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A BOREL TOPOS

by Donovan H. VAN OSDOL

INTRODUCTION.

Workers in the field of functional analysis concern themselves primarily, if not completely, with subsets of complete separable metric spaces. The collection of open and closed subsets does not afford a sufficiently general class of subsets however, since it is not closed under countable unions or countable intersections. This observation leads naturally to consideration of the collection of Borel subsets (the smallest σ-algebra containing all open subsets). This collection serves well for many applications, but also falls short of our expectations: the continuous image of a Borel subset need not be a Borel subset. Those subsets (with the subspace topology) which are the continuous image of a Borel subset are called analytic spaces. The collection of analytic spaces is not closed with respect to the function-space construction (for example the space $C(N^\infty)$ of continuous, real-valued functions on a countable product of the natural numbers, endowed with the topology of compact convergence, is not analytic [2, Cor. Page 12]).

It is the purpose of this paper to suggest a category which «contains» the standard Borel spaces (Borel subsets of some separable complete metric space), to which all Borel spaces map nicely, and which is closed under the usual set-theoretic constructions. This last requirement means that our category should be an elementary topos in the sense of Lawvere and Tierney (a general reference for which is [3]); that is, the category should be an intuitionistic model of set theory. We will define the category and outline the state of our knowledge about it. The research is still at a rudimentary stage, and no new results in functional analysis are presented here.

1. THE DEFINITION AND COMPARISON.

Let $\mathcal{S}$ be the category whose objects are the Borel subsets of the unit interval $[0, 1]$ and whose morphisms are inclusion functions $f: X \to Y$. It is a fact that any standard Borel space is Borel-isomorphic to an object in $\mathcal{S}$; see, e.g., [4]. We define a pretopology [1] on $\mathcal{S}$ as follows: for X in $\mathcal{S}$, $Cov(X)$ is the set of all countable covering families

$$\{ f_i: Y_i \to X \mid \cup f_i[Y_i] = X, \ i \text{ is in } N \}$$

in $\mathcal{S}$. It is obvious that these families of covers form a pretopology on $\mathcal{S}$, and hence that $(\mathcal{S}, Cov)$ defines a site.

Let $\mathcal{E}$ be the category of sheaves of sets on the site $(\mathcal{S}, Cov)$ [1]. An object of $\mathcal{E}$ is thus a contravariant functor $F: \mathcal{S} \to \mathcal{S}ets$ such that for each $\{ Y_i \}$ in $Cov(X)$ the obvious diagram

$$F(X) \longrightarrow \Pi F(Y_i) \rightrightarrows \Pi F(Y_i \cap Y_j)$$

is an equalizer; a morphism $\phi: F \to G$ in $\mathcal{E}$ is simply a natural transformation of functors. This category $\mathcal{E}$ is then a topos in the sense of [1], and in particular is an elementary topos in the sense of Lawvere and Tierney [3]. It is our candidate for a «good» category in which to do functional analysis.

Let $\mathcal{B}$ be the category of all Borel spaces and Borel maps, let $i: \mathcal{S} \to \mathcal{B}$ be the inclusion, and let $\Phi : \mathcal{B} \to \mathcal{E}$ be the functor defined by

$$\Phi(X) = \mathcal{B}(i-, X).$$

That Φ actually has its values in $\mathcal{E}$ is clear: for $\{ Y_i \}$ in $Cov(Z)$,

$$\mathcal{B}(Z, X) \longrightarrow \Pi \mathcal{B}(Y_i, X) \rightrightarrows \Pi \mathcal{B}(Y_i \cap Y_j, X)$$

is an equalizer (because the covers are restricted to be countable). The full subcategory of $\mathcal{B}$ determined by the standard Borel spaces is mapped fully and faithfully by Φ to $\mathcal{E}$; use the usual Yoneda-type argument. We will see below that Φ, and Φ restricted to standard Borel spaces, preserves all limits.

There is a kind of geometric realization $\Psi : \mathcal{E} \to \mathcal{B}$, by which we mean simply that $\Phi : \mathcal{B} \to \mathcal{E}$ has a left adjoint. To see this, notice first that $\mathcal{B}$ has coequalizers and coproducts. If T is a set and X is in $\mathcal{B}$, let $T \cdot X$ be the Borel space whose underlying set is $T \times X$, and whose Borel structure is the σ-algebra

$$\{ \bigcup_{t \in T} \{t\} \times B_t \mid B_t \text{ is a Borel set in } X \}.$$

Then for F in $\mathcal{E}$, $\Psi(F)$ is defined by the coequalizer diagram:

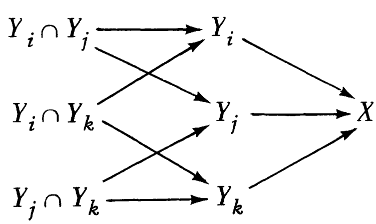
$$\coprod_{\substack{f : X \to Y \\ \text{in } \mathcal{S}}} F(Y) \cdot X \rightrightarrows \coprod_{Y} F(Y) \cdot Y \longrightarrow \Psi(F)$$

where the f-th injection of the top (resp. bottom) map is the Y-th injection following $F(Y) \cdot f$ (resp. the X-th injection following $F(f) \cdot X$). That this definition of Ψ on objects extends to morphisms is clear, thus giving a functor $\Psi : \mathcal{E} \to \mathcal{B}$. The verification that Ψ is a left adjoint of Φ can be safely left to the reader. Notice that $\epsilon : \Psi \circ \Phi \to \mathcal{B}$ is a surjection since it is induced by evaluation.

It follows that $\Phi : \mathcal{B} \to \mathcal{E}$ preserves limits; it also preserves some colimits (exactly which, it is not clear). For example, suppose

$$\{ Y_i \mid i \text{ is in } N \}$$

is a countable family of Borel subsets of the Borel space X whose union is X. One can easily see that



is a colimit diagram (the arrows are simply inclusions) in $\mathcal{B}$. Now Φ preserves this colimit. For given a compatible family $\phi_i : \Phi(Y_i) \to F$ in $\mathcal{E}$

$(0 \leq i < \infty)$, we can define $\psi : \Phi(X) \to F$ as follows. For B in $\mathcal{S}$ and f in

$$\Phi(X)(B) = \mathfrak{B}(B, X),$$

first let $B_i = f^{-1}[Y_i]$ and let $f_i : B_i \to Y_i$ be the restriction of f to B_i. Notice that

$$\{ B_i \to B \mid i \text{ is in } N \}$$

is in $Cov(B)$, and that the sequence

$$< \phi_i B_i(f_i) \mid i \text{ is in } N >$$

is in $\Pi F(B_i)$; a check of the two images of this sequence in $\Pi F(B_i \cap B_j)$ shows them to be equal. Thus since F is a sheaf, there is a unique $\psi B(f)$ in $F(B)$ whose restriction to $F(B_i)$ is $\phi_i B_i(f_i)$, $0 \leq i < \infty$. The composite

$$\Phi(Y_i) \longrightarrow \Phi(X) \xrightarrow{\psi} F$$

is clearly ϕ_i, and both naturality and uniqueness of ψ are easily proved.

One can think of $\mathcal{S}$ as the category of «models» for $\mathfrak{B}$, much as the standard simplices are the models for CW-complexes in topology. Then $\Phi : \mathfrak{B} \to \mathcal{E}$ can be thought of as the singular-simplex functor, and $\Psi : \mathcal{E} \to \mathfrak{B}$ as geometric realization. It is not known whether pushing this analogy further will yield interesting results in functional analysis (or topos theory).

II. SPECIAL OBJECTS.

Having devoted some attention to the comparison $\Phi : \mathfrak{B} \to \mathcal{E}$ and the embedding of standard Borel spaces into $\mathcal{E}$ induced by Φ, we turn now to $\mathcal{E}$ itself. The terminal object 1 in $\mathcal{E}$ is simply the constant sheaf whose value at any X in $\mathcal{S}$ is a one point set. The «truth values» in $\mathcal{E}$ are the subobjects of 1 ; it is important to have alternate descriptions of the truth values, so we examine that first.

Let U be a subobject of 1 in $\mathcal{E}$. Then U is completely determined by the set of all X in $\mathcal{S}$ with $U(X) = 1$; let us denote this set by $OB(U)$; since U is a functor, if X is in $OB(U)$ and Y is a Borel subset of X, then Y is in $OB(U)$. Since U is a sheaf, if X is in $OB(U)$ and Y_0, Y_1,...

are Borel subsets of $[0,1]$ whose union is X then Y_i is in $OB(U)$ for $0 \leq i < \infty$. A set A of objects in $\mathcal{S}$ which satisfies these two conditions will be called a saturated system. We claim that the saturated systems are in one-to-one correspondence with subobjects of 1 via OB. To see this, suppose A is a saturated system and define $TV(A)\colon \mathcal{S}^{op} \to \mathcal{S}ets$ by

$$TV(A)(X) = 1 \text{ if } X \text{ is in } A, \ \emptyset \text{ otherwise.}$$

It is readily verified that $TV(A)$ is a sheaf, and that it is a subobject of 1 in $\mathcal{E}$. Clearly OB and TV are inverse functions. Now any subset M of $[0,1]$ uniquely determines a truth value; let A be the saturated system of all Borel sets in $[0,1]$ which are contained in M. If $M' \neq M$ is a subset of $[0,1]$ and A' its associated saturated system, then $A \neq A'$ (since points are Borel sets) and hence $TV(A) \neq TV(A')$. However there are truth values which do not arise from subsets of $[0,1]$ in this way: for example, the set of all countable subsets of the irrational numbers in $[0,1]$ is a saturated system not induced by a subset of $[0,1]$ as above.

It is important to understand Ω, the subobject classifier in $\mathcal{E}$. A general reference for this is [3]. For X in $\mathcal{S}$, a sieve R on X is a set of Borel subsets of X with the property that if Y is a Borel set in $[0,1]$ and Y is contained in some member of R then Y itself is in R (i.e., a sieve on X is a subfunctor of $\mathcal{S}(-,X)$). For a sieve R on X, let $j(R)$ be the sieve

$$\{ Z \mid Z \text{ is a Borel subset of } X \text{ and } \{ Z \cap Y \mid Y \in R \} \text{ contains a member of } Cov(Z) \}.$$

That is, $j(R)$ consists of all Borel subsets of X which are (countably) covered by the restriction of R to them. Then $\Omega(X)$ is the set of fixed points of j; that is,

$$\Omega(X) = \{ R \mid R \text{ is a sieve on } X, \text{ and } \{ Z \cap Y \mid Y \in R \} \text{ contains a countable cover of } Z \text{ implies } Z \text{ is in } R \}.$$

For example, the set R of all countable subsets of the Borel set X is in $\Omega(X)$. The restriction map $\Omega(X) \to \Omega(Y)$ is given by mapping R to

$$\{ Z \cap Y \mid Z \text{ is in } R \}.$$

We now want to show that $\mathcal{E}$ is not a Boolean topos, and hence the set theory it gives is not classical. The initial object 0 in $\mathcal{E}$ is classified by $false: 1 \to \Omega$, where $false\ X: 1(X) \to \Omega(X)$ takes the unique element of $1(X)$ to the sieve $\{\emptyset\}$. The negation operator $not: \Omega \to \Omega$ is by definition the classifying map of $false$, so that $not\ X: \Omega(X) \to \Omega(X)$ takes the sieve R to the sieve

$$\{\ Y \mid Y \text{ is a Borel subset of } X \text{ and } Y \cap Z = \emptyset \text{ for all } Z \text{ in } R\ \}.$$

To see that $not \circ not \neq$ the identity on Ω, let $R \in \Omega([0,1])$ be the sieve of all countable sets in $[0,1]$. Then $not[0,1](R) = \{\emptyset\}$, and

$$not[0,1] \circ not[0,1](R) = \{\ X \mid X \text{ is a Borel subset of } [0,1]\ \}.$$

Finally, in any Grothendieck topos (such as $\mathcal{E}$) the natural-number object N is the sheaf associated to the constant presheaf whose value is the natural numbers N (or equally well, N is a countable coproduct of 1). It is essentially obvious that in $\mathcal{E}$ this associated sheaf is

$$\Phi(N) = \mathcal{B}(-, N),$$

i.e. the «locally» constant N-valued functions. The rational-number object in $\mathcal{E}$ is similarly $\mathcal{B}(-, Q)$. But an alternate description of the (Dedekind) real-number object in $\mathcal{E}$ is still not known; it could well be $\mathcal{B}(-, R)$ also. This, and many other, questions will have to be answered before further progress can be made.

1. A. GROTHENDIECK & J.L. VERDIER, Théorie des topos (SGA 4, Exposés I-VI), 2nd edition, *Lecture Notes in Math.* 269-270, Springer (1972).

2. J.P.R. CHRISTENSEN, *Topology and Borel structure*, North-Holland Notas Mat. 51, 1974.

3. P.T. JOHNSTONE, *Topos Theory*, Academic Press, L.M.S. Monographs 10, 1977.

4. G.W. MACKEY, *Unitary group representations in Physics, Probability, and Number theory*, Benjamin/Cummings Math. Lecture Notes Series 55, 1978.

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