context sheaves on a locale H turn out to be precisely *symmetric Cauchy-complete B-categories* for a suitable bicategory B constructed out of H.

This idea arose in conversations with Stefano Kasangian and Renato Betti in Milan. The necessary B-category theory was developed with Betti. I present here only the basic idea; developments will appear elsewhere.

1. CATEGORIES BASED ON A BICATEGORY (see [1])

The theory of categories with hom taking values in a bicategory, rather than a monoidal category (= bicategory with one object) seems to be very little developed. I have only some unpublished notes of R. Betti. However, most of what we need for this lecture is a simple translation of [3]. For our application we need only consider the case where the base bicategory B is locally partially-ordered; i.e., $B(a, b)$ is a poset for all a, b in B. We need also to assume that all these posets are co-complete and that suprema are preserved by composition in B.

DEFINITIONS. A *B-category* X is a set X with a function $e: X \to obj.B$ and a function $d: X \times X \to morph.B$ satisfying:

(i) $d(x_1, x_2): e(x_1) \to e(x_2)$,

(ii) $1_{e(x)} \leq d(x, x)$,

(iii) $d(x_2, x_3) . d(x_1, x_2) \leq d(x_1, x_3)$.

(Draw a picture: X is a space lying over B.)

A *B-functor* f from X to Y is a function $f: X \to Y$ satisfying:

(i) $e(f(x)) = e(x)$,

(ii) $d(x_1, x_2) \leq d(fx_1, fx_2)$.

EXAMPLE. Let H be a locale. Form a bicategory B from H as follows:

objects of B: opens u in H,

arrows from u to v : elements $w \leq u \wedge v$,

2-cells: order in H,

composition of arrows: intersection.

Notice that $B = Relations(H)$.

From a sheaf F on H we can form a B-category $L(F)$ as follows:

$L(F)$ = set of partial sections of F,

$e: L(F) \to obj. B: \; s \mapsto domain\ of\ s$,

$d: L(F) \times L(F) \to morph. B: \; (s, t) \mapsto \vee \{u;\ s|u = t|u\}$.

Notice that $L(F)$ has the property that if

$$s, t \in L(F) \quad \text{and} \quad d(s, t) = e(s) = e(t),$$

then $s = t$. Call such a B-category *skeletal*.

Notice that the bicategory $B = Span(H)$ of this example has the property that B^{op} (arrows reversed) $= B$. This property allows us to say that a B-category X *is symmetric* if

$$d(x_1, x_2) = d(x_2, x_1) \quad \text{for all} \quad x_1, x_2 \in X.$$

Clearly $L(F)$ is symmetric and in fact L is a fully-faithful functor

$$L: Sheaves(H) \to skeletal\ symmetric\ B\text{-}categories.$$

2. CAUCHY-COMPLETENESS

To express Lawvere's notion of Cauchy-completeness we need to define bimodules. A *bimodule* ϕ from X to Y (denoted $\phi: X \longrightarrow\!\!\!\!\!\!\!+\;\; Y$) is a function $\phi: X \times Y \to morph.\ B$ satisfying (for all $x, x' \in X$, $y, y' \in Y$)

(i) $\phi(x, y): e(x) \to e(y)$,

(ii) $\phi(x, y) . d(x', x) \leq \phi(x', y)$,

(iii) $d(y, y') . \phi(x, y) \leq \phi(x, y')$.

As usual a B-functor $f: X \to Y$ yields a pair of bimodules

$$f^*: X \nrightarrow Y \quad \text{and} \quad f_*: Y \nrightarrow X$$

defined by

$$f^*(x,y) = d(fx,y) \quad \text{and} \quad f_*(y,x) = d(y,fx).$$

Further f^* and f_* are *adjoint* in the sense that

(i) $d(x,x') \leq \exists y\,[f_*(y,x').f^*(x,y)]$

(where we write $\exists y$ for the supremum (over y) in $B(x,x')$) and

(ii) $\exists x\,[f^*(x,y').f_*(y,x)] \leq d(y,y')$.

Then a B-category Y *is Cauchy-complete* if every adjoint pair of bimodules $\phi, \psi : X \rightleftarrows Y$ arises from a functor $X \to Y$.

3. SHEAVES

We now have the definitions required to state the result.

THEOREM. *If H is a locale, then Sheaves(H) is equivalent to the category of skeletal symmetric Cauchy-complete Rel(H)-categories.*

PROOF. We want to see

(a) that L lands in Cauchy-complete B-categories, and

(b) that every *skeletal* Cauchy-complete symmetric B-category is isomorphic to $L(F)$ for some sheaf F.

For each element $u \in H$ we can define a B-category $\hat{u}$ with one element $*$ and with $e(*) = u$, $d(*,*) = u$. Then, in testing Cauchy-completeness of Y, we need only consider adjoint pairs of bimodules from $\hat{u}$ to Y for each $u \in H$.

To prove (a) consider an adjoint pair of bimodules $\phi(s), \psi(s)$ ($s \in L(F)$) from $\hat{u}$ to F. Then condition (i) of adjointness says that: $u_s = \phi(s) \wedge \psi(s)$ ($s \in L(F)$) is a cover of u. Condition (ii) says that $s|u_s$ ($s \in L(F)$) is a compatible family of sections, and so there is a section $s_0 \in F(u)$ such that

$$s_0 | u_s = s | u_s \quad \text{for all } s \in L(F).$$

Now it is clear that for a general s,

$$d(s_0, s) = \bigvee_t d(s, t|u_t).$$

From property (ii) of adjunction:

$$\phi(s) \wedge \psi(t) \wedge \phi(t) \leq d(s,t) \leq d(s,t \mid u_t)$$

and so by (i)

$$\phi(s) \leq \bigvee_t d(s,t \mid u_t) = d(s_0, s).$$

From property (iii) of bimodules

$$\phi(s) \geq d(s,t) \wedge \phi(t) \geq d(s,t \mid u_t), \text{ and so } \phi(s) \geq d(s_0, s).$$

Hence,

$$\phi(s) = \psi(s) = d(s_0, s).$$

That is, the pair of bimodules arises from a functor.

To prove (b) consider a *skeletal* Cauchy-complete symmetric B-category Y. We need to be able to define the restriction of an element y over u to $v \leq u$. But this restriction comes from the fact that the adjoint pair of bimodules

$$\phi(y') = \psi(y') = v \wedge d(y, y') : \hat{v} \rightleftarrows Y$$

is given by a functor. We need also to have the glueing together of a compatible family of elements $(y_\alpha)_\alpha$ with $\bigvee_\alpha e(y_\alpha) = u$. In this case the required section comes from the representation of the bimodules

$$\phi(y') = \psi(y') = \bigvee_\alpha d(y_\alpha, y') : \hat{u} \rightleftarrows Y$$

as a functor.

REFERENCES

1. J. BENABOU, Introduction to bicategories, *Lecture Notes in Math.* 47, Springer (1967), 1-77.

2. D. HIGGS, A category approach to boolean-valued set theory, unpub. manus.

3. F. W. LAWVERE, Metric spaces, generalized logic and closed categories, *Rend. Sem. Mat. e Fis. di Milano* 43 (1974), 135-166.

4. M.P. FOURMAN & D.S. SCOTT, Sheaves and Logic, *Lecture Notes in Math.* 753, Springer (1979).

Pure Mathematics Department
University of Sydney
SYDNEY. AUSTRALIE